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I. INTRODUCTION

Noether’s theorem states that every differentiable symmetry of the action corresponds to a conserved quantity. Considering classical electrodynamics, two symmetries immediately come to mind: Poincaré invariance (Lorentz transformations and translations) and gauge invariance. In the absence of sources, Poincaré invariance implies the conservation of energy-momentum and angular momentum of the field, but in the presence of electric charges and currents, these quantities are no longer conserved. In this article, we will derive the corresponding equations using Noether’s theorem.

This paper uses the Einstein summation convention where greek indices take on values between 0 and 3 and latin indices take on values between 1 and 3. The metric signature is (+, −, −, −).

We consider the action from classical Electrodynamics in the Gaussian unit system

\[ S = -\frac{1}{c} \int A_\mu j^\mu \, dt \, dV - \frac{1}{16\pi} \int F_{\mu\nu} F^{\mu\nu} \, dt \, dV \]  

with the Lagrangian

\[ \mathcal{L} = -\frac{1}{c} A_\mu j^\mu - \frac{1}{16\pi} F_{\mu\nu} F^{\mu\nu}, \]

where the electromagnetic field tensor \( F \) is defined as \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \). We will make use of the various derivatives of the Lagrangian (equations A2, A3 and A5) which are calculated in appendix A1.

II. CONSERVATION OF CHARGE

It is well known that the continuity equation for the electric charges follows directly from Maxwell’s equations. Here, we derive the continuity equation from the gauge invariance of the action.

We want to show that \( \partial_\mu j^\mu = 0 \) everywhere. We do this by showing that this equation holds on all compact subsets of \( \mathbb{R}^4 \). Let \( \Omega \subset \mathbb{R}^4 \) be a compact set, and \( \chi : \mathbb{R}^4 \to \mathbb{R} \) an arbitrary smooth function with \( \text{supp} \chi \subset \Omega \) (\( \text{supp} \) denotes the support of a function). Consider the variation of the 4-potential due to a gauge transformation

\[ \delta A_\mu = \partial_\mu \chi. \]  

A formal description of such variations is discussed in appendix A2. The variation of the Lagrangian is then

\[ \delta \mathcal{L} = -\frac{1}{c} (\partial_\mu \chi) j^\mu = -\frac{1}{c} \partial_\mu (\chi j^\mu) + \frac{1}{c} \chi \partial_\mu j^\mu. \]  

This implies the following expression for the variation
of the action

\[ c\delta S = \int_{\mathbb{R}^4} \delta \mathcal{L} \, d^4x \]

\[ = \lim_{r \to \infty} \frac{1}{r^4} \int_{\mathbb{B}(r)} \left[ -\partial_\mu (\chi j^\mu) + \chi \partial_\mu j^\mu \right] \, d^4x \]

\[ = - \lim_{r \to \infty} \frac{1}{r^4} \int_{\partial \mathbb{B}(r)} \chi j^\mu n_\mu \, dS \]

\[ + \lim_{r \to \infty} \frac{1}{r^4} \int_{\mathbb{B}(r)} \chi \partial_\mu j^\mu \, d^4x \]

\[ = \frac{1}{c} \int_{\mathbb{R}^4} \chi \partial_\mu j^\mu \, d^4x \]

where the surface integral over the boundary of \( \mathbb{B}(r) \), a sphere of radius \( r \) with normal \( n \), vanishes for sufficiently large \( r \) due to \( \text{supp} \chi \subset \Omega \), which is compact and thus bounded (by the Heine-Borel theorem). At the same time we have

\[
\delta \mathcal{L} = \frac{\partial \mathcal{L}}{\partial A_\mu} \delta A_\mu + \frac{\partial \mathcal{L}}{\partial A_{\nu,\mu}} \delta_j^{\mu} \]

\[ = \frac{\partial}{\partial x^\mu} \left( \frac{\partial \mathcal{L}}{\partial A_{\nu,\mu}} \right) \delta A_\nu + \frac{\partial}{\partial x^\mu} \left( \frac{\partial \mathcal{L}}{\partial A_{\nu,\mu}} \right) \delta A_\nu \]

