Exercises for $\text{QM}_{\text{Extended}}$, Winter Term 2019, Sheet 1

1) Natural units - part 1

(a) Express the following quantities in the elementary particle physics natural units (i.e. in proper eV units using $\hbar = c = 1$): atomic radius (1 $\mathring{A}$), nucleon radius (1 fm = typical size of atomic nuclei), graviational acceleration at the earths surface.

(b) Determine, in natural units the inverse life times (also called decay width $\Gamma = \tau^{-1}$) of the neutron (n) and the muon (μ^-). Compare the results with the masses of these particles in natural units.

2) Natural units - part 2

The following quantities are written down employing natural units. Recover the proper factors of $\hbar$, c and ϵ_0 appearing in SI units.

- Fine structure constant: $\alpha = \frac{e^2}{4\pi} = \frac{1}{137.036}$
- Coulomb potential of electron and proton with relative distance r: $V(r) = -\frac{\alpha}{r}$
- Fine structure formula for the binding energy levels of hydrogen: (e is the electron charge)
$$E_{n,j=\ell\pm\frac{1}{2},m,\ell} = -\frac{m_e e^4}{2(4\pi)^2 n^2}\left[1 - \frac{e^2}{(4\pi)^2 n^2}\left(\frac{3}{4} - \frac{n}{j+1}\right)\right]`$$
- Hydrogen Bohr radius: $a = \frac{4\pi}{m_e e^2}$
- Electron Compton wave length: $\lambda_e = \frac{1}{m_e}$
- Total cross section (in area units) for the scattering process $e^+e^- \to \mu^+\mu^-$ at very high energies: (E_{cm} is the e^+e^- center-of-mass energy)
$$\sigma = \frac{4\pi\, e^4}{3(4\pi)^2 E_{\text{cm}}^2}$$

3) Hermitian operators

Construct the quantum mechanical operator to the classic observable $\vec{\mathbf{x}}\cdot\vec{\mathbf{p}}$ (location times momentum). Construct the quantum mechanical operator to the spatial angular momentum $\vec{\mathbf{L}} = \vec{\mathbf{x}} \times \vec{\mathbf{p}}$. What is different in the two cases? (In this exercise you have to become aware of what the tricky issue is.)

4) Heisenberg picture

Let $X_H(t)$ be the Heisenberg picture location operator of a free particle with mass μ in one dimension. Determine the commutator

$$[X_H(t), X_H(t=0)]\,.$$

5) Schrödinger equation in momentum space representation

Start from the abstract (i.e. representation independent) Schrödinger equation for a spinless particle with mass μ in 3 dimensions in a Coulomb potential $V_c(|\vec{x}|) = -\frac{\alpha}{|\vec{x}|}$ and derive the corresponding momentum space Schrödinger equation.

Hint: Use the Dirac notation involving the eigenstates of the $\vec{X}$ and $\vec{P}$ operators to derive the ansatz for the required calculations. For the determination of the Coulomb potential V_c in momentum space it is useful to do the calculation with the Yukawa potential $V(r) = -\frac{\alpha}{r}e^{-mr}$ $(m > 0)$ and take the limit $m \to 0$ afterwards.

6) Two-state system

Consider a box devided into a right and a left compartment by a thin partition containing a particle. We neglect spatial variations within each half of the box. If the particle is known to be on the right (or left) side with certainty, the state is represented by the position eigenket $|R\rangle$ (or $|L\rangle$). The most general state can then be written as

$$|\alpha\rangle = |R\rangle\langle R|\alpha\rangle + |L\rangle\langle L|\alpha\rangle\,.$$

The particle can tunnel through the partion in such a way that the Hamilton operator of the particle has the form

$$H = \Delta(|L\rangle\langle R| + |R\rangle\langle L|)\,,$$

where Δ is positive and has the dimension of energy. Work out the exercises below using the $\{|L\rangle, |R\rangle\}$ as the basis assuming that both ket states are orthonormal.

(a) Argue why it makes sense to assume that $|L\rangle$ and $|R\rangle$ are orthogonal.

(b) Determine the normalized energy eigenkets and the corresponding energy eigenvalues.

(c) Determine the time-dependent state vector $|\alpha(t)\rangle$ assuming that $|\alpha(t=0)\rangle = |\alpha\rangle$.

(d) Write down the coupled differential equations for $a_L(t) \equiv \langle L|\alpha(t)\rangle$ and $a_R(t) \equiv \langle R|\alpha(t)\rangle$. (Make yourself clear that this is nothing else than the Schrödinger euqation for $|\alpha(t)\rangle$.)

(e) Suppose that at $t = 0$ the particle is with certainty in the right compartment, what is the probability to observe the particle in the left compartment as a function of time for $t > 0$?

(f) The box system just considered, is mathematically equivalent to a corresponding spin system. Describe the physical (i.e. experimental) set up of this spin system.