Quantum Mechanics (T2) Lecture Exam Question Sheet 2019

The following problems show you the type of problems you can expect for the lecture exam. In the exam you have to work on 5 different problems.

For the lecture exam you are NOT allowed to use any helping equipments.

1) Axioms of quantum theory I

Consider the case of a *finite-dimensional* space of states $\mathcal{H}$. Here the space of states $\mathcal{H}$ is a unitary vector space, i.e. a vector space over the field of complex numbers, equipped with a scalar product. Use bra and ket notation when suitable.

1. What are the defining properties of a scalar product?
2. How can pure states and linear operators A be represented assuming that $\mathcal{H}$ is n-dimensional? What is the explicit form of the scalar product of two states ? What is the definition of the adjoint operator $A^\dagger$ to a given linear operator A?
3. How are (i) Hermitian operators, (ii) projection operators, (iii) unitary operators defined?
4. How is an observable (observable quantity) represented in the mathematical formalism of quantum theory? What is the correspondence principle?
5. What are the mathematical equivalents of the possible measurement values of an observable?
6. What are the eigenvalues of a projection operator?
7. Write down the *general* mathematical definition of a state in quantum theory.

2) Axioms of quantum theory II

Consider the case of a *finite-dimensional* space of states $\mathcal{H}$. Here the space of states $\mathcal{H}$ is a unitary vector space, i.e. a vector space over the field of complex numbers, equipped with a scalar product. Use bra and ket notation when suitable.

1. Write down the *general* mathematical definition of a state in quantum theory.
2. What is a pure state? What is a mixed state? Which type of state is more general?
3. What is a complete orthonormal system?
4. What is the "spectral representation" of an observable A.
5. Explain the meaning of the "completeness relation".
6. The system shall be in a pure normalized state $|\phi\rangle$ and one makes a measurement of observable A. Assume that all eigenvalues of A are non-degenerate. What kind of outcome can you expect for a single measurement?

7. In which state is the system after the measurement?
8. What is the mean value for many measurements on identical copies of the system in the state $|\phi\rangle$? What is the standard deviation for these measurements?
9. What is the Hamilton operator? How does the state $|\phi(t=0)\rangle\rangle$ given at time $t=0$ evolve in time? What is the time evolution operator? Assume that the Hamiltonoperator is time-independent.

3) Axioms of quantum theory III

Consider the case of a *finite-dimensional* space of states $\mathcal{H}$. Here the space of states $\mathcal{H}$ is a unitary vector space, i.e. a vector space over the field of complex numbers, equipped with a scalar product. Use bra and ket notation when suitable.

1. What are the defining properties of a density operator? What is its physical meaning?
2. What additional property does the density operator have in case of a pure state?
3. Write down the form of the density operators for a state with maximal mixing.
4. Write down the form of the density operator for the state if the system is in thermodynamical equilibrium with an external heat bath with absolute temperature T.
5. Explain the content of the Neumann's Projection Theorem. How does the density operator of the system change after the measurement of a general observable A?
6. Explain "Schrödinger Picture" and "Heisenberg Picture" for a system where the Hamilton operator is time-independent.
7. How does the von-Neumann equation for the time-evolution $\rho(t)$ of a density operator in the Schr"odinger picture look like? ($\dot{\rho}(t) = \ldots ?$) Also write down the relation between $\rho(t)$ and $\rho(0)$ in case of a time-independent Hamilton operator.
8. In the Heisenberg picture an observable is represented by a time-dependent operator $A(t)$. How does the Heisenberg equation look like? Also write down the relation between $A(t)$ and $A(0)$ in case of a time-independent Hamilton operator.

4) Uncertainty relations

1. Write down the *general* mathematical definition of a state ω in quantum theory.
2. Let there be a Hermitian operator A and a state ω. What is the definition of the mean square deviation of A in the state ω?
3. What is the general definition of the uncertainty relation for two hermitian operators A, B, if the system is in the state ω?

4. Write down the uncertainty relation for the special case of position and momentum operators in one spatial dimension.
5. Verify the result of part 4 by evaluating the formula from part 3 concretely.
6. The Hamilton operator of the one-dimensional harmonic oscillator is

$$H = \frac{P^2}{2m} + \frac{m\nu^2}{2} X^2 \,,$$

where ν is the frequency. Determine the ground state energy of the harmonic oscillator by means of the uncertainty relation for X and P. You can assume that the ground state is a state with minimal uncertainty.