\[ = \frac{\partial}{\partial x^\mu} \left( \frac{\partial \mathcal{L}}{\partial A_{\nu,\mu}} \delta A_\nu \right) \]

developed due to the Euler-Lagrange equations for fields. Computing the variation of the action from this expression, one obtains

\[ c\delta S = \lim_{r \to \infty} \int_{\mathbb{B}(r)} \frac{\partial}{\partial x^\mu} \left( \frac{\partial \mathcal{L}}{\partial A_{\nu,\mu}} \delta A_\nu \right) \, d^4x \]

\[ = \lim_{r \to \infty} \int_{\partial \mathbb{B}(r)} \frac{\partial \mathcal{L}}{\partial A_{\nu,\mu}} \partial_\nu \chi n_\mu \, dS = 0 \]

since \( \chi \) is smooth and has compact support. Comparing with equation (5), we have

\[ \int_{\Omega} \chi \partial_\nu j^\mu \, d^4x = 0 \]

for all smooth functions \( \chi \) with compact support in \( \Omega \). By the Du Bois-Reymond lemma (the fundamental lemma of calculus of variations), this implies that \( \partial_\nu j^\mu = 0 \) on \( \Omega \). Since the compact set \( \Omega \subset \mathbb{R}^4 \) was arbitrary, this implies \( \partial_\nu j^\mu = 0 \) everywhere.

III. NON-DYNAMICAL FIELDS AS SOURCES

For the further discussion, we consider a certain form of Noether’s theorem which allows for non-dynamical fields (here: \( j^\mu \)). This means that our model contains only the equations of motion for the electromagnetic field and not those of the source fields \( j^\mu \). We thus obtain the tensors for energy-momentum and angular-momentum of the field only and expect these quantities not to be conserved (only the total quantities are expected to be conserved in general).

We consider a symmetry transformation of the fields \( A_\mu \) and \( j^\mu \), which changes the Lagrangian by a divergence \( \delta \mathcal{L} = \partial_\mu D^\mu \). At the same time we have

\[ \delta \mathcal{L} = \frac{\partial \mathcal{L}}{\partial j^\mu} \delta j^\mu + \frac{\partial \mathcal{L}}{\partial A_\mu} \delta A_\mu + \frac{\partial \mathcal{L}}{\partial A_{\nu,\mu}} \delta A_{\nu,\mu} \]

\[ = \frac{\partial \mathcal{L}}{\partial j^\mu} \delta j^\mu + \frac{\partial \mathcal{L}}{\partial A_\mu} \delta A_\mu \]

by the same argument as before. (We have only used the equations of motion for the electromagnetic field, not those of the source fields.) Comparing the two expressions for \( \delta \mathcal{L} \), we find

\[ \frac{\partial}{\partial x^\mu} \left( \frac{\partial \mathcal{L}}{\partial A_{\nu,\mu}} \delta A_\nu - D^\mu \right) + \frac{\partial \mathcal{L}}{\partial j^\mu} \delta j^\mu = 0. \]

We see that the non-dynamical 4-current density acts as a source of the quantity in the parenthesis.

In the next two sections, we will examine two instances of this equation where the 4-current density acts as a source of the electromagnetic energy-momentum tensor and radiates angular momentum via the electromagnetic field.

