Exercises for $\text{QM}_{\text{Extended}}$, Winter Term 2019, Sheet 3

1) Strong isospin

in 1932 Werner Heisenberg made the hypothesis that the proton and neutron particles are members of a spin-$\frac{1}{2}$ representation as far as a new internal isospin symmetry of the strong interactions is concerned. He also assumed that this symmetry was based on the group SU(2) and that the proton and neutron are members of a SU(2) doublet of the form

$$\begin{pmatrix} |\frac{1}{2}, +\frac{1}{2}\rangle_I \\ |\frac{1}{2}, -\frac{1}{2}\rangle_I \end{pmatrix} = \begin{pmatrix} |p\rangle \\ |n\rangle \end{pmatrix}$$

The three isospin vector operators was given the name $\mathbf{I_i}$ (i=1,2,3). The revolutionary point was that there was no theory for the strong interaction at that time. Only models existed. Nevertheless many powerful predictions could be obtained when using the mathematics that follows from the hypothesis. Heisenberg's approach is still used today in many branches of physics.

(a) Construct the isospin-1 (triplet) and isospin-0 (singlet) states and write the isospin-1 states in column form as above. Discuss obvious objections that one may have against the hypothesis of this SU(2) symmetry. The isospin-0 singlet state turns out to immediately form a single particle (i.e. a bound state made from a proton and neutron) called the "deutron" (symbol: d)

(b) The hypothesis was so successful that physicists took (and still take) it very seriously. It was found that the three pions π^0 (electrically neutral), π^+ (charge of the proton) and π^- (charge of the electron) are members of a isospin-1 triplet of the form

$$\begin{pmatrix} |1, +1\rangle_I \\ |1, 0\rangle_I \\ |1, -1\rangle_I \end{pmatrix} = \begin{pmatrix} |\pi^+\rangle \\ |\pi^0\rangle \\ |\pi^-\rangle \end{pmatrix}$$

Use the Wigner-Eckart theorem to predict the relations between the matrix elements for the processes

(a) $p + p \longrightarrow d + \pi^+$ (two protons turn into a deuteron and a π^+ particle)

(b) $p + n \longrightarrow d + \pi^0$

(c) $n + n \longrightarrow d + \pi^-$

For example, you can write the matrix element of the first process in the form $\langle d, \pi^+|O_s|p, p\rangle \equiv (\langle d| \otimes \langle\pi^+|)O_s(|p\rangle \otimes |p\rangle)$, where O_s is some strong interaction operator that is a scalar under isospin rotations. Which relation between rates for these three processes can you expect if you assume that kinematical effects coming from the small mass differences between the particles can be neglected? (Recall the difference between an amplitude and a measurable quantity).

2) Projection Theorem

Prove the "Projection theorem" discussed in class by using the Wigner-Eckart theorem.

3) Recursion relation for Clebsch-Gordon coefficients

(a) Prove the recursion relation

$$\begin{aligned}&\sqrt{j(j+1)-m(m\pm 1)}\,\langle j_1 j_2; m_1 m_2 | j_1 j_2; j, m\pm 1\rangle \\ &= \sqrt{j_1(j_1+1)-m_1(m_1\mp 1)}\,\langle j_1 j_2; m_1\mp 1, m_2 | j_1 j_2; j, m\rangle \\ &\quad + \sqrt{j_2(j_2+1)-m_2(m_2\mp 1)}\,\langle j_1 j_2; m_1, m_2\mp 1 | j_1 j_2; j, m\rangle\end{aligned}$$

by using the known relations for how the $J_\pm$ angular momentum operators act on $|jm\rangle$ angular momentum states. You may call the upper relation J_+ relation and the lower one J_- relation.

(b) Illustrate (i.e. explain in words) how the recursion relation can be used to determine in principle all Clebsch-Gordon coefficients for "adding" a spin-j_1 and a spin-j_2 representation. Do not forget that some choices of m, m_1, m_2 only one term on the RHS is non-zero and that the relations is only useful when $m = m_1 + m_2$.

4) Rotation matrices for irreducibls spin representations

(a) Derive the relation

$$D^{(j)}_{mm'}(\vec{\theta}) = \sum_{m_1=-j_1}^{j_1}\sum_{m_1'=-j_1}^{j_1}\sum_{m_2=-j_2}^{j_2}\sum_{m_2'=-j_2}^{j_2} \langle j_1 j_2; m_1 m_2 | j_1 j_2; jm\rangle \langle j_1 j_2; m_1' m_2' | j_1 j_2; jm'\rangle \times D^{(j_1)}_{m_1 m_1'}(\vec{\theta})\, D^{(j_2)}_{m_2 m_2'}(\vec{\theta})$$

(b) Use the relation from (a) to (re)derive the spin-1 angular momentum operator matrices $T^{(1)}_{1,2,3}$ from the spin-1/2 angular momentum operator matrices $T^{(1/2)}_{1,2,3}$.

(c) Use the relation from (a) to derive the spin-1 rotation matrix for $\vec{\theta} = (0,0,\theta)$ from the corresponding spin-1/2 rotation matrix and check the result from the direct computation (which is not hard for a rotation around the z-axis).

5)* Spherical tensor operators

[Star exercise to earn additional points.] Show that the relation

$$T_{\ell,m} = \sum_{m_1=-\ell_1}^{\ell_1}\sum_{m_2=-\ell_2}^{\ell_2} \langle \ell_1 \ell_2; m_1 m_2 | \ell_1, \ell_1; \ell m\rangle\, T_{\ell_1,m_1} \otimes T_{\ell_2,m_2}$$

for spherical tensor operators is correct in the sense that the RHS transforms like $T_{\ell,m}$ under rotations

$$U(\vec{\theta}) T_{\ell,m} U^\dagger(\vec{\theta}) = \sum_{m=-\ell}^{\ell} T_{\ell,n}\, D^{(\ell)}_{nm}(\vec{\theta})\,.$$

To proceed, use the previous equality for the $T_{\ell_{1,2},m_{1,2}}$ and two of the relation shown on page (9) of the lecture notes.