5) Calculations with observables

Let $\mathcal{H} = \mathbb{C}^3$ with the usual scalar product $\langle \chi | \psi \rangle = \sum_{k=1}^{3} \chi_k^* \psi_k$. Furthermore let a certain observable be represented by the matrix

$$A = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 1/2 \end{pmatrix} \qquad \left[|\phi\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right] \qquad \text{[examples for illustration !]}$$

1. What are the possible values a_1, a_2, a_3 for a measurement of this observable?
2. Determine the corresponding orthonormalized eigenvectors e_1, e_2, e_3.
3. Determine the probability for the measurement of the values a_1, a_2, a_3 when the system is in the state $|\phi\rangle$.
4. Determine the projectors P_1, P_2, P_3 onto the eigenspaces that correspond to the different eigenvalues.
5. Determine the spectral representation of A.
6. For every possible measurement value a_1, a_2, a_3 determine the probability to get this value, if the system is in a state described by the density operator $\rho = \mathbb{1}_3/3$. Determine the expectation value for measurement of A in this state.
7. Calculate the matrix $\exp(-i\alpha A)$ with α being a real number.

6) Particle at a potential step and in a trap

Consider a particle with mass m moving in one dimension in the potential ($V_0 > 0$!)

$$V(x) = \begin{cases} V_0 & x \geq 0 \\ 0 & x < 0 \end{cases} \,. \tag{1}$$

1. Write down the Hamilton operator and the Schrödinger equation for the particles wave function. Write down the Schrödinger equation for the wave function of a stationary state.

2. Use the stationary Schrödinger equation to determine what boundary conditions the wave function needs to satisfy.

3. Determine all solutions of the stationary Schrödinger equation satisfying the boundary conditions. Which different physical cases have to be distinguished? Can the solutions for the stationaly states be normalized?

4. Write down the definition for the probability current for a stationary state and show that the results for the probability current are independent of the position x for positive as well as negative x. Determine the probability current for $x < 0$ and $x > 0$ for the case that the particle has an energy E with $0 < E < V_0$.

5. Consider the same particle in the modified potential of the form

$$V(x) = \begin{cases} V_0\,, & x \geq 0\,, \\ 0 & -a < x < 0 \\ \infty\,, & x < -a \end{cases}\,, \tag{2}$$

for $a > 0$. For what energy range is it in principle possible to obtain stationary states that are bound states, i.e. that can be normalized? Determine the energy eigenvalues for the possible bound states. What is the probability to detect the particle in the interval $x \in [-a, 0]$ for the sationary bound state with the smallest possible energy?

7) Particle in a Delta function potential

Let the wave function of a particle in one dimension be given by

$$\psi(x) = \mathcal{N}\exp(-a|x|) = \mathcal{N}(\, e^{ax}\theta(-x) + e^{-ax}\theta(x)\,), \quad a > 0.$$

(The function $\theta(x)$ is the Heaviside step function.)

1. Calculate the norm $\mathcal{N}$ of the wave function.

2. Calculate the first and second derivative of $\psi(x)$.

3. Show that the wave function $\psi(x)$ is, for a suitable choice of the parameter a, an energy eigenfunction of the Hamilton operator

$$H = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} - \lambda\,\delta(x), \quad (\lambda > 0)$$

What is the necessary choice for a and what is the result for the energy eigenvalue E?

4. How is the probability current for a given wave function $\Phi(x,t)$ defined? Determine the probability current for the energy eigenfunction $\psi(x,t)$ from part 3 for $|x| > 0$ and argue that the wave function has to be interpreted as a bound state.

5. Assume the particle experiencing the delta-potential is in the state with the wave function

$$\phi(x) = \mathcal{N}' \begin{cases} 1 + x\,, & -1 < x \leq 0 \\ 1 - x\,, & 0 < x \leq 1 \\ 0\,, & \text{elsewhere} \end{cases}\,.$$

Determine the average energy obtained in many repeated energy measurements on the system in this state.

6. What is the probability to receive the energy value E from part 3 in an energy measurement? Hint: The relation $\pm\, x\, e^{\pm ax} = \frac{d}{da} e^{\pm ax}$ is quite useful for the calculation of the integrals.