A. Conservation of energy-momentum

Consider a translation of both the source fields and the electromagnetic field in the direction of an arbitrary 4-vector \( n \), coupled with a gauge transformation by a function \( \chi \). The variations of the fields are then given by

\[ \delta j^\mu = \delta \rho_j n^\rho + \delta \mu \chi \]

\[ \delta A_\mu = \delta \rho A_\mu n^\rho + \delta \mu \chi \]

A derivation of the variations of the fields due to translations is given in appendix A2b. We choose \( \chi = -A_\mu n^\mu \) to make the variation of the 4-potential gauge invariant, since we then have

\[ \delta A_\mu = \delta \rho A_\mu n^\rho + \delta \mu \chi \]

\[ = (\delta \rho A_\mu - \delta \mu A_\mu) n^\rho = F_{\mu \nu} n^\rho. \]

This gives us the following variations

\[ \delta j^\mu = \delta \rho_j n^\rho \]

\[ \delta A_\mu = F_{\mu \nu} n^\rho. \]

Using the continuity equation \( \partial_\mu j^\mu = 0 \), we find the variation of the Lagrangian density to be a divergence

\[ \delta \mathcal{L} = \delta \rho \mathcal{L} n^\rho - \frac{1}{c} (\partial_\mu \chi) j^\mu = \delta \mu D^\mu, \]

1 Whether the gauge transformation is performed prior or after the translation does not change the variation, which depends only on terms of first order in the variation parameter.
where the first term stems from the translation (due to the chain rule) and the second one originates from the gauge transformation. We have set

$$D^\mu = \left( \mathcal{L}_\rho - \frac{1}{c} A_\rho j^\mu \right) \eta^\rho.$$  

Plugging everything into equation 10, we obtain

$$\frac{\partial}{\partial x^\mu} \left( F^{\rho \mu} + \frac{1}{16\pi} F^{\alpha \beta} F_{\alpha \beta} \delta^\mu_\rho \right) n^\rho$$

$$+ \frac{\partial}{\partial x^\mu} \left( \frac{1}{c} A_{\rho j^\mu} - \frac{1}{c} A_{\mu j^\rho} \right) n^\rho$$

$$- \frac{1}{c} A_{\mu} \partial \rho j^\mu n^\rho = 0$$  

(16)

The terms in parentheses in the first line define the energy-momentum tensor of the electromagnetic field

$$T_{\nu}^\mu = \frac{1}{4\pi} F^{\rho \mu} F_{\rho \nu} + \frac{1}{16\pi} F^{\rho \sigma} F_{\rho \sigma} \delta^\mu_\nu.$$  

(17)

Since the vector \( n \) is arbitrary, we can deduce

$$\frac{\partial}{\partial x^\mu} \left( T_{\nu}^\mu + \frac{1}{c} A_{\nu} j^\mu - \frac{1}{c} A_{\mu} j^\nu \right)$$

$$- \frac{1}{c} A_{\mu} \partial \rho j^\mu = 0$$  

(18)

The terms apart from the energy-momentum tensor can be simplified using the continuity equation:

$$\partial_\mu (A_{\nu} j^\nu - A_{\rho} j^\mu)$$

$$= \partial_\mu A_{\nu} j^\nu - \partial_\mu A_{\rho} j^\mu - A_{\rho} \partial_\mu j^\mu$$  

(19)

Finally, we arrive at the result

$$\partial_\mu T^{\mu \nu} + f^\nu = 0.$$  

(20)

where the 4-force density is \( f^\nu = \frac{1}{2} F^{\mu \nu} j_\nu \). Consequently, the divergence of the energy-momentum tensor of the electromagnetic field vanishes in the absence of charges and currents.

B. Conservation of angular momentum

We now consider the variations of the fields due to a Lorentz transformation (as derived in appendix A.2c) coupled with a gauge transformation by a function \( \chi \).

$$\delta j^\mu = \partial_\rho j^\mu a^\rho \alpha x^\sigma - a^\mu \alpha j^\nu$$

$$\delta A_\mu = \partial_\rho A_\mu a^\rho \alpha x^\sigma - a^\mu \alpha A_\nu + \partial_\mu A = \partial_\rho A_\mu a^\rho \alpha x^\sigma - a^\mu \alpha A_\nu + \partial_\mu \chi$$  

(21)

