Exercises for $\text{QM}_{\text{Extended}}$, Winter Term 2019, Sheet 4

1) Lorentz transformation matrix for a 4-tensors I

One important aspect of the Lorentz transformation matrix $\Lambda^{\mu}{}_{\nu}(S \to S')$ is that it itself is a 4-tensor. This means that one may freely raise and lower its indices (depending on the context one uses it) and determine the form it has from the point of view of a third inertial frame S'' by Lorentz transforming $\Lambda^{\mu}{}_{\nu}(S \to S')$ itself. This can be used to construct the Lorentz transformation matrix $\Lambda^{\mu\nu}(S \to S')$ (we raised the ν index just to simplify the notation) by the following consideration: Let us assume that we know the explicit form of one particular (!) 4-vector q^{ν} in system S (called q_s^{ν}) and in system S' (called $q_{s'}^{\nu}$). One can then consider the Lorentz transformation matrix to be a function of q_s^{ν} and $q_{s'}^{\nu}$, $\Lambda^{\mu\nu}(S \to S') = \Lambda^{\mu\nu}(q_s, q_{s'})$. Because $\Lambda^{\mu\nu}(S \to S')$ is a 4-tensor itself it must therefore be a combination of all possible irreducible tensors related to the Lorentztransformation. This means that $\Lambda^{\mu\nu}(q_s, q_{s'})$ can be written as a linear combination of the rank 2 tensors $\eta^{\mu\nu}$, $q_s^{\mu} q_s^{\nu}$, $q_{s'}^{\mu} q_s^{\nu}$, $q_s^{\mu} q_{s'}^{\nu}$, $q_{s'}^{\mu} q_{s'}^{\nu}$.

(a) Determine $\Lambda^{\mu\nu}(q_s, q_{s'})$ using the condition that the metrix tensor in invariant under Lorentz transformations. Note that you can $\Lambda^{\mu}{}_{\nu}(S \to S')$ from that by simply lowering the index ν.

(b) Calculate $\Lambda^{\mu}{}_{\nu}(q_s, q_{s'})$ from (a) for a rotation by an angle θ around the z-axis by using suitable (and simple) 4-vectors q_s^{μ} and $q_{s'}^{\mu}$ and compare to the result you expect.

(c) Calculate $\Lambda^{\mu}{}_{\nu}(q_s, q_{s'})$ from (a) for the boost in z-direction discussed in class by using suitable (and simple) 4-vectors q_s^{μ} and $q_{s'}^{\mu}$ and show that it provides the correct result.

2) Phase transformation symmetry and Noether current

According to the Noether theorem there is a conserved quantity for each continuous symmetry of a theory. It turns out that the well known statement that probability is conserved is related to the symmetry of the Schrödinger theory with respect to space-time-independent phase transformations of the particles wave function $\psi(x) \to e^{-i\lambda}\,\psi(x)$ with λ being some real constant. Recall (see Chapter 2.11 of the QM1 lecture) that the continuity equation which expresses probability conservation has the form

$$\partial_\mu j^\mu(x) = \partial_t \rho(x) - \vec{\nabla}\vec{j}(x) = 0\,, \quad \text{where} \quad j^\mu(x) \equiv \Big(\rho(x), \vec{j}(x)\Big)\,.$$

(a) Show that the Lagrange density $\mathcal{L}(x)$ for the Schrödinger equation for a single non-relativistic particle of mass m in a potential is symmetric with respect to the constant phase transformations $\psi(x) \to e^{-i\lambda}\,\psi(x)$ (and the corresponding transformation for the complex conjugated (cc) wave function $\psi^*(x) \to e^{i\lambda}\,\psi^*(x)$). Notice that this symmetry is local in space and time, i.e. is is true for $\mathcal{L}(x)$ at any space-time point x.

(b) Consider the infinitesimal version $\psi(x) \to \psi(x) + \delta\psi(x)$, $\delta\psi(x) = -i\lambda\psi(x)$ (and the corresponding cc wave function). From this variation you can of course also formally derive the Euler-Lagrange equations using the variational principle for the action $S = \int dx^4 \mathcal{L}(x)$.

But because you know from (a) that the Lagrange density $\mathcal{L}(x)$ is invariant locally at the x, you can formally derive the form of the Noether current $j^\mu(x)$. Write down the variation of $\mathcal{L}(x)$ (i.e. not of the action S!) and follow the variational calculations we did to derive the Euler-Lagrange equations. This way you cannot do away with the total derivative term. Show that this (together with the validity of the Euler-Lagrange equations) can be used to identify a locally conserved 4-current $J^\mu(x)$ satifying $\partial_\mu J^\mu(x) = 0$. Show that $J^\mu(x)$ agrees with $j^\mu(x)$ discussed in the QM1 lecture up to an overall factor.

