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TRAVELLING WAVES FOR GAMES IN ECONOMICS AND BIOLOGY

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1. INTRODUCTION

An important class of problems for models based on systems of reaction-diffusion equations leads to so-called 'bistable' situations, that is situations where the reaction system has two stable equilibria (A and C, say) and an unstable saddle (B, say). In principal it is often relatively easily to estimate the basins of attraction of A,C which commonly have the stable manifold of the saddle as boundary. However, the situation is a great deal more difficult for the corresponding reaction-diffusion system, as it is complicated by the presence of spatial boundaries and the size of the diffusion coefficients - among other factors. An approach to this problem for the scalar case, which resolves at least some of problems, is to take the spatial domain to be $\mathbb{R}$ and to interpret the attractivity of the equilibria in terms of the direction of the wave (strictly we are concerned with wave-fronts, but we shall follow normal practice and always refer to it as a wave or travelling wave) from A to C. This approach is reviewed in [1], where the term 'dominance' for the equilibrium towards which the travelling wave moves is used. Much may be said in this case, and it is known that for an important class of examples with two stable equilibria 0,1 separated by an unstable equilibrium α ($0<\alpha<1$), the dominance of 1 is equivalent to the fact that its basin of attraction contains functions of compact support, and so is 'infinitely larger' than that of 0. Dominance, in both interpretations, is a very important concept in applications as it describes the ability of species or types to 'invade'.

For two equations, corresponding to two types, much less is known. Most of the literature is concerned with the difficult problem of existence and uniqueness, see [2, 3, 4] for example, but there is little known about the questions of determining the direction of travelling waves. The purpose of the present investigation is to see what progress may be made in devising techniques for answering this question. In comparing equilibria, we shall keep the term 'dominance' to mean that an equilibrium 'wins' in the travelling-wave sense. However, for two equations, the ratio of the diffusion coefficients is often crucial, and we introduce the term 'uniform dominance' to mean that an equilibrium is dominant for all values of this ratio. This serves to distinguish cases where the direction of the wave may not be changed by modifying the diffusion coefficients.

The investigation is based on a particular class of equations which arises in evolutionary game theory, namely bi-matrix games. These have been used to model asymmetric conflicts in both biology (like Dawkins' battle of the sexes, see [5, p.139]) and economics (two-person normal-form games, see [6]). We shall concentrate here on the economics game as we believe that it is important to develop and analyse models with a spatial structure, an idea which does not appear to have been extensively followed.

A brief description of the background for the reaction equations is as follows. There are two types of players $\mathcal{A}$ and $\mathcal{B}$. $\mathcal{A}$ players have n options and choose to play option i with probability

p_i while $\mathcal{B}$ players have m options and choose option j with probability q_j. The pay-off to an $\mathcal{A}$ player, when it plays option i and its opponent plays j, is a_{ij} and the pay-off to the $\mathcal{B}$ player is b_{ji}. Thus the expected pay-off to an $\mathcal{A}$ player when it plays option i is $(Aq)_i$, where $A=(a_{ij})$. Adding an evolutionary-dynamic perspective to this game leads to the so-called replicator dynamic, see e.g. [5, p.141],

$$\dot{p}_i = p_i[(Aq)_i - p^T Aq] \quad (1 \le i \le n),$$
$$\dot{q}_j = q_j[(Bp)_j - q^T Bp] \quad (1 \le j \le m),$$

where a dot denotes d/dt and t is time. We consider the simplest case when $m=n=2$ and write $p_1=u$, $p_2=1-u$, $q_1=v$, $q_2=1-v$. The above equations are now

$$\begin{aligned} \dot{u} &= u(1-u)[(a_1+a_2)v-a_1], \\ \dot{v} &= v(1-v)[(b_1+b_2)u-b_1] \end{aligned} \tag{1.1}$$

where $a_1=a_{22}-a_{12}$, $a_2=a_{11}-a_{21}$, $b_1=b_{22}-b_{12}$, $b_2=b_{11}-b_{21}$. Both (0,0) and (1,1) are stable equilibria for the system (1.1) and also Nash equilibria for the underlying game when a_1, a_2, b_1, b_2 are all positive (as will be assumed from here onwards). This game is equivalent to a so-called coordination game: Players $\mathcal{A}$ and $\mathcal{B}$ obtain pay-offs a_i and b_i (respectively) if both choose the same option i $(i=1,2)$, and receive nothing if they choose different options. Economists have devoted a considerable amount of effort to finding criteria for deciding between such equilibria. The most complete 'equilibrium selection theory' is due to [6] who introduced the concept of risk-dominance. In the above special class of 2×2 (coordination) games,