Choosing the gauge function \( \chi = -A_\rho a^\rho \alpha x^\sigma \), we have

$$\delta j^\mu = \partial_\rho j^\mu a^\rho \alpha x^\sigma - a^\mu \alpha j^\nu$$

$$\delta A_\mu = F_{\rho \sigma} a^\rho \alpha x^\sigma - a^\mu \alpha j^\nu$$  

(22)

The variation of the Lagrangian is

$$\delta \mathcal{L} = \partial_\mu \mathcal{L} a^\mu \alpha x^\sigma - \frac{1}{c} \partial_\mu \chi j^\mu.$$  

(23)

The two terms arise, again, from the chain rule and the gauge transformation, respectively. Using the continuity equation \( \partial_\mu j^\mu = 0 \) and the vanishing trace of the coefficients \( a^\mu \alpha = a^\nu \alpha \sigma \delta^\sigma_\nu = a^\mu \alpha \partial_\mu x^\sigma = 0 \), this can be rewritten as a divergence

$$\delta \mathcal{L} = \partial_\mu \mathcal{L} a^\mu \alpha x^\sigma + \mathcal{L} [a^\mu \alpha \partial_\mu x^\sigma - \frac{1}{c} \partial_\mu \chi j^\mu - \frac{1}{c} \partial_\mu j^\mu]$$  

(24)

We can read off the vector \( D^\mu \) as

$$D^\mu = \mathcal{L}_\rho a^\rho \alpha x^\sigma - \frac{1}{c} \partial_\mu \chi j^\mu$$

$$= \left( \frac{1}{16\pi} F^{\rho \sigma} F_{\rho \sigma} - \frac{1}{c} A_{\alpha j^\mu} \right) a^\rho \alpha x^\sigma$$

$$+ \frac{1}{c} A_{\rho} a^\rho \alpha x^\sigma j^\mu$$  

(25)

Plugging everything into equation 10, we find

$$\partial_\mu \left( \frac{1}{4\pi} F^{\rho \mu} F_{\rho \nu} x^\sigma + \frac{1}{16\pi} F^{\rho \sigma} F_{\rho \sigma} \delta^\mu_\nu \right) a^\rho \alpha x^\sigma$$

$$+ \partial_\mu \left[ \frac{1}{c} A_{\rho} a^\rho \alpha x^\sigma j^\mu - \frac{1}{c} A_{\rho} a^\rho \alpha j^\mu \right]$$

$$- \frac{1}{c} A_{\mu} \partial_\rho j^\mu a^\rho \alpha x^\sigma + \frac{1}{c} A_{\rho} a^\rho \alpha j^\mu = 0$$  

(26)

We can recognize the first term as the divergence of \( T_{\nu}^\mu a^\rho \alpha x^\sigma \). The second and third line form one single coefficient of \( a^\rho \alpha x^\sigma \), which we can simplify (here without the factor \( 1/c \)) using the continuity equation

$$\partial_\mu [A_{\rho} a^\rho \alpha x^\sigma \delta^\rho_\nu - A_{\nu} a^\rho \alpha j^\sigma] - A_{\rho} \partial_\mu j^\sigma + A_{\nu} \partial_\mu j^\nu$$

$$= \partial_\mu A_{\rho} a^\rho \alpha j^\nu - \partial_\mu A_{\rho} a^\rho \alpha x^\sigma$$

$$= \partial_\mu A_{\rho} a^\rho \alpha j^\nu - \partial_\mu A_{\rho} a^\rho \alpha x^\sigma + A_{\rho} \partial_\mu j^\nu$$

$$= F_{\rho \sigma} j^\rho x^\sigma + A_{\rho} j^\rho \delta^\rho_\nu$$  

(27)

The last term gives zero when contracted with \( a^\rho \alpha \), due to its vanishing trace. Consequently, we have

$$\left[ \partial_\mu (T_{\mu \nu} x^\sigma) + \frac{1}{c} F_{\mu \nu} a^\rho \alpha x^\sigma \right] a_{\rho \sigma} = 0$$  

(28)

