Exercises for $\mathrm{QM}_{\mathrm{Extended}}$, Winter Term 2019, Sheet 10

1) Anticommuting observables

A quantum mechanical state $|\psi\rangle$ is known to be a simultaneous eigenstate of the Hermitian operators A and B, which are known to anticommute, i.e. $\{A, B\} = 0$. What can you say about the eigenvalues of A and B for state $|\psi\rangle$? Illustrate your point using the parity operator Π (being defined such that $\Pi = \Pi^\dagger$ for state $|\psi\rangle$) and the momentum operator.

2) Parity transformation for arbitrary angular momentum states

Proof that for any angular momentum state $|j, m\rangle$ (i.e. $j = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$) the parity operator acts in the form

$$\Pi\,|j, m\rangle \;=\; \eta_j\,|j, m\rangle\,,$$

where the complex phase η_j may depend on j, but is independent of $m = -|j|, \ldots, j$. Use the algebraic properties of the angular momentum states and the parity properties of angular momentum operators.

3) Selection rules for a spin-1/2 particle in a spherically symmetric potential

Consider a spin-1/2 particle bound to a spherically summetric potential at the origin and assume the theorem of exercise (2).

(a) Write down the spin-angular function $\tilde{Y}_{j=1/2,m=1/2}(\theta, \phi)$ (see the lecture notes of Chapter 2, page 13).

(b) Compute $(\vec{S}\cdot\vec{X})\,\tilde{Y}_{j=1/2,m=1/2}(\theta, \phi)$ explicitly term of the other $\tilde{Y}_{j,m}(\theta, \phi)$ spin-angular functions, where $\vec{S}$ is the spin-1/2 operator and $\vec{X}$ the location operator.

(c) Show that the result from (b) is understandable from the point of view of the transformation properties of the operator $(\vec{S}\cdot\vec{X})$ with respect to the rotation and the parity transformations.

4) Transformations

(a) Let $\psi(\vec{x}, t)$ be the wave function of a spinless particle corresponding to a plane particle wave in three dimensions. Show that $\tilde{\psi}(\vec{x}, t) = \psi^*(x, -t)$ is the wave function for the plane particl wave with the momentum direction reversed. Be very accurate in your argumentation and carry out all mandatory calculations.

(b) Let $\chi(\vec{n})$ the two-component (spin-1/2) eigenspinor of the spin operator $\vec{S}\cdot\vec{n}$ with eigenvalue $+\frac{1}{2}$, where $\vec{n} = \vec{n}(\theta, \phi)$ is the unit vector pointing in (θ, ϕ)-direction. Write down the explicit form of $\chi(\vec{n})$ in terms of the angles θ and ϕ (see the Quantum Mechanics 1 lecture). Verify that $\tilde{\chi}(\vec{n}) = -i\sigma_2\chi^*(\vec{n})$ is the two-component eigenspinor with the spin-direction reversed.