Exercises for $\text{QM}_{\text{Extended}}$, Winter Term 2019, Sheet 6

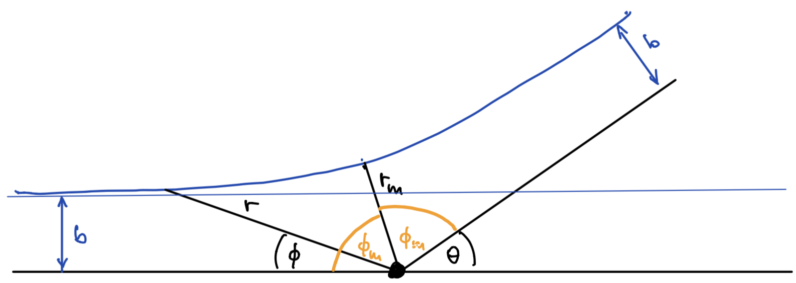
1) Classic potential scattering

Consider classic potential scattering of a particle with kinetic energy E and mass m off a repulsive potential $V = V(r)$ that only depends on the radius r. Use the notations and polar coordinates where the center of the potential the origin as a shown in the sketch. Use energy and angular momentum conservation to show that the relation between the angle of closest approach ϕ_m and the closest distance to the center r_m reads

$$\phi_m = b \int_0^{1/r_m} \frac{\mathrm{d}u}{\sqrt{1 - b^2 u^2 - \frac{V(1/u)}{E}}}\,, \tag{1}$$

with $u = 1/r$. Which equation does r_m have to satisfy and explain the reason. This calculation from classical mechanics is a classic and can be found in many books.

Hint: To start one should realize that the scattering trajectory of the particle is fully embedded in the so-called scattering plane which also contains the center of the potential. So it is best to use cylindrical coordinates in the scattering plane w.r. to the center of the potential using ϕ as polar angle and r as the distance to the origin. Write down the velocity of the scattered particle that is located at the point $\vec{x}(r(t), \phi(t)) = r(t)\,\hat{n}_r$ noting that its differential has the form $d\vec{x}(r, \phi) = \hat{n}_r\, dr + r\,\hat{n}_\phi\, d\phi$, where $\hat{n}_r$ and $\hat{n}_\phi$ are the unit vectors in r and ϕ directions, respectively, at the point $\vec{x}$. One can now write down the expressions for the conserved angular momentum $\vec{L}$ and the total energy E.



2) Rutherford scattering

Derive from the integral in Eq. (1) the relation between the scattering angle θ and the impact parameter b for Coulomb scattering discussed in class. Use the sketch to relate ϕ_m to the scattering angle θ and determine r_m from the constraint that the denominator in the integrant vanishes.

3) Measurement of the total cross section

In class we mostly discussed that the total cross section σ_{tot} can be determined from a measurement of the differential cross section $\mathrm{d}\sigma_{\text{tot}}/\mathrm{d}\Omega$ which is then subsequently integrated over all solid angles. Another method is using a beam of particles that hits a very thin foil made of the scattering centres. For simplicity assume that the beam direction is perpendicular to the foil's surface. In this case σ_{tot} can be determined from the reduction of the particle intensity in the transmitted beam behind the foil. Show that the total cross section can be determined from the formula

$$\sigma_{\text{tot}} = \frac{1}{\rho\, s} \ln\left(\frac{I(0)}{I(s)}\right), \tag{2}$$

where $I(x)$ is the particle intensity (=number of particles per unit area perpendicular to the beam direction per unit time) of the particle beam after distance x *in the foil*, s is the thickness of the foil and ρ the density (=number of scattering centres per unit volume) of scattering centres of the foil. So $I(0)$ is the beam intensity before it hits the foil. Why is the assumption of a thin foil relevant?

Hint: Determine an expression for $I(x + \delta x)$ assuming that $I(x)$ is known using the considerations discussed in class concerning the alternative interpretation of the total cross section. Derive from that relation the differential equation for $I(x)$ in x.

4) Gaussian integral

Prove the Gaussian integral formula

$$I(a, b) = \int_{-\infty}^{\infty} dx \, \exp(-ax^2 + bx) = \sqrt{\frac{\pi}{a}} \exp\left(\frac{b^2}{4a}\right), \quad a, b \in \mathbb{C}, \quad \mathrm{Re}(a) > 0, \tag{3}$$

where $\sqrt{a} \equiv \sqrt{|a|} e^{i\theta/2}$ for $a = |a|e^{i\theta}$ and $-\pi/2 < \theta < \pi/2$.

First convince yourself about the formula is correct by using a program such as `Mathematica` to test it for complex values of a and b.

To proceed show the relation first for $I(a, 0)$ by using that the expression

$$[I(a, 0)]^2 = \int_{-\infty}^{\infty} dx \, \exp(-ax^2) \int_{-\infty}^{\infty} dy \, \exp(-ay^2) \tag{4}$$

can be calculated rather easily using polar coordinates. Then use the Cauchy integral theorem to determine the relation between $I(a, 0)$ and the integral

$$\int_{-\infty}^{\infty} dx \, \exp(-a(x - c)^2) \quad \text{for} \quad c \in \mathbb{C}. \tag{5}$$

From there it is straightforward to determine $I(a, b)$.