Since this equation has to be satisfied for all antisymmetric \( a_{\rho \sigma} \), the antisymmetric part of the bracket must vanish. Thus, we finally arrive at

$$\partial_\mu (x^\sigma T_{\mu \nu} - x^\nu T_{\mu \sigma}) + x^\sigma f^\nu - x^\rho f^\rho = 0.$$  

(29)

The term in the parenthesis defines the (orbital) angular momentum density

$$M^{\mu \rho} = x^\mu T_{\mu \rho} - x^\sigma T_{\rho \sigma}$$  

(30)

and the above equation now reads

$$\partial_\mu M^{\mu \rho} + x^\mu f^\nu - x^\nu f^\mu = 0.$$  

(31)

Thus, the divergence of the angular momentum tensor vanishes in the absence of charges and currents.
The field energy in a given volume

\[ Q = \frac{1}{c} \int_J \delta^0 \, dV. \]  

(32)

where \( \varrho = \delta^0/c \) is the charge density. The time derivative of the charge in \( \mathcal{V} \) is then given by

\[ \frac{d}{dt} Q = \int_{\mathcal{V}} \partial_\nu j^\nu \, dV = \int_{\mathcal{V}} \left( \partial_\nu j^\nu - \partial_k j^k \right) \, dV = -\oint_{\partial\mathcal{V}} j^k \, dA_k \]  

(33)

due to the divergence theorem. The charge contained in \( \mathcal{V} \) can thus only change by means of an electric current through the boundary, which is given by the surface integral of the current density \( j^\nu \) over the surface \( \partial\mathcal{V} \).

\[ \frac{d}{dt} P^\mu = -\int_{\mathcal{V}} j^\mu \, dV - \oint_{\partial\mathcal{V}} \sigma^{ki} \, dA_k \]  

(38)

The electromagnetic momentum in a given volume thus changes due to the Lorentz force applied to charges and due to radiation through the boundary surface. The density of the momentum radiation is given by the Maxwell stress tensor \( \sigma \) with components \( \sigma^{ik} = T^{ik} \). Note that some authors define \( \sigma \) with the opposite sign.

Written in matrix form, the components of the electromagnetic energy-momentum tensor are given by

\[ (T^{\mu\nu}) = \begin{pmatrix} W & S_{1/c} & S_{2/c} & S_{3/c} \\ S_{1/c} & \sigma_{11} & \sigma_{12} & \sigma_{13} \\ S_{2/c} & \sigma_{21} & \sigma_{22} & \sigma_{23} \\ S_{3/c} & \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix} \]  

(39)

where we lower indices of spatial tensors with the spatial metric coefficients \( \delta_{ij} \), which are the negative of the spatial components of the four-dimensional metric \( \eta_{ij} \).

The electromagnetic energy content of a region thus changes due to work done on the charges inside the region and the energy flux through the surface area via the energy flux density or Poynting vector \( S \) with the components \( S^k = cT^{k0} \).

b. Momentum The quantity associated with spatial translations is the momentum \( P \) with the components \( P^\mu \). The quantities \( T^{0i}/c \) form the momentum density of the electromagnetic field and due to the symmetry of the energy-momentum tensor, the energy flux density \( S \) is equal to the momentum density multiplied by \( c^2 \). The equation for the change of electromagnetic momentum in the region \( \mathcal{V} \) is

\[ \frac{d}{dt} P^i = \int_{\mathcal{V}} f^i \, dV - \oint_{\partial\mathcal{V}} \sigma^{ki} \, dA_k \]  

(40)

In complete analogy to the preceding discussion, introduce the doubly indexed quantity

\[ M^{\mu\nu} = \frac{1}{c} \int_{\mathcal{V}} M^{\mu\nu 0} \, dV, \]  

which is (as we saw) related to Lorentz-transformations. The time derivative thereof is given by

\[ \frac{d}{dt} M^{\mu\nu} = \int_{\mathcal{V}} (x^\mu f^\nu - x^\nu f^\mu) \, dV - \oint_{\partial\mathcal{V}} M^{\nu k} \, dA_k \]  