8) Hydrogen atom

Given is the position space wave function $\psi_{200}(\vec{x})$ in a hydrogen atom in a state with quantum numbers $(n, l, m) = (2, 0, 0)$:

$$\psi_{200}(\vec{x}) = R_{20}(r)\, Y_{00}(\theta, \phi),$$

$$R_{20}(r) = \mathcal{N}\,(1 - r/2a)\, \exp(-r/2a), \quad Y_{00}(\theta, \phi) = \sqrt{\frac{1}{4\pi}},$$

where $r = |\vec{x}|$ is the distance to the nucleus, $R_{20}(r)$ the radial wave function, $Y_{00}(\theta, \phi)$ a spherical harmonic function, $a = \hbar/m_e \alpha c$ the Bohr radius, m_e the mass of the electron, and α the fine-structure constant.

1. What is the unit (or dimension) of the parameter a?
2. Calculate the normalization constant $\mathcal{N}$.
3. Draw the radial wave function $R_{20}(r)$ and the squared radial wave function $|R_{20}(r)|^2$ in a sketch as a function of the radius r, including all relevant assignments on the x-axis (also extremal points).
   Draw also a sketch of the radial probability distribution $r^2\, R_{20}(r)^2$ as a function of r (no calculation of extremal points needed here).
4. What is the unit (or dimension) of the normalized wave function $\psi_{200}(\vec{x})$?
5. Which unit (or dimension) does a general normalized wave function $\psi(\vec{x})$ in three dimensions have and why?
6. Calculate the expectation values of the radial position operator $|\vec{X}|$ and the Coulomb potential $V(\vec{X}) = -e^2/|\vec{X}|$.
7. Calculate the standard deviation $\Delta|\vec{X}|$.

Hints: $\int_0^\infty \mathrm{d}x\, x^n e^{-bx} = n!/b^{n+1}, \quad n \in \mathbb{N}_0, \quad b \in \mathbb{C}$ mit $\mathrm{Re}(b) > 0\,;$

$\vec{P}^2 = \frac{1}{r^2}\vec{L}^2 - \hbar^2\left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}\right)$ ($\vec{P}^2$-Operator in Kugelkoordinaten),

$\vec{L}^2 = -\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]$ ($\vec{L}^2$-Operator),

$\int \mathrm{d}^3\vec{x} = \int_0^\infty \mathrm{d}r\, r^2 \int_0^\pi \mathrm{d}\theta\, \sin(\theta) \int_0^{2\pi} \mathrm{d}\phi = \int_0^\infty \mathrm{d}r\, r^2 \int_{-1}^{1} \mathrm{d}\cos(\theta) \int_0^{2\pi} \mathrm{d}\phi$

9) Harmonic oscillator

The Hamilton operator of the one-dimensional harmonic oscillator is:

$$H = \frac{P^2}{2m} + \frac{m\omega^2}{2}X^2.$$

The usual ladder operators have the form $a = \alpha X + i\beta P$ and $a^\dagger = \alpha X - i\beta P$, where α, β are real numbers. The number operator is given by $N = a^\dagger a$.

1. Write down the possible energy eigenvalues of H.
2. What are the eigenvalues of the number operator N?
3. What is the commutation relation between a and $a^\dagger$? What condition for α and β follows?
4. It is possible to rewrite the Hamilton operator as $H = \hbar\omega(a^\dagger a + 1/2)$. Determine α and β from this form.
5. Calculate the commutator $[N, a^\dagger]$.
6. Let the state $|\phi\rangle$ be an energy eigenstate with eigenvalue E. Calculate the energy eigenvalue of $a^\dagger|\phi\rangle$.

10) Electrically charged harmonic oscillator

Consider a particle in one dimension (x-direction) with mass m and electric charge q. This particle is trapped inside a potential of the form $V_{\text{ho}}(x) = \frac{m\omega^2}{2}x^2$. In addition there is a homogeneous time-independent electric field E in the x-direction, so that there is an additional potential $V_{\text{E}}(x) = -qxE + c$, where c is a constant with the dimension of energy.

1. Write down the total potential energy $V_{\text{tot}}(x) = V_{\text{ho}}(x) + V_{\text{E}}(x)$ of the particle assuming that $V_{\text{tot}}(0) = 0$. Write down the Hamilton operator $H(X, P)$ for the system and the Schrödinger equation for the particles wave function.
2. Show that you can define a shifted position operator of the form $X' = X + b\mathbb{1}$, where b is a constant with dimenion of distance, such that the Hamilton operator $H(X', P)$ takes the form of a regular harmonic oscillator with an additional energy shift but without an electric field. Determine the constant b.
3. Which important relation do X and P satisfy? Does X' satisfy this relation as well? (Only then you can consider also X' being a location operator.
4. Give the general form of the normalized energy eigenfunctions $\tilde{\phi}_n(x') = \langle x|\tilde{n}\rangle$ and the corresponding energy eigenvalues $\tilde{E}_n$.
5. In the case that the electric field is very small, one can calculate the modifications of the eigenenergies and the wave function due to the electric field using time-independent perturbation theory. In time-independent perturbation theory on can then write the eigenenergies $\tilde{E}_n$ and eigenstates $|\tilde{n}\rangle$ as

