Exercises for $\mathrm{QM}_{\mathrm{Extended}}$, Winter Term 2019, Sheet 2

1) Adding angular momenta

Consider a system of two particles with spin. Particle 1 has spin $s_1 = \frac{1}{2}$ and particle 2 has spin $s_2 = 2$, where $\hbar^2 s_i(s_i+1)$ stands for the eigenvalue of the spin operator $\vec{S}_i^2$ $(i = 1, 2)$. Assume now that the combined system has spin $s = \frac{5}{2}$, where $s(s+1)$ is the eivenvalue of the spin operators $\vec{S}^2$ where $\vec{S} = \vec{S}_1 + \vec{S}_2$. The spin in the z-direction of the combined system is $m = \frac{1}{2}$, where m is the eivenvalue of the spin operator $S_z = S_{1,z} + S_{2,z}$. Which values of m_2 (eigenvalues of spin operator $S_{2,z}$) are in this state possible for particle 2? Determine the probabilities that each of these values is obtained in a measurement. The following relation is useful:

$$|\ell, \tfrac{1}{2}; \ell + \tfrac{1}{2}, m\rangle \;=\; \sqrt{\frac{\ell + m + \frac{1}{2}}{2\ell + 1}}\, |\ell, m - \tfrac{1}{2}\rangle \otimes |\tfrac{1}{2}, +\tfrac{1}{2}\rangle \;+\; \sqrt{\frac{\ell - m + \frac{1}{2}}{2\ell + 1}}\, |\ell, m + \tfrac{1}{2}\rangle \otimes |\tfrac{1}{2}, -\tfrac{1}{2}\rangle$$

2) Angular momentum algebra

(a) Let $\vec{\mathbf{J}}$ be an angular momentum operator. Use the commutation relations of the angular momentum operators to show that $\vec{\mathbf{J}}^{\mathbf{2}} = \mathbf{J}_-\mathbf{J}_+ + \mathbf{J_z} + \mathbf{J_z^2} = \mathbf{J}_+\mathbf{J}_- - \mathbf{J_z} + \mathbf{J_z^2}$

(b) Show that $\mathbf{J}_\pm|j, m\rangle = \sqrt{j(j+1) - m(m \pm 1)}|j, m{\pm}1\rangle$ for normalized standard angular momentum states $|j, m\rangle$.

(c) Show that any component of $\vec{\mathbf{J}}$ commutes with any scalar product $\vec{\mathbf{A}} \cdot \vec{\mathbf{B}}$ of two vector operators $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$

3) Orbital angular momentum states and spherical harmonic functions

The (unnormalized) wave function of a particle in 3 dimension has the form

$$\psi(\vec{x}) = (x + y + 3z)e^{-r/r_0} \,,$$

where r_0 is some distance.

(a) Is ψ an eigenfunction of total orbital angular momentum $\vec{\mathbf{L}}^{\mathbf{2}}$? If so, what is the ℓ-value? If not, what are the possible values of ℓ that may be obtained when $\vec{\mathbf{L}}^{\mathbf{2}}$ is measured?

(b) What are the probability for the particle to be found in the various m_ℓ states?

4) Two spins in a magnetic field

The spin-dependent Hamiltonian of an electron-positron system in the presence of a uniform magnetic field B in the z-direction can be written in the form

$$\mathbf{H} = A\,\vec{\mathbf{S}}^{(\mathbf{e^-})} \cdot \vec{\mathbf{S}}^{(\mathbf{e^+})} + \frac{eB}{m}\left(\mathbf{S}_\mathbf{z}^{(\mathbf{e^-})} - \mathbf{S}_\mathbf{z}^{(\mathbf{e^+})}\right) .$$

Assume that the electron-positron system is in the state $|\psi\rangle = |\frac{1}{2}, +\frac{1}{2}\rangle_{(e^-)} \otimes |\frac{1}{2}, -\frac{1}{2}\rangle_{(e^+)}$.

(a) Is this an eigenstate of $\mathbf{H}$ in the limit $A \to 0$ when $\frac{eB}{m} \neq 0$? If it is, what is the energy eigenvalue? If it is not, what is the expectation value of $\mathbf{H}$?

(b) Solve the same problem when $\frac{eB}{m} \to 0$ but $A \neq 0$.

5) Angular momentum matrices for spin-$\frac{3}{2}$

(a) Determine the spin-$\frac{3}{2}$ angular momentum matrices $\mathbf{T}_{\mathbf{1,2,3}}^{(\mathbf{3/2})}$ assuming the (standard) convention that a general state

$$|\psi\rangle = \sum_{m=-3/2}^{3/2} c_m |\tfrac{3}{2}, m\rangle$$

is written as the vector

$$|\psi\rangle \simeq \begin{pmatrix} c_{3/2} \\ c_{1/2} \\ c_{-1/2} \\ c_{-3/2} \end{pmatrix}$$

(b) Write down the rotation matrix $U(\theta\,\vec{e_z})$ for the spin-3/2 states in this convention? How does $U(\theta\,\vec{e_z})$ for the spin-1/2 states? How does the matrix look for $\theta = 2\pi$? What does this mean physically?

6)* Transformation law for density matrices under spatial rotations

This exercise already appeared in the QM1 lecture, Chapter 5.2, and was also discussed once more on an exercise sheet. You do not have to hand it in, but recap the QM1 lecture notes and your old exercise solutions. Exercise (b) just requires to obtain the proper interpretation.

Let $U(\theta\,\vec{e_z})$ be the spin-1/2 rotation matrix that actively rotates a state by an angle θ around the z-axis.

(a) Show:

$$U(\theta\,\vec{e_z})(\vec{n}\cdot\vec{\sigma})U(\theta\,\vec{e_z})^\dagger = \vec{n}'\cdot\vec{\sigma}, \qquad \vec{n} \in \mathbb{R}^3,$$

where $\vec{n}' = R(\theta\,\vec{e_z})\vec{n}$ and $R(\vec{\alpha})$ is the spatial rotation matrix that rotates vectors by an angle $|\vec{\alpha}|$ around the axis $\vec{\alpha}/|\vec{\alpha}|$.

(b) Use the result from (a) to determine the transformation law for the density matrix

$$\rho = \frac{1}{2}(\mathbb{1}_2 + \vec{n}\cdot\vec{\sigma}), \qquad |\vec{n}| = 1.$$

under a rotation given by $U(\theta\,\vec{e_z})$, and argue why $U(\theta\,\vec{e_z})$ indeed has the physical interpretation described above.