Exercises for T2, Summer term 2019, Sheet 3

1) Natural units

Express the following quantities in units of eV (with respective power) by using natural units $\hbar = c = k = 1$: typical atomic radius (1 $\AA$), typical radius of nucleons (1 fm), Compton wavelength of the electron, gravitational acceleration g on the surface of the earth, temperature inside the ITER Tokamak that is necessary to initiate nuclear fusion. Use the internet to find numbers that you dont know by heart.

2) Bohr's atomic model

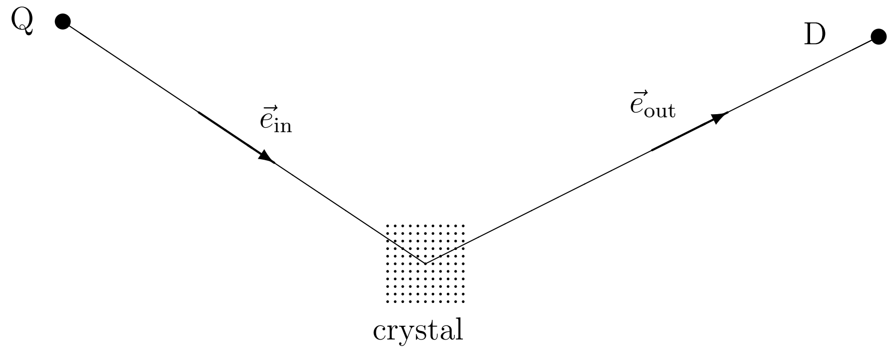
One can derive the quantised binding energy levels of the hydrogen atom $E_{\text{bind,n}} = -e^4 m_e/(2\hbar^2 n^2)$, $n = 1, 2, 3, \ldots$, (valid in the non-relativistic approximation) from Bohr's atomic model. The model uses de Broglie's postulate that particles have wave character where their wave length λ is related to the particles momentum $\vec{\mathbf{p}} = m\vec{\mathbf{v}}$ through the relation $\lambda = h/|\vec{\mathbf{p}}|$ and assumes in addition that the electron travels only on certain allowed, but otherwise classical orbits around the proton. Do this calculations once yourself.

(a) Assume that the potential energy between electron and proton (which you assume to have infinite mass) has the form $E_{\text{pot}} = -e^2/r$, where r is the electrons distance from the proton and e is the proton (or minus the electron) electric charge. Assume that the Coulomb attractive force acts as the centripetal force which keeps the electron with momentum $\vec{\mathbf{p}}$ on a circular orbit with radius r and determine from this a relation between r and the electron momentum $\vec{\mathbf{p}}$.

(b) Use de Broglie's postulate together with the argument that only n complete (i.e. n times 360^0 or 2π) oscillations, $n = 1, 2, 3, \ldots$, can fit onto the circumference of an orbit to derive the expression for the possible binding energies $E_{\text{bind}} = E_{\text{kin}} + E_{\text{pot}}$. Determine the allowed radii of the electron orbits and the corresponding electron angular momenta along the rotation axis.

(d) Discuss the limitations of the thoughts on which the calculations is based. Is this all conclusive?

3) Scattering of neutrons



The atoms of a crystal lattice are located at the points $\vec{x}_{\vec{n}} = a\vec{n}$, $\vec{n} \in \mathbb{Z}^3$, $n_i = -N, -N+1, \ldots, N$ $(i = 1, 2, 3)$. Furthermore there is a source Q ($R_{\rm Q} \gg Na$) at the point $\vec{x}_{\rm Q} = -R_{\rm Q}\,\vec{e}_{\rm in}$ ($|\vec{e}_{\rm in}| = 1$) which emits neutrons with momentum p. In addition a neutron-detector ($R_{\rm D} \gg Na$) is located at $\vec{x}_{\rm D} = R_{\rm D}\,\vec{e}_{\rm out}$ ($|\vec{e}_{\rm out}| = 1$). The amplitude $\langle \text{D out}|\text{Q in}\rangle$, that the detector D detects a neutron which originated from Q is of the form

$$\langle \text{D out}|\text{Q in}\rangle \sim \sum_{\vec{n}} \frac{e^{ip|\vec{x}_{\rm Q}-\vec{x}_{\vec{n}}|/\hbar}}{|\vec{x}_{\rm Q}-\vec{x}_{\vec{n}}|} W_{\vec{n}} \frac{e^{ip|\vec{x}_{\rm D}-\vec{x}_{\vec{n}}|/\hbar}}{|\vec{x}_{\rm D}-\vec{x}_{\vec{n}}|}.$$

(multiple scattering is omitted). Calculate this expression assuming that $W_{\vec{n}}$ is the same for all atoms. Use a suitable approximation for $|\vec{x}_{\rm Q} - \vec{x}_{\vec{n}}| = \sqrt{(\vec{x}_{\rm Q} - \vec{x}_{\vec{n}})^2}$ which should refect that $|\vec{x}_{\rm Q}| = R_{\rm Q} \gg |\vec{x}_{\vec{n}}|$. (analogous for $|\vec{x}_{\rm D} - \vec{x}_{\vec{n}}|$.) By following that strategy you should get an expression like

$$\langle \text{D out}|\text{Q in}\rangle \sim \sum_{\vec{n}} e^{ipa\vec{\Delta}\cdot\vec{n}/\hbar} = \prod_{i=1}^{3} \sum_{n_i=-N}^{N} e^{ipa\Delta_i n_i/\hbar}, \quad \vec{\Delta} = \vec{e}_{\rm in} - \vec{e}_{\rm out}$$

This means it is necessary to calculate a geometric series of the type

$$s(\alpha) = \sum_{n=-N}^{N} e^{i\alpha n}$$

Now show that this equals

$$s(\alpha) = \frac{\sin \alpha(N+\frac{1}{2})}{\sin\frac{\alpha}{2}}$$

Discuss the behavior of this function. For which values of α are there distinct extrema? Show that in the case of neutron scattering

$$\frac{pa}{2\pi\hbar}\vec{\Delta} = \vec{\nu} \in \mathbb{Z}^3$$

leads to distinct maxima of the interference pattern. This is Laue's condition for interference for the simple cubic lattice. It is also possible to write it in the form

$$\frac{a}{2\pi}\left(\vec{k}_{\rm in} - \vec{k}_{\rm out}\right) = \vec{\nu} \in \mathbb{Z}^3$$