(41)

This antisymmetric tensor can be decomposed into a polar vector \( \mathbf{N} \) with components \( M^{0i}/c \) and an antisymmetric spatial tensor with components \( M^{ij} \). This spatial antisymmetric tensor is dual to an axial vector \( \mathbf{M} \). Symbolically, we write this as \( \langle M^{\mu\nu} \rangle = (c\mathbf{N}, \mathbf{M}) \). This is completely analogous to the decomposition of the electromagnetic field tensor into the electric and magnetic fields, written as \( \langle F^{\mu\nu} \rangle = (\mathbf{E}, -\mathbf{B}) \).
a. Angular Momentum

By our derivation, the spatial components \( M^{ik} \) are related to spatial rotations. The components of the antisymmetric tensor \( M^{ik} \) can be combined to form the pseudovector of angular momentum \( M \) with components

\[
M_i = \frac{1}{2} \varepsilon_{ijk} M^{jk} \quad \text{such that} \quad M^{ij} = \varepsilon^{ijk} M_k, \tag{42}
\]

where \( \varepsilon_{ijk} \) is the permutation symbol. The rate of change of the angular momentum evaluates to

\[
\frac{d}{dt} M_i = - \int \varepsilon_{ijk} x^j f^i \, dV - \oint_{\partial V} \varepsilon_{ijk} x^j \sigma^{kl} \, dA_k \tag{43}
\]

So the angular momentum of the electromagnetic field in a given volume changes due to the torque applied to charges inside the volume and the radiation of momentum through the surface of the volume.

b. Mass Moment

The components \( M^{0i} \) form the vector of dynamical mass moment \( \mathbf{N} \) by

\[
N^i = \frac{1}{c} M^{0i} = \int_{V} (t T^{0i}/c - x^i W/c^2) \, dV = t P^i - (\varepsilon^i/c^2) R^i \tag{44}
\]

where \( R \) is the center of energy given by

\[
R^i = \frac{1}{\varepsilon} \int_{V} x^i W \, dV. \tag{45}
\]

The rate of change of the mass moment is given by

\[
\frac{d}{dt} N^i = - \int_{V} (t f^i - x^i f^0/c^2) \, dV - \oint_{\partial V} (x^i S^k/c^2) \, dA_k \tag{46}
\]

We see that mass moment of the electromagnetic field is transferred to charges via the Lorentz force and is radiated through the Lorentz force, and is radiated through the surface area. The radiation term consists of two parts: the radiation of momentum (described by the stress tensor \( \sigma \)) and the energy flux (described by the Poynting vector \( S \) divided by \( c^2 \)).

To summarise, we write the components of the angular momentum tensor in matrix form.

\[
(M^{\mu\nu}) = \begin{pmatrix}
0 & cN_1 & cN_2 & cN_3 \\
-cN_1 & 0 & M_3 & -M_2 \\
-cN_2 & -M_3 & 0 & M_1 \\
-cN_3 & M_2 & -M_1 & 0
\end{pmatrix} \tag{47}
\]

V. CONCLUSION

We have derived the conservation of charge and the laws governing the energy-momentum tensor and the angular momentum tensor of the electromagnetic field in the presence of charges and currents from the gauge symmetry and Poincaré symmetry of the action. Gauge transformations, which were chosen such that the variations became gauge independent, directly lead us to the symmetric and gauge independent energy-momentum tensor, instead of the canonical energy-momentum which is neither gauge invariant nor symmetric.

The presented derivation suggests the interpretation of the found quantities, for example by relating energy to translations in time and momentum to translations in space. This lead us to a consistent description of all dynamical quantities found.

