**Exercises for QM$_{\text{Extended}}$, Winter Term 2019, Sheet 8**

## 1) Optical theorem and Neumann series

Consider the scattering of a spinless particle off the stationary potential $V(\mathbf{x})$. Show that when one applies the optical theorem to the Neuman series, one obtains the relation

$$\sigma_{\text{tot}}^{\text{Born approx.}} = \frac{4\pi}{k}\,\text{Im}\Big[\,f(\theta=0)|^{2^{\text{nd}}\text{order in V}}\Big]\,.$$

This means that the optical theorem relates different orders of the perturbation series to each other.

**Hint:** Use the momentum space Feynman rules discussed in class and one of the relations from the previous exercise sheet.

## 2) Green's function of the time-dependent free particle Schrödinger equation

(a) Derive the (3 different) expressions for the Green's function $G_0(x,x') = G_0(t,\mathbf{x}|t',\mathbf{x}')$ for the time-dependent Schrödinger equation for a free particle in 3 dimensions defined by

$$\left[\,i\frac{\partial}{\partial t} + \frac{\nabla_{\mathbf{x}}^2}{2\mu}\,\right] G(t,\mathbf{x};|t',\mathbf{x}') = i\,\delta(t-t')\,\delta^{(3)}(\mathbf{x}-\mathbf{x}')\,,$$

which we discussed in class (see the three equalities (■) on page (29)). Due to causality we are only interested in the Green's function that describes forward time evolution. Use the result of exercise (2) on sheet 2 to argue which $i\epsilon$-presciption is necessary.

(b) Assume that you know the particles wave function $\Psi(t,\mathbf{x})$ on all locations $\mathbf{x}$ at a particular time $t=t'$. Show that

$$\Psi^{(+)}(t,\mathbf{x}) = \int d^3\mathbf{x}'\,G(t,\mathbf{x}|t',\mathbf{x}')\,\Psi(t',\mathbf{x}')$$

is a solution of the time-dependent Schrödinger equation for times $t>t'$ and that $\Psi^{(+)}(t',\mathbf{x}) = \Psi(t',\mathbf{x})$ is valid.

(c) Show that

$$(\Psi^{(-)}(t,\mathbf{x}))^* = \int d^3\mathbf{x}'\,\Psi^*(t',\mathbf{x}')\,G(t',\mathbf{x}'|t,\mathbf{x})$$

is the Hermitian adjoint of a solution of the time-dependent Schrödinger equation for times $t>t'$ and that $\Psi^{(-)}(t',\mathbf{x}) = \Psi(t',\mathbf{x})$ is valid.

**Hint:** It may be possible that the integral

$$\int_0^\infty dk\,k^n\,e^{-r^2k^2} = \frac{1}{2r^{n+1}}\,\Gamma\left(\frac{n+1}{2}\right)$$

arises, where $\Gamma$ is the Gamma function, with $\Gamma(3/2) = \sqrt{\pi}/2$. Recall in your mind how you may derive that relation yourself.

### 3) S-matrix element off a time-dependent potential

Determine the S-matrix element $S_{\mathbf{k_f},\mathbf{k_i}}$ in Born approximation for a particle with mass $\mu$ that scatters of a time-dependent potential of the form $\delta H(t,\mathbf{x}) = \frac{V_0}{r}\, e^{-mr} \sin(wt)$. Determine the expression for the full differential cross section $\frac{d\sigma}{d\Omega}$ in the Born approximation.

### 4) Time evolution operator for an arbitrary time-dependent Hamiltonian

Let $H = H(t)$ be an explicitly time-dependent Hamilton operator in the Schrödinger picture. You may *not* assume any particular form of the Hamilton operator, so you cannot split it into a free particle Hamilton operator plus some interaction potential. Construct the forward time evolution operator $G(t,t_0)$ that evolves any state given at some time $t_0$ to the corresponding state at time $t > t_0$ consistent with the Schrödinger equation,

$$|\psi(t)\rangle = G(t,t_0)|\psi(t_0)\rangle\,.$$

Recall that the forward time evolution operator for a time-independent Hamilton operator $H$ has the form $G(t,t_0) = \Theta(t-t_0)\exp[-iH(t-t_0)]$. The expression you have to derive here is generalization of this expression.

**Hint:** Use the time-dependent Schrödinger equation to derive an operator-valued differential equation for $G(t,t_0)$ (i.e. $i\frac{\partial}{\partial t}G(t,t_0) = \ldots$). Solve the differential equation iteratively by deriving a perturbative series in the number of times the Hamilton operator appears. Use the fact that obviously $G(t_0,t_0) = \mathbb{1}$ and construct the series by integration, assuming that $t - t_0$ is very small.