$$\begin{aligned} |\tilde{n}\rangle &= |n^{(0)}\rangle + |n^{(1)}\rangle + |n^{(2)}\rangle + \dots\,, \\ \tilde{E}_n &= E_n^{(0)} + E_n^{(1)} + E_n^{(2)} + \dots\,, \end{aligned}$$

where the $E_n^{(0)}$ and $|n\rangle^{(0)}$ refer to the eigenenergies and eigenstates in the absence of the electric field (i.e. $V_{\rm tot}(x) = V_{\rm ho}(x)$) and the $E_n^{(i)}$ and $|n\rangle^{(i)}$ to the respective quantities accounting for the i-th order effect of the electric field. From time-independent perturbation theory one can derive the formulae

$$\begin{aligned} E_n^{(1)} &= \langle n^{(0)}|V_E|n^{(0)}\rangle\,, \qquad |n^{(1)}\rangle = \sum_{m\neq n} \frac{|m^0\rangle\langle m^{(0)}|}{E_n^{(0)} - E_m^{(0)}} V_E|n^{(0)}\rangle\,, \\ E_n^{(2)} &= \langle n^{(0)}|V_E|n^{(1)}\rangle\,. \end{aligned}$$

Determine $E_n^{(1)}$ and $\phi_n^{(1)}$ to first order in the electric field. Show that the result is consistent with the exact solution of part 4.

6. Determine $E_n^{(2)}$. Show that the result is compatible with the result given in part 4.

Hints: The normalized eigenfunctions of the regular harmonic oscillator with potential $V_{\rm ho}(x) = \frac{m\omega^2}{2}x^2$ without electric field are

$$\phi_n(x) = \langle x|n^{(0)}\rangle = (2^n n!\sqrt{\pi}\tilde{x})^{-1/2} \exp(-\tfrac{1}{2}(\tfrac{x}{\tilde{x}})^2) H_n(\tfrac{x}{\tilde{x}})\,,$$

where the H_n are the Hermite-Polynomials and $\tilde{x} = (\frac{\hbar}{\omega m})^{1/2}$. The ladder operators $a = \frac{1}{\sqrt{2}}(\frac{X}{\tilde{x}} + i\frac{\tilde{x}}{\hbar}P)$ and $a^\dagger = \frac{1}{\sqrt{2}}(\frac{X}{\tilde{x}} - i\frac{\tilde{x}}{\hbar}P)$ satisfy the relations $a|n^{(0)}\rangle = \sqrt{n}|(n-1)^{(0)}\rangle$ and $a^\dagger|n^{(0)}\rangle = \sqrt{n+1}|(n+1)^{(0)}\rangle$. Useful relations among the Hermite polynomials are $H_n'(x) = 2n\,H_{n-1}(x)$ and $xH_n(x) = nH_{n-1}(x) + \frac{1}{2}H_{n+1}(x)$.

11) Particle in an impenetrable box

Consider a particle in one dimension that is confined inside an impenetrable box, but can otherwise move freely inside the box in the interval $[-L, L]$. The box potential has the form

$$V(x) = \begin{cases} 0\,, & -L \le x \le L \\ \infty\,, & |x| > L \end{cases}\,.$$

1. Write down the Hamilton operator H for the system.

2. Write down the Schrödinger equation for the particles wave function and the boundary conditions that the wave function has to satisfy.

3. Determine the energy eigenfunctions $\phi_n(x)$ and the corresponding energy eigenvalues E_n, by solving the Schrödinger equation and accounting for the boundary conditions.

4. Normalize the wave functions $\phi_n(x)$ such that the condition

$$\int_{-L}^{L} dx\,|\phi_n(x)|^2 = 1 \qquad \text{is satisfied.}$$

5. Let $\psi(x)$ be a non-normalized wave function of the particle. Write down the expression for the probability that the lowest possible value is obtained in an energy measurement.

6. Calculate that probability for $\psi(x) = \Theta(L^2 - x^2)$.

12) Bra and Ket formalism and completeness relation.