(0,0), which is the pure strategy 1, risk-dominates
(1,1), which is the pure strategy 2, if and only if,

$$a_1 b_1 > a_2 b_2. \tag{1.2}$$

Recently [7] has given a dynamic justification for this criterion. The idea is to introduce spatial structure and to compare the concept of risk-dominance with the above-mentioned dominance of equilibria determined by the direction of the travelling wave. In [7] best-response dynamics are assumed. The piece-wise linearity of this model simplifies the analysis considerably and it is shown that the two concepts of dominance, that is risk-dominance and dominance in the travelling-wave sense, agree exactly. Thus the risk-dominant equilibrium can be justified as being the ultimately successful strategy that can withstand all invasion attempts. Here we use the replicator dynamic (1.1) and endeavour to develop a truly evolutionary approach to equilibrium selection. We show that in this more complex model, the two concepts no longer give the same criterion. The replicator dynamic also plays an important role in economic game theory as a dynamic model for imitation, see [8]. In detail, the equations we consider here, with spatial variable x, are

$$\begin{aligned} \frac{\partial u}{\partial t} &= u(1-u)[(a_1+a_2)v-a_1] + d_1\frac{\partial^2 u}{\partial x^2}, \\ \frac{\partial v}{\partial t} &= v(1-v)[(b_1+b_2)u-b_1] + d_2\frac{\partial^2 v}{\partial x^2} \end{aligned} \tag{1.3}$$

on $\mathbb{R}\times(0,\infty)$. This pair is known, see Section 2, to have a travelling-wave solution of the form $(u(x-ct), v(x-ct))$ where c is the constant wave speed and $u(\cdot), v(\cdot)$ are increasing with $u(-\infty)=v(-\infty)=0$, $u(\infty)=v(\infty)=1$. Clearly, if $c>0$ (respectively $c<0$) (0,0) (respectively (1,1)) wins in the sense that the travelling wave is towards that equilibrium, or equivalently, (0,0) (respectively (1,1)) is dominant. Such reaction-diffusion equations for games were first considered in [9] and [10]. One could assume $d_1=d_2$ (both players equally mobile) as in [7], but as will be seen in the following, this restriction is neither essential nor helpful here. The principal aim is then to devise techniques for finding the sign of c.

The plan of the paper is as follows. In Section 2 the background for the travelling-wave problem is set up and some basic results on sgn(c) are obtained. Section 3 describes an asymptotic method which gives detailed and explicit results when the reaction rates a_1, a_2 are large. A conjecture (which gives the qualitative picture for all values of the parameters) suggested by these results is described. In Section 4 the important bargaining game is considered and certain key results obtained by a perturbation method. Section 5 contains some summarising remarks and indicates where further work is required.

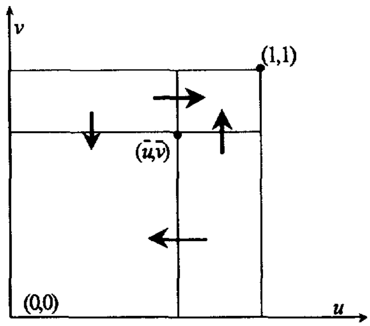

Fig. 2.1 Phase plane for the reaction system arising from Eqn (2.1)

2. THE TRAVELLING WAVE

For the study of travelling waves it is convenient to re-write the system (1.3) in a form which makes the key role of the interior equilibrium point apparent:

$$\frac{\partial u}{\partial t} = Au(1-u)(v-\bar{v}) + d_1\frac{\partial^2 u}{\partial x^2},$$
$$\frac{\partial v}{\partial t} = Bv(1-v)(u-\bar{u}) + d_2\frac{\partial^2 v}{\partial x^2}, \tag{2.1}$$

where $\bar{u} = b_1/(b_1+b_2)$, $\bar{v} = a_1/(a_1+a_2)$, $A = (a_1+a_2)$ and $B = (b_1+b_2)$. A and B may be regarded as rate constants for the interactions. Set $u(1-u)(v-\bar{v}) = f(u,v,\bar{v})$ and $v(1-v)(u-\bar{u}) = g(u,v,\bar{u})$. The reaction system has stable equilibria at (0,0) and (1,1), a saddle at $(\bar{u},\bar{v})$ and also equilibria at (0,1) and (1,0). These last two play no role and will not be mentioned again. The phase plane is shown in Fig 2.1. We look for a monotone travelling wave, by convention increasing, from (0,0) to (1,1). Thus solutions will be of the form $(u(x-ct), v(x-ct))$ with $u(-\infty) = v(-\infty) = 0$ and $u(\infty) = v(\infty) = 1$, where, for example, $u(\infty) = \lim_{z\to\infty} u(z)$. To fit in with the standard theory of reaction-diffusion equations, it is convenient to consider the following related pair of equations