VI. REFERENCES

Landau and Lifshitz provide in [1] an excellent overview of electrodynamics and discusses all dynamical quantities introduced in this paper. In [2] Pauli discusses both the historical introduction of the energy-momentum tensor of the field due to Minkowski and a variational approach similar to ours. In [3] Rindler introduces the energy-momentum tensor of the field by considering the forces acting on charge distributions. The method of obtaining a gauge invariant energy-momentum tensor from Noether’s theorem by considering transformations together with a coupled gauge transformation was discussed before e.g. in [4].

Appendix A: Appendix

1. Derivatives of the Lagrangian

We compute the derivatives of the Lagrangian

\[
\mathcal{L} = -\frac{1}{c} A_{\mu} j^\mu - \frac{1}{16\pi} F^{\mu\nu} F_{\mu\nu}, \tag{A1}
\]

with respect to the fields. The field tensor is \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = A_{\nu,\mu} - A_{\mu,\nu} \). We can immediately read off

\[
\frac{\partial \mathcal{L}}{\partial j^\mu} = \frac{1}{c} A_\mu \tag{A2}
\]

\[
\frac{\partial \mathcal{L}}{\partial A_\mu} = -\frac{1}{c} j^\mu \tag{A3}
\]

For the derivative with respect to \( A_{\mu,\nu} \) we first compute

\[
\frac{\partial F_{\mu\sigma}}{\partial A_{\mu,\nu}} = \frac{\partial A_{\sigma,\mu}}{\partial A_{\mu,\nu}} = \frac{\partial A_{\mu,\sigma}}{\partial A_{\mu,\nu}} = \frac{\partial A_{\mu,\nu}}{\partial A_{\mu,\nu}} = \delta_{\mu}^\sigma \delta_{\nu}^\rho - \delta_{\mu}^\rho \delta_{\nu}^\sigma \tag{A4}
\]

But since \( F^{\mu\nu} F_{\mu\nu} \) is a quadratic form in the components of \( F \), we have

\[
\frac{\partial (F^{\mu\sigma} F_{\mu\sigma})}{\partial A_{\mu,\nu}} = 2 F^{\mu\rho} \frac{\partial F_{\rho\sigma}}{\partial A_{\mu,\nu}} = 2 F^{\mu\rho} (\delta_{\rho}^\sigma \delta_{\sigma}^\nu - \delta_{\rho}^\nu \delta_{\nu}^\sigma) = 2 F^{\mu\nu} - 2 F^{\mu\mu} = -4 F^{\mu\nu}
\]
which gives us the result
\[
\frac{\partial L}{\partial A_{\mu \nu}} = \frac{1}{4\pi} F_{\mu \nu}
\]  
(A5)

2. Variations of the fields

We define the variation of a function with respect to some variational parameter \( \epsilon \) in the following way. Let \( f(\epsilon, x, y, ...) \) be any analytic expression in \( \epsilon \) and an arbitrary number of other arguments. We then set
\[
\delta f(x, y, ...) = \left[ \frac{d}{d\epsilon} f(\epsilon, x, y, ...) \right]_{\epsilon=0} (A6)
\]
\( \delta f(x, y, ...) \) is thus the second coefficient of the Taylor expansion of \( f(\epsilon, x, y, ...) \) around \( \epsilon = 0 \)
\[
f(\epsilon, x, y, ...) = f(x, y, ...) + \epsilon \delta f(x, y, ...) + \mathcal{O}(\epsilon^2) (A7)
\]
Note that this variation obeys a chain rule. For brevity, we will not write the argument \( \epsilon \) explicitly in our formulas.

a. Gauge transformations

Given any smooth function \( \chi \), we can construct a one-parameter family of gauge transformations by
\[
\bar{A}_\mu(x) = A_\mu(x) + \epsilon \partial_\mu \chi(x) (A8)
\]
which trivially satisfies
\[
\delta A_\mu = \partial_\mu \chi(x). (A9)
\]

b. Translations

We compute the variations of arbitrarily valued tensor fields due to translations. Let \( n \) be an arbitrary vector and define the one-parameter family of coordinate transformations
\[
\begin{align*}
x^\mu &= \bar{x}^\mu + \epsilon n^\mu \\
\bar{x}^\mu &= x^\mu - \epsilon n^\mu (A10)
\end{align*}
\]
Since the coordinates are not rotated, the components of tensor fields transform trivially and we can restrict ourselves to scalar fields, which transform in the following way.
\[
\delta f(x) = \partial_\rho f(x)n^\rho + \mathcal{O}(\epsilon^2) \quad (A11)
\]
So the variation of any field \( f \) is simply
\[
\delta f(x) = \partial_\rho f(x)n^\rho. \quad (A12)
\]
c. Lorentz transformations