Let X ($\vec{X}$) und P ($\vec{P}$) be the location and momentum operators in one dimension (three dimensions). Let the state $|\psi\rangle$ in 3 dimensions have the configuration space wave function $\psi(\vec{x}) \equiv \langle\vec{x}|\psi\rangle$ and the momentum space wave function $\tilde{\psi}(\vec{p}) \equiv \langle\vec{p}|\psi\rangle$.

1. Determine the following matrix elements in one dimension:
   $\langle x|X|p\rangle,\ \langle p|X|x\rangle,\ \langle x|X|x'\rangle,\ \langle p|X|p'\rangle.$
2. Determine the following matrix elements in three dimensions:
   $\langle\vec{x}|\vec{P}|\vec{x}'\rangle,\ \langle\vec{p}|\vec{P}|\vec{x}'\rangle.$
3. Determine the following matrix elements in three dimensions (m real and positive):
   $\langle\vec{x}|\frac{1}{|\vec{X}|}e^{-m|\vec{X}|}|\vec{x}'\rangle,\ \langle\vec{x}|\frac{1}{|\vec{X}|}e^{-m|\vec{X}|}|\psi\rangle,\ \langle\vec{p}|\frac{1}{|\vec{X}|}e^{-m|\vec{X}|}|\psi\rangle$.
4. Write down the abstract form (i.e. in terms of the representation independent location and momentum operators) of the Hamilton operator **H** and the Schrödinger equation for an arbitrary ket-state $|\phi(t)\rangle$ of a particle in 3 dimensions in the presence of a Coulomb potential which is centered at $\vec{x} = 0$.
5. Determine **by calculation** the form of the Schrödinger equation of 4. in configuration and momentum space representation. Here you may use the abbreviations $\phi(t,\vec{x}) \equiv \langle\vec{x}|\phi(t)\rangle$ and $\tilde{\phi}(t,\vec{p}) \equiv \langle\vec{p}|\phi(t)\rangle$.

13) Spin measurements

Initially, let there be a spin-1/2 system in a pure spin state $\chi_{z,+}$, where the spin is pointing in the z-direction.

1. For which spin operators is $\chi_{z,+}$ an eigenstate and what are the corresponding eigenvalues?
2. Write down the density matrix for the spin state $\chi_{z,+}$.
3. What are the probabilities to get the values $\pm\hbar/2$ if a measurement of the spin in x-direction is taken? What are the corresponding probabilities for a measurement of the spin in y-direction? What are the respective expectation values?
4. Let's suppose you get the value $-\hbar/2$ for a measurement of the spin in x-direction. Assume that the spin is still there and not destroyed by the measurement. In which state is the system after the measurement?
5. Now take another measurement of the spin in the $\vec{n}$ direction on this spin. The vector $\vec{n}$ is a unit vector which is rotated with respect to the x direction by the angle ϕ around the z axis. What is the probability to get the values $\hbar/2$ for this measurement?
6. Determine the expectation value for that spin measurement in the $\vec{n}$ direction.

Hints: The spin operators have the form

$$\vec{\mathbf{S}} = \frac{\hbar}{2}\begin{pmatrix}\sigma_1\\ \sigma_2\\ \sigma_3\end{pmatrix}, \qquad \sigma_1 = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \sigma_2 = \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}, \sigma_3 = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},$$

14) Density operator and rotation for spin-$\frac{1}{2}$ states

1. What is the general form of a density operator for a spin-1/2 state? What conditions should this operator satisfy? Under which condition does the density operator describe a pure state?
2. Explain what we mean by (degree of) polarization and how polarization is related to the states (and the corresponding form of the density operator).
3. What is the relation between the components of the spin-1/2 operator and the operator $U(\theta\,\vec{n})$ for spatial rotations on spin-1/2 states around an axis $\vec{n}$ (with $|\vec{n}| = 1$) by the angle θ? What mathematica property do the operators $U(\theta\,\vec{n})$ have?
4. Show:
$$U(\theta\,\vec{e_z})(\vec{n}\cdot\vec{\sigma})U(\theta\,\vec{e_z})^\dagger = \vec{n}'\cdot\vec{\sigma}, \qquad \vec{n}\in\mathbb{R}^3,$$
where $\vec{n}' = R(\theta\,\vec{e_z})\vec{n}$ and $R(\vec{\alpha})$ is the spatial rotation matrix that rotates vectors by an angle $|\vec{\alpha}|$ around the axis $\vec{\alpha}/|\vec{\alpha}|$.
5. Use the result from (a) to determine the transformation law for the density matrix
$$\rho = \frac{1}{2}(\mathbb{1}_2 + \vec{n}\cdot\vec{\sigma}), \qquad |\vec{n}| = 1.$$
under a rotation given by $U(\theta\,\vec{e_z})$.

