Exercises for QM$_{\text{Extended}}$, Winter Term 2019, Sheet 9

1) Time evolution operator in the interaction picture

In class we discussed that the time evolution operator in the interaction picture for a system with the Schrödinger picture Hamilton operator $H(t) = H_0 + \delta H(t)$ can be written in the form

$$U_I(t_1, t_0) = \mathbf{T} \exp\Big\{ -i \int_{t_0}^{t_1} dt \, \delta H_I(t) \Big\} ,$$

where $\mathbf{T}$ is the time ordering operator and $\delta H_I(t) = U_0(t_0, t)\delta H(t) U_0^\dagger(t, t_0)$, with $U_0(t, t') = e^{-iH_0(t-t')}$, is the interaction picture "interaction" term of the full Hamiltonian. For a given time-dependent Schrödinger picture state $|\psi(t)\rangle$, the interaction picture state is given by $|\psi(t)\rangle_I = U_0^\dagger(t, t_0)|\psi(t)\rangle$.

(a) Derive the Schrödinger equation for the interaction picture states $|\psi(t)\rangle_I$ from its definition in terms of the Schrödinger picture state and show that the state $|\psi(t)\rangle_I = U_I(t, t_0)|\psi(t_0)\rangle_I$ has the correct time t dependence.

(b) Show that

$$[U_I(t_1, t_0)]^\dagger = \bar{\mathbf{T}} \exp\Big\{ -i \int_{t_1}^{t_0} dt \, \delta H_I(t) \Big\}$$

is indeed the inverse of $U_I(t_1, t_0)$ as claimed in the lecture. Here $\bar{\mathbf{T}}$ is the anti-time ordering operator. Do this by showing that $U_I(t_1, t_0)[U_I(t_1, t_0)]^\dagger = \mathbb{1}$ to second order in (i.e. two powers of) $\delta H_I(t)$, which means one order beyond the Born approximation.

2) Linear system of first order differential equations

One can use the time-evolution formalism involving the time-ordered exponentials known from quantum theory to solve a linear system of first order differential equations (which is actually precisely what you do when solving the time-dependent Schrödinger equation). Consider an n-dimensional vector $\mathbf{f}(t) = (f_1(t), \ldots, f_n(t))^T$ of functions in time which satisfy the first order differential equation

$$\frac{d}{dt}\mathbf{f}(t) = \mathbf{A}(t) \cdot \mathbf{f}(t) \qquad \text{with} \qquad \mathbf{f}(0) = \mathbf{a} = (a_1, \ldots, a_n) ,$$

where $\mathbf{A}(t)$ is a time-dependent diagonalizable $n \times n$ matrix and where t is the variable of the problem, which you may interpret t as time at this point.

(a) Show that

$$\mathbf{f}(t) = \mathbf{O}(t) \cdot \mathbf{a} \quad \text{with} \quad \mathbf{O}(t) \equiv \mathbf{T} \exp\Big\{ \int_0^t dt \, A(t) \Big\}$$

is the solution.

(b) Consider this problem for a coupled linear system of first order differential equation involving two functions $f_1(t)$ and $f_2(t)$ with $\mathbf{A}(t) = \mathbf{A} = \sigma_1$ (first Pauli matrix). This is a really easy problem, which you can first solve exactly by finding a new basis where the two differential equations decouple (i.e. $\mathbf{A}$ becomes diagonal). Subsequantly, calculate $\mathbf{O}(t)$ from the time-ordered exponential (which you can also do exactly to all orders in the expansion of the exponential function due to some valuable property of the Pauli matrices) and show that the solutions agree.

3) Path integral representation of the time-dependent Green's function

Show that the general phase-space path-integral representation of the one-dimensional time-dependent configuration space Greens function ($t_f > t_i$)

$$G(t_f, x_f; t_i, x_i) = \langle x_f, t_f | x_i, t_i \rangle = \tilde{N} \int [\mathcal{D}p(t)][\mathcal{D}x(t)] \exp\left\{ i \int_{t_i}^{t_f} dt \left[p(t)\dot{x}(t) - H\big(p(t), x(t)\big) \right] \right\}$$

can be further simplified to the expression

$$\langle x_f, t_f | x_i, t_i \rangle = N \int [\mathcal{D}x(t)] \exp\left\{ i \int_{t_i}^{t_f} dt \, L\big(x(t), \dot{x}(t)\big) \big] \right\}$$

Use the discretized versions of the respective path-integral as discussed in class (see lecture notes pages 4 and 5)

4) Path integral computation of the time-dependent Green's function of a free particle

The time-dependent retarded Green's function of a free particle with mass μ in one dimension has the form ($t_f > t_i$)

$$G_0(t_f, x_f; t_i, x_i) = \langle x_f, t_f | x_i, t_i \rangle = \sqrt{\frac{\mu}{2\pi i (t_f - t_i)}} \exp\left\{ \frac{i\mu (x_f - x_i)^2}{2(t_f - t_i)} \right\}.$$

(a) Confirm this result by explicitly computing the path-integral representation of the Green's function in terms of the Lagrangian L in the discretrized way. This is a reading+doing exercise since the computation is already shown on page 6 of the lecture notes. Carry out the calculations yourself by carefully applying the Gauss integrals and taking care of the algebra and the induction that is mandatory when doing the computations with discretized slices in time.

(b) Show that the term in the exponential $(i\mu(x_f - x_i)^2)(2(t_f - t_i))$ is exactly equal to i times the action S_{classic} of a classic free particle evolving along the path $x(t)$ which satisfies the boundary conditions $x(t_i) = x_i$ and $x(t_f) = x_f$