3) Lagrange density for the electromagnetic field in Coulomb gauge I

Start from the Lagrangian

$$\mathcal{L} \;=\; -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - j_\mu A^\mu$$

accounting for the effects of the electric current j^μ.

(a) The generalized momentum conjugates of the vector fields $A^\mu(x)$ are defined as

$$\pi^\mu \;=\; \frac{\partial \mathcal{L}}{\partial(\partial_0 A^\mu)}, .$$

Calculate the generalized momentum conjugates for all four comments of A^μ. What do the results mean physically?

(b) Since A^0 is not a physically independent degree of freedom, it can be expressed in terms of other quantities. Use the Maxwell equations $\partial_\mu F^{\mu\nu} = j^\mu$ to derive the equation of motion for A^0 and show that, applying the Coulomb gauge condition $\vec{\nabla}\vec{A}(x) = 0$, a solution for A^0 is

$$A^0(t,\vec{x}) \;=\; \frac{1}{4\pi} \int \mathrm{d}^3\vec{x}\, \frac{\rho(t,\vec{x}')}{|\vec{x}-\vec{x}'|}\,,$$

where the 4-current has the form $j^\mu(t,\vec{x}) = (\rho(t,\vec{x}), \vec{j}(t,\vec{x}))$.

(c) Derive the equation of motion for $\vec{A}(t,\vec{x})$.

4) Lagrange density for the electromagnetic field in Coulomb gauge II

Calculate the explicit form of the Lagrangian $L = \int \mathrm{d}^3\mathbf{x}\, \mathcal{L}(t,\mathbf{x})$ in Coulomb gauge and show that the result can be written in the form

$$L \;=\; \int \mathrm{d}^3\mathbf{x} \Big[\frac{1}{2}\, \vec{E}_v^2(t,\vec{x}) \;-\; \frac{1}{2}\, \vec{B}^2(t,\vec{x}) \;+\; \vec{j}(t,\mathbf{x})\cdot\vec{A}(t,\mathbf{x})\Big] \;-\; \frac{1}{8\pi} \int \mathrm{d}^3\mathbf{x}\, \mathrm{d}^3\mathbf{x}'\, \frac{\rho(t,\mathbf{x})\rho(t,\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\,,$$

where the magnetic is defined as usual, $\vec{B} = \vec{\nabla}\times\vec{A}$, and you have to use the explicit form of A^0 derived in exercise (3). Recall that here the electric field is defined as $\vec{E}_v \equiv -\partial_t \vec{A}$ and only accounts for the dynamic electromagnetic wave contributions of the electric field, while the static contribution is made explicit in terms of a potential term.

5) Classic Hamilton function for a nonrelativistic electron and proton in the presence of an electromagnetic field

Assume now that you have an electron and a proton located at the generalized coordinates $\mathbf{q}_e(t)$ and $\mathbf{q}_p(t)$, so that the electric current adopts the form

$$j^\mu(t,\mathbf{x}) \;=\; -\,e\,\delta^{(3)}(\mathbf{x}-\mathbf{q}_e(t))\,(1,\dot{\mathbf{q}}_e(t)) + e\,\delta^{(3)}(\mathbf{x}-\mathbf{q}_p(t))\,(1,\dot{\mathbf{q}}_p(t))\,,$$

and add to the Lagrangian of exercise (4) the (non-relativistic) classic Lagrange function for a free electron and a free proton, expressed as a function of the $\mathbf{q}_i$ and their generalized velocities. Convince yourself that the current j^μ is indeed conserved. Use canonical Lagrange formalism to show that the Hamilton function for this system can be written in the form

$$\begin{aligned} H \;&=\; \int \mathrm{d}^3\vec{x}\,\Big[\frac{1}{2m_e}\Big(\vec{p}_e(t)+e\vec{A}(t,\vec{q}_e(t))\Big)^2 \;+\; \frac{1}{2m_p}\Big(\vec{p}_p(t)-e\vec{A}(t,\vec{q}_p(t))\Big)^2 \\ &\qquad -\frac{1}{4\pi}\frac{e^2}{|\vec{q}_e(t)-\vec{q}_p(t)|} \;+\; \frac{1}{2}\,\vec{E}_v^2(t,\vec{x}) \;+\; \frac{1}{2}\,\vec{B}^2(t,\vec{x})\Big]\,, \end{aligned}$$

where $\mathbf{p}_e$ and $\mathbf{p}_p$ are the generalized momenta of the electron and positron, respectively. To get to this result one has to discard the so-called self-energy contributions. What is the physical interpretation of these self-energies. How does the Hamilton function look for a infinitely heavy proton at the origin?