Exercises for $\text{QM}_{\text{Extended}}$, Winter Term 2019, Sheet 5

1) Lorentz transformation matrix for 4-tensors II

Lorentz transformations for boosts and rotations (i.e. orthochonous and proper Lorentz transformations) can be written down in the standard "exponential of generators"-form

$$\Lambda^{\mu}{}_{\nu} = \exp\left[-i\,\frac{\omega_{\alpha\beta}}{2}(\tilde{J}^{\alpha\beta})^{\mu}{}_{\nu}\right]$$

where $(\tilde{J}^{\alpha\beta})_{\mu\nu} = i(\delta^{\alpha}{}_{\mu}\delta^{\beta}{}_{\nu} - \delta^{\alpha}{}_{\nu}\delta^{\beta}{}_{\mu}) = i(\eta^{\alpha}{}_{\mu}\eta^{\beta}{}_{\nu} - \eta^{\alpha}{}_{\nu}\eta^{\beta}{}_{\mu})$. Note the location of the index μ and that you need the metric tensor to lower or raise indices. Also note that the notation of the μ-ν indices inside the exponential means that the exponentiation happens on these indices, so e.g. $(A^{\mu}{}_{\nu})^2$ actually means $A^{\mu}{}_{\sigma}A^{\sigma}{}_{\nu}$. The choice for $\omega_{\alpha\beta}$ determines which kind of transformation you have. Here $\omega_{\alpha\beta}$ has three parameters for spatial rotations and three for boosts in the form $(i,j = 1,2,3)$

$$\omega_{\alpha\beta} = \begin{pmatrix} 0 & \omega_{0j} \\ \omega_{i0} & \omega_{ij} \end{pmatrix}.$$

(a) A spatial rotation by angle θ around the axis in direction of the unit vector $\vec{n}$ has $\omega_{ij} = \theta_k\epsilon_{kij}$ with $\vec{\theta} = \theta\vec{n}$ and where we use the convention $\epsilon_{123} = 1$. All other entries of $\omega_{\alpha\beta}$ are zero. Determine $\Lambda^{\mu}{}_{\nu}$ for a rotation by the angle θ around the z-axis, i.e. $\vec{\theta} = (0,0,\theta)$.

(b) For a boost along the x-axis (active transformation) we have $\omega_{01} = -\omega_{10} = \beta$, where β is called the *rapidity parameter* and does <u>not</u> represent a Lorentz index in this context. Determine $\Lambda^{\mu}{}_{\nu}$ and the relation between β and the velocity of the boosted system v.

For (a) and (b) you should check the correctness of your calculations with the well-known results.

2) Gauge transformation and gauge invariance

Show by explicit calculations that the Lagrangian density $\mathcal{L}(x) = \psi^*(x)[iD_t + \frac{1}{2m}\vec{D}^2]\psi(x)$ for a wave-function describing a particle with electric charge q is invariant under the gauge transformations $A^{\mu}(x) \to A^{\mu}(x) + \partial^{\mu}\lambda(x)$, $\psi(x) \to \exp(-iq\lambda(x))\psi(x)$. Also show the correctness of the transformation rules $D_t \to \exp(-iq\lambda(x))D_t\exp(+iq\lambda(x))$ and $\vec{D} \to \exp(-iq\lambda(x))\vec{D}\exp(+iq\lambda(x))$. Use the definitions of the covariant derivatives discussed in class.

3) Time evolution after a gauge transformation

Consider a gauge transformation with the gauge function $\lambda(\vec{x},t) = \Delta t$, where Δ is a constant. How do wave-functions for a particle with electric charge q that are solutions of the Schrödinger equation transform under this gauge transformation? It is quite obvious from observing the Schrödinger equation that the energy eigenvalues of energy eigenfunctions get modified by the gauge transformation. What does this mean and how should one interpret this issue in the context that it is generally said that gauge transformation leave the physics invariant?

Suppose that you have determined the ONB of energy eigenstates for a particular choice for $A^\mu(x)$. Now you carry out a gauge transformation. Are the gauge-transformed states in the ONB still energy eigenstates? Consider the cases $\lambda(\vec{x},t) = \Delta t$, $\lambda(\vec{x},t) = \lambda(t)$, $\lambda(\vec{x},t) = \lambda(\vec{x})$,

4) Electron in a homogeneous magnetic field I

Consider an electron ($q_{e^-} = -e$) that moves is homogeneous magnetic field in the z-direction, $\vec{B} = B\,\vec{n}_3$, where $\vec{n}_3$ is the unit vector in z-direction. Adopt the gauge where $\vec{A}(\vec{x}) = -\frac{1}{2}\,\vec{x}\times\vec{B}$.