15) Angular momentum and representations I

Consider the abstract and general angular momentum operators J_k ($k = 1,2,3$) and the orbital angular momentum operator L_k ($k = 1,2,3$).

1. Write down: $[J_k, J_l]$.
2. The components of the orbital angular momentum operator L_k can be written as functions of the position operators X_l and the momentum operators P_m, $L_k = \epsilon_{klm}X_lP_m$. Calculate the commutator $[L_k, P_m]$ by means of the fundamental commutation relations for X_l, P_n without using any special representation.
3. Calculate $[L_k, L_l]$.
4. What eigenvalues are in principle possible for the operator $\vec{J}^2$? What eigenvalues are possible for J_3 for a given fixed eigenvalue of $\vec{J}^2$?

5. In position space representation the components of the orbital angular momentum operator have the form $L_k = \epsilon_{klm} x_l (-i\hbar \nabla_m)$. Complete the statements:

   (1) The spherical harmonics $Y_{lm}(\theta, \varphi)$ are simultaneous eigenfunctions of the operators .... and .... with the corresponding eigenvalues .... and ....

   (2) $L_+ Y_{lm}(\theta, \varphi) =$ .... $(L_+ = L_1 + iL_2)$

6. Consider two irreducible angular momentum representations with quantum numbers j_1 and j_2 with respect to the total angular momentum operator $\vec{J}^2$. What are the dimensions of each of these two irreducible representations? Consider taking the direct product of these two irreducible representations. What is the dimension of this direct product representation? Is the result in general also an irreducible representation?

16) Angular momentum and representations II

Consider the properties of the abstract and general angular momentum operators J_k ($k = 1, 2, 3$ or $k = x, y, z$).

1. Write down the SU(2) commutation relations of the angular momentum operators $J_{x,y,z}$.

2. Prove the following statement: If an operator A commutes with two components of the angular momentum operators $\vec{J}$ (lets say J_x and J_y) then A also commutes with the third component.

3. The components of the orbital angular momentum operator L_k can be written as functions of the position operators X_l and the momentum operators P_m, $L_k = \epsilon_{klm} X_l P_m$. Calculate the commutator $[L_n, \vec{L}^2]$ by means of the fundamental commutation relations for X_l, P_n without using any special representation. For efficiency it is useful when you first calculate $[L_n, X_k]$, $[L_n, P_k]$ and $[L_n, L_k]$ then use these results for $[L_n, \vec{L}^2]$.

4. In classic physics one can write the spatial angular momentum $\vec{L}$ either as $\vec{x} \times \vec{p}$ or as $-\vec{p} \times \vec{x}$. Write down the resulting two possibilities to define quantum mechanical angular momentum operators using the correspondence principle (i.e. the usual way how to write down a quantum measurement operator). Explain why this could in principle lead to an ambiguity in the definition of the angular momentum operators and why - in the case of the angular momentum operator - both expressions turn out to be equivalent.

5. Let the set of states $\{|j, m\rangle \mid m = -j, -j+1, \ldots, j-1, j\}$ for one of $j = 0, \frac{1}{2}, 1, \ldots$ be a standard basis of eigenstates forming an irreducible representation of the angular momentum operators $\vec{J}^2$ and J_z. Show by calculation that the standard deviation of measurements of the angular momenta J_x and J_y in each of these eigenstates $|j, m\rangle$ has the form

$$\Delta J_x = \Delta J_y = \hbar \sqrt{\frac{j(j+1) - m^2}{2}}\,.$$

To proceed it is useful to first consider the expectation values of the raising and lowering operators $J_{\pm} = J_x \pm iJ_y$ and of $J_+^2 + J_-^2$, and in addition the relation

$$J_{\pm}|j,m\rangle = \hbar\sqrt{j(j+1) - m(m \pm 1)}|j,m \pm 1\rangle$$

to show that the mean values $\langle J_{x,y}\rangle$ vanish and that $\langle J_x^2\rangle = \langle J_y^2\rangle$. You may then consider the expectation value $\langle J_x^2 + J_y^2\rangle$.