Exercises for $\mathrm{QM}_{\mathrm{Extended}}$, Winter Term 2019, Sheet 7

1) Scattering off the Yukawa potential

Determine the Born approximation of the differential and total cross section for the scattering of a spinless particle off the so-called Yukawa potential, which has the form

$$V(r) \;=\; \frac{V_0}{r}\, e^{-mr}\,.$$

(Which physical interpretation may the parameter m have?) Derive from the result the Born approximation of the quantum mechanical version of Rutherford's differential cross section formula for Coulomb potential scattering (where $V(r) = 1/(4\pi)\,(q_1 q_2)/r$). Compare to the classical scattering formula discussed in class (where you should consider using trigonometric additional formulae). Think about (and discuss in class) what we learn from the comparison.

2) Integrals with $i\epsilon$ prescription I

For the following two exercises you may have to look back into your math classes. The algebra and the physics content of these relations is important for many manipulations quantum theory. In the following ϵ is an infinitesimally small positive variable, which is always much much smaller than any given quantity that arises in the same context.

Show that for real x and x' the relation

$$\frac{1}{x - x' - i\epsilon} \;=\; i\pi\,\delta(x - x') \;+\; \mathcal{P}\frac{1}{x - x'} \tag{1}$$

holds, where $\mathcal{P}$ is the principal value of the singular intergral. You can assume for simplicity the case $x' = 0$ (Why is this still general?) and consider an integration over a test function $f(x)$ which does not have any poles in the vicinity of the real axis. Try to use the Cauchy integral theorem. How does the RHS of the relation look for $1/(x - x' + i\epsilon)$?

3) Integrals with $i\epsilon$ prescription II

Show that for any real z the relation

$$\frac{1}{2\pi i}\int_{-\infty}^{+\infty} dx \frac{e^{ixz}}{x - i\epsilon} \;=\; \Theta(z) \tag{2}$$

holds, where $\Theta(z)$ is the Heaviside step function. Determine the result for the case when the sign in front of the $i\epsilon$ term is reversed.

4)* Dispersion relation

The function $f(x) = e^{-\sqrt{-x}}$ is an analytic function in the complex x plane except for a cut along the positive real x axis. Cauchy's integral theorem states that for any positively oriented closed path γ in the complex x-plane that is fully witin the region where $f(x)$ is analytic (i.e. the path γ does not cross the cut) and for a complex number z within the closed path γ the following relation holds:

$$f(x) = \frac{1}{2\pi i} \oint\limits_{\gamma} \frac{f(z)}{z-x}\, \mathrm{d}x\,.$$

(a) Use `Mathematica` to draw 3D pictures (using the `Plot3D` command) of $\mathrm{Re}[f(x)]$ and $\mathrm{Im}[f(x)]$ in the complex x plane for $-2 < \mathrm{Re}[x] < 2$ and $-2 < \mathrm{Im}[x] < 2$. Convince yourself that $\mathrm{Im}[f(x)]$ has a cut along the positive real axis, while $\mathrm{Re}[f(x)]$ does not.

(b) Calculate $\mathrm{Im}[f(x+i\epsilon)]$ for positive real x and show that $\mathrm{Im}[f(x+i\epsilon)] = -\mathrm{Im}[f(x-i\epsilon)]$. From now on we *define* $\mathrm{Im}[f(x)] \equiv \mathrm{Im}[f(x+i\epsilon)]$ for positive real x. (Recall that ϵ is infinitesimally positive and that you should set it to zero once it has done its job of telling you how to calculate.)

(c) Consider Cauchy's theorem for the positively oriented path γ shown in the figure below, which consists of the patches γ_c and γ_r. The distance of path γ_c to the real axis is just ϵ. In the limit $r \to \infty$ and because ϵ is infinitesimal any given complex number is inside the path γ. Show that the contribution of path γ_r does not contribute for $r \to \infty$.

(d) Use the results from (b) and (c) to show that for any complex x that is not on the positive real axis (i.e. the distance to the real axis is finite) the relation

$$f(x) = \int\limits_0^\infty \frac{\mathrm{Im}[f(z+i\epsilon)]}{z-x+i\epsilon}\mathrm{d}z = \int\limits_0^\infty \frac{\mathrm{Im}[f(z)]}{z-x}\mathrm{d}z$$

holds. You do not need to use the specific form of $f(x)$ at this point. Why can you drop the $i\epsilon$ term in the denominator?

(e) Consider now that x is real. Show that the following relations, called *"dispersion relations"*, hold (use previously known results!) Again, you do not need to use the specific form of $f(x)$ at this point.

$$f(x+i\epsilon) = \int\limits_0^\infty \frac{\mathrm{Im}[f(z)]}{z-x-i\epsilon}\mathrm{d}z \quad \text{and} \quad \mathrm{Re}[f(x)] = \mathcal{P}\int\limits_0^\infty \frac{\mathrm{Im}[f(z)]}{z-x}\mathrm{d}z$$

(f) Use `Mathematica` (employing the `NIntegrate` command) and show for a few cases that the relations in (d) and (e) are indeed valid numerically for $f(x) = e^{-\sqrt{-x}}$. Use some small finite value for ϵ.