Exercises for $\text{QM}_{\text{Extended}}$, Winter Term 2019, Sheet 12

1) Lagrangian for a Dirac particle

The Lagrangian density for a classic massive and charged Dirac particle has the form

$$\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m)\psi\,,$$

where $\bar{\psi} = \psi^\dagger\gamma^0$ is the adjoint Dirac field and one can consider ψ and $\bar{\psi}$ as independent.

(a) Use the Euler-Lagrange equations to derive the equation of motion for ψ and $\bar{\psi}$ and show that you correct forms of the Dirac equation.

(b) Determine the generalized momenta $\bar{\pi}$ for ψ and π for $\bar{\psi}$. (The bar superscript concerning complex conjugation is the usual convention.)

(c) Show that the Hamiltonian has the form

$$H = \int d^3\vec{\mathbf{x}}\,\psi^\dagger\Big[-i\vec{\alpha}\cdot\vec{\nabla} + \beta m\Big]\psi\,,$$

where α^i and β are the Dirac matrices discussed in class.

(d) Calculate the Hamiltonian (by doing the integration over $\vec{\mathbf{x}}$) for the general particle/antiparticle field

$$\psi(x) = \sum_s \int d^3\vec{\mathbf{k}}\,\Big[\,a(\vec{\mathbf{k}},s)\,\psi^{(+)}_{\vec{\mathbf{k}},s}(x) + b^*(\vec{\mathbf{k}},s)\,\psi^{(-)}_{\vec{\mathbf{k}},s}(x)\Big]\,.$$

Identify what is problematic about this result compared to the result obtained for the Klein-Gordon particle.

2) Dirac Matrices

(a) Verify the following relations for the γ matrices. Note that for a number of cases you just need to use their representation-independent property $\{\gamma^\mu,\gamma^\nu\} = 2g^{\mu\nu}$:

(1) $(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0$,

(2) $\gamma_\alpha\gamma^\mu\gamma^\alpha = -2\gamma^\mu$,

(3) $\gamma_\alpha\gamma^\mu\gamma^\nu\gamma^\alpha = 4g^{\mu\nu}$

(4) $\mathrm{Tr}[\gamma^\mu\gamma^\nu] = 4g^{\mu\nu}$

(5) $a\!\!\!/ b\!\!\!/ = a\cdot b - i\sigma_{\mu\nu}a^\mu b^\nu$, where $a\cdot b \equiv a_\mu b^\mu$, $\sigma_{\mu\nu} \equiv \frac{i}{2}[\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu]$

(b) Verify the following properties of the momentum space Dirac spinors, where we have $p^\mu = (E_p, \vec{\mathbf{p}})$:

(1) $(\not{p} - m)\, u(\vec{\mathbf{p}}, s) \;=\; 0\,, \quad (\not{p} + m)\, v(\vec{\mathbf{p}}, s) \;=\; 0$

(2) $\bar{u}(\vec{\mathbf{p}}, s)\, (\not{p} - m) \;=\; 0\,, \qquad \bar{v}(\vec{\mathbf{p}}, s)\, (\not{p} + m) \;=\; 0$

3) Lorentz Transformation

The general form for a Lorantz transformation matrix for a Dirac spinor reads $S(\Lambda(\vec{\theta}, \vec{\beta})) = \exp[-\frac{i}{2}\omega_{\alpha\beta}S^{\alpha\beta}]$ where $S^{\alpha\beta} = \frac{i}{4}[\gamma^\alpha, \gamma^\beta]$ and $(i, j = 1, 2, 3)$

$$\omega_{\alpha\beta} = \begin{pmatrix} 0 & \omega_{0j} \\ \omega_{i0} & \omega_{ij} \end{pmatrix}$$

with $\omega_{0j} = -\omega_{j0} = \beta_j$ for boosts and $\omega_{ij} = \theta_k \epsilon_{kij}$ for rotations in analogy to 4-vector Lorentz transformation matrix treated before. Show that the expression can be rewritten in the form

$$S(\Lambda(\vec{\theta}, \vec{\beta})) = \exp\Big[\frac{1}{2}\vec{\beta}\cdot\vec{\alpha} - \frac{i}{2}\gamma^5\vec{\theta}\cdot\vec{\alpha}\Big],$$

where $\alpha^k = \gamma^0\gamma^k$

4)* Rotation

(a) Show that the rotation matrix for a Dirac spinor $S(\Lambda(\vec{\theta}, 0))$ has the form $(\vec{\theta} = \theta\vec{\mathbf{n}})$

$$S(\Lambda(\vec{\theta}, 0)) \;=\; \begin{pmatrix} \mathbf{s}(\vec{\theta}\,) & 0 \\ 0 & \mathbf{s}(\vec{\theta}\,) \end{pmatrix},$$

where $\mathbf{s}(\vec{\theta}\,) = \exp(-\frac{i}{2}\vec{\theta}\vec{\sigma}) = \cos(\frac{\theta}{2})\,\mathbf{1} - i\sin(\frac{\theta}{2})\,\vec{\mathbf{n}}\vec{\sigma}$ is the rotation matrix for Pauli spinors.

(b) Let $S(\Lambda(\vec{\theta}, 0))$ act on a particle spinor $u(\vec{\mathbf{p}}, s)$ and call the result $u(\vec{\mathbf{p}}', s')$. Determine the momentum $\vec{\mathbf{p}}'$ in terms of $\vec{\mathbf{p}}$ and the Pauli spinor $\chi(s')$ in terms of $\chi(s)$.

5)* Lorentz Transformation

Show that the boost matrix for a Dirac spinor $S(\Lambda(0, \vec{\beta}))$ discussed in class has the form $(\vec{\beta} = \beta\vec{\mathbf{n}})$

$$S(0, \Lambda(\vec{\beta})) \;=\; \cosh\left(\frac{\beta}{2}\right)\mathbf{I} + \sinh\left(\frac{\beta}{2}\right)\vec{\mathbf{n}}\,\vec{\alpha} = \cosh\left(\frac{\beta}{2}\right)\left(\mathbf{I} + \tanh\left(\frac{\beta}{2}\right)\vec{\mathbf{n}}\,\vec{\alpha}\right)$$

Recall from sheet 5 that for a boost that turns a particle with mass m at rest to a particle with momentum $\vec{\mathbf{p}}$ we have $\cosh(\beta) = E_p/m$. Show that

$$\cosh\left(\frac{\beta}{2}\right) = \sqrt{\frac{E_p + m}{2m}}\,, \qquad \tanh\left(\frac{\beta}{2}\right) = \frac{|\vec{\mathbf{p}}|}{E_p + m}\,,$$

and calculate the results of the boost matrix $S(0, \Lambda(\vec{\beta}))$ acting on a particle spinor $u(0, s)$ at rest. Show that you are indeed obtaining $u(\vec{\mathbf{p}}, s)$.