We derive the general formula for the variations of scalar, vector and covector valued fields due to Lorentz transformations.

Let \( (a^\alpha) \) be an element of the Lie algebra of the Lorentz group and consider the one-parameter family of Lorentz transformations obtained from this element by exponentiation.
\[
\begin{align*}
x^\mu &= L^\mu_{\nu} \bar{x}^\nu + \epsilon a^\rho x^\rho + \mathcal{O}(\epsilon^2) \\
\bar{x}^\mu &= (L^{-1})^\mu_{\nu} x^\nu - \epsilon a^\rho x^\rho + \mathcal{O}(\epsilon^2) \quad (A13)
\end{align*}
\]
As is well known, we have to require that \( a_{\mu\nu} = -a_{\nu\mu} \) for this to define a Lorentz transformation. (The generators of the Lorentz group are antisymmetric matrices.) For a scalar field \( f \) we then have
\[
\delta f(\bar{x}) = f(x) = f(\bar{x} + \epsilon \bar{x} + \mathcal{O}(\epsilon^2))
\]
\[
= f(\bar{x}) + \epsilon \partial_\rho f(\bar{x})a^\rho \bar{x}^\sigma + \mathcal{O}(\epsilon^2). \quad (A14)
\]
A vector field \( V \) transforms as
\[
\delta V^\mu(\bar{x}) = \frac{\partial \bar{V}^\mu}{\partial V^\nu} V^\nu(x)
\]
\[
= (\delta^\mu_{\nu} - \epsilon a^\rho_{\nu} + \mathcal{O}(\epsilon^2))
\times (V^\nu(x) + \epsilon \partial_\rho V^\nu(x)a^\rho \bar{x}^\sigma + \mathcal{O}(\epsilon^2))
\]
\[
= V^\mu(x)
\]
\[
+ \epsilon [\partial_\rho V^\nu(a^\rho \sigma_{\nu}) - a^\rho_{\nu} V^\nu(a^\rho \sigma)] + \mathcal{O}(\epsilon^2), \quad (A15)
\]
while for a covectorfield \( \alpha \) we have
\[
\delta \alpha_\mu(\bar{x}) = \frac{\partial \bar{\alpha}_\mu}{\partial x^\nu} \alpha_\nu(x)
\]
\[
= (\delta^\mu_{\nu} + \epsilon a^\rho_{\nu} + \mathcal{O}(\epsilon^2))
\times (\alpha_\nu(x) + \epsilon \partial_\sigma \alpha_\nu(x)a^\rho \bar{x}^\sigma + \mathcal{O}(\epsilon^2))
\]
\[
= \alpha_\mu(x) + \epsilon [\partial_\rho \alpha_\nu(x) + a^\rho_{\sigma} \alpha_\nu(x)] + \mathcal{O}(\epsilon^2) \quad (A16)
\]
So the variations of scalar, vector and covector valued fields are of the following form.
\[
\delta f(x) = 0
\]
\[
\delta V^\mu(x) = \partial_\rho V^\mu(x)a^\rho \sigma x^\sigma - a^\rho_{\nu} V^\nu(x) \quad (A17)
\]
\[
\delta \alpha_\mu(x) = 0
\]
\[
\delta \alpha_\mu(x) = \partial_\rho \alpha_\nu(x)a^\rho \sigma x^\sigma + a^\rho_{\nu} \alpha_\sigma(x)
\]