(a) Write down the Hamilton operator and determine the Heisenberg evolution equation for the physical kinematic momentum $m_e\dot{\vec{X}}_H$ and the Lorentz force $m_e\ddot{\vec{X}}_H$.

(b) Solve the Heisenberg equations and show that the evolution of a small wave packet corresponds to a helix in the z-direction.

(c) Argue (in terms of a rigorous derivation) why all elements of an ONB of energy eigenfunctions can always be factorized in (i.e. written as a product of) the momentum eigenfunctions $\frac{1}{\sqrt{2\pi}}e^{ip_3x_3}$ of the P_3 operator and functions of x_1 and x_2, which are eigenfunctions of the Hamilton operator with the P_3 terms dropped (which we call H_{xy}).

(d) Show that one can rewrite $H_{xy} = \omega(a^\dagger a + \frac{1}{2})$ in terms of the operator $a = \sqrt{\frac{m_e}{2\omega}}(\dot{X}_1 - i\dot{X}_2)$ with $\omega = \frac{Be}{m_e}$ and show that a and $a^\dagger$ satisfy commutation relations in analogy to the ladder operator of the harmonic oscillator. Argue (in a rigorous way) why the possible energy eigenvalues are $E_n = \omega(n+\frac{1}{2})$ with $n \in \mathbb{N}_0$ and acting with $a^\dagger$ on an energy eigenstate to the eigenvalue E_n gives an eigenstate to the eigenvalue E_{n+1}.

5)* Electron in a homogeneous magnetic field II

(e) From the information of (d) one cannot yet determine all energy eigenstates because they are degenerate. Argue (with plausibility arguments) why this must be the case. Show that one cannot use the two kinematic momenta $m\dot{X}_1$ and $m\dot{X}_2$ as a complete set of observables to classify all energy eigenstates by their respective eigenvalues.

(f) Show that the solution of the Heisenberg equations for $\dot{X}_{1,H}(t)$ and $\dot{X}_{2,H}(t)$ can be written as $\dot{x}_H(t) = \exp(-\omega\,\tau\,t)\,\dot{x}_H(0)$, where $x_H(t) \equiv \begin{pmatrix} X_{1,H}(t) \\ X_{2,H}(t)\end{pmatrix}$ and $\tau = \begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix}$ and that from that one derive that $x_H(t) = \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} + \frac{1}{\omega}\tau\,\dot{x}_H(t)$, where $C_{1,2}$ are two time-independent operators that we can consider to be the center of the helix in the x-y-plane. We use the last equality as the definition for the operators $C_{1,2}$. Show that $C_{1,2}$ commute with $\dot{X}_{1,H}(t)$ and $\dot{X}_{2,H}(t)$ and thus also with H_{xy}, but that they do not commute with each other. It turns out that one can use $R^2 \equiv C_1^2 + C_2^2$ (i.e. the operator for the squared distance of the helix center to the z-axis) as the second operator we need to form a complete set of observables to classify all energy eigenstates.

(g) Show that one can write $R^2 = c^\dagger c + r_0^2$ in terms of the operator $c = C_1 + iC_2$, where $r_0 = 1/\sqrt{m_e\omega}$, and that $[c, c^\dagger] = 2r_0^2$. Show that r_0 is the smallest possible eigenvalue of the R^2 operator, that all possible eigenvalues have the form $r_n^2 = (2n+1)r_0^2$, $n \in \mathbb{N}_0$ and that acting with $c^\dagger$ on an eigenstate to the eigenvalue r_n^2 gives an eigenstate to the eigenvalue r_{n+1}^2.

(h) Which condition must the state $|E_0, r_0^2\rangle$ satisfy and how can you obtain (yet unnormalized energy eigenstates) $|E_n, r_{n'}^2\rangle$ from $|E_0, r_0^2\rangle$ using $a^\dagger$ and $c^\dagger$?

(i) Write the operators $a, a^\dagger, c, c^\dagger$ in terms of the kanonical $X_{1,2}$ and $P_{1,2}$ operators and subsequently in position space representation using the variables $x_\pm = \frac{1}{\sqrt{2}}(x_1 \mp i x_2)$ instead of x_1 and x_2. Determine the normalized wave function $\psi_{n=0,n'=0}(x_+, x_-)$ for the ground state.

(j) How do you have to modify the formula from (h) such that the $|E_n, r_{n'}^2\rangle$ are normalized when staring from the normalized state $|E_0, r_0^2\rangle$?