$$\frac{\partial u}{\partial t} = f(u,v,\bar{v}) + D\frac{\partial^2 u}{\partial x^2},$$
$$\frac{\partial v}{\partial t} = g(u,v,\bar{u}) + E\frac{\partial^2 v}{\partial x^2}, \tag{2.2}$$

where $D = d_1/A$, $E = d_2/B$. As we shall see, the sign of the wave speed for the system (2.2) (but not its magnitude) is identical with that of (2.1). The substitution of $(u(x-ct), v(x-ct))$ into (2.2) yields the following pair of ordinary differential equations,

$$D\ddot{u} + c\dot{u} + f = 0, \quad E\ddot{v} + c\dot{v} + g = 0. \tag{2.3}$$

These in turn are written as the following system of travelling-wave equations:

$$\begin{aligned} \dot{u} &= p, \\ D\dot{p} &= -cp - f, \\ \dot{v} &= q, \\ E\dot{q} &= -cq - g. \end{aligned} \tag{2.4}$$

Here a dot denotes differentiation with respect to the travelling-wave variable $z = x - ct$. The travelling wave is then a heteroclinic connection joining the saddle points (0,0,0,0) and (1,0,1,0) in $\mathbf{R}^4$. The existence and uniqueness theory for such a system is by no means trivial. However, it is known [4] that there is a unique connection for exactly one value of c and that this is stable when regarded as a solution of (2.2). Furthermore, c is a continuous function of $\bar{u}, \bar{v}, D, E$. It is easy to deduce by a homotopy argument that, as asserted above, the sign of c for the systems (2.1) and (2.2) is the same. (The essential point is that in the critical case $c = 0$ the equations (2.3) are the same for both.) Two basic theorems yielding information on the wave speed follow. The first depends crucially on the special symmetries of f and g.

THEOREM 2.1. The wave speed c is zero in each of the following cases:

(i) the interior equilibrium $(\bar{u}, \bar{v})$ lies on the diagonal $\bar{u} + \bar{v} = 1$ and $D = E$,

(ii) $\bar{u} = \bar{v} = ½$ (for any D and E).

PROOF

(i) With $u = U, v = 1 - V$, the system (2.3) becomes

$$E\ddot{U} + c\dot{U} + U(1-U)(\bar{u} - V) = 0,$$
$$E\ddot{V} + c\dot{V} + V(1-V)(\bar{u} - U) = 0,$$

where $U(-\infty) = 0, U(\infty) = 1, V(-\infty) = 1, V(\infty) = 0$. The wave is now from (0,1) to (1,0). The form of the equations is identical, and so (by interchanging U and V) the result follows from uniqueness.

(ii) Set $u = 1 - U, v = 1 - V$ and replace z by $-z$ to obtain

$$D\ddot{U} - c\dot{U} + U(1-U)(V - ½) = 0,$$
$$E\ddot{V} - c\dot{V} + V(1-V)(U - ½) = 0.$$

But uniqueness again holds, so $c = -c$ and the result is proved.

In general, for pairs of reaction-diffusion systems, the wave speed depends upon the ratio of the diffusion coefficients. Remarkably, in (ii) above, this is not the case. In the following theorem and proof, the travelling-wave problem defined by the system (2.4) is referred to as $P(\bar{u}, \bar{v})$.

THEOREM 2.2. Fix D and E. If $c = 0$ for $P(u_0, v_0)$ then $c < 0$ for $P(u_1, v_1)$ whenever $u_1 < u_0, v_1 < v_0$.

PROOF. The system is quasi-monotone non-decreasing and subsolution arguments are therefore directly valid, see [1, 11]. We deduce that the assumption $c \geq 0$ for $P(u_1, v_1)$ leads to a contradiction.

Let $(\phi(z), \psi(z))$ be the (stationary) solution of $P(u_0, v_0)$, and let $(u(x-ct), v(x-ct))$ be the solution of $P(u_1, v_1)$. It is straightforward to check from the linearization of the system (2.4) that if $c \geq 0$ then $u(z) \geq \phi(z)$ and $v(z) \geq \psi(z)$ as $z \to \pm\infty$. It is therefore possible to re-define u, v (by a translation if necessary) and obtain $u(z) \geq \phi(z)$, $v(z) \geq \psi(z)$ for all $z \in \mathbf{R}$.

We next show that if $\theta < 0$ and if $|\theta|$ is small enough then $(\phi(x - \theta t), \psi(x - \theta t))$ is a subsolution for $P(u_1,v_1)$. With $\underline{u}(x,t)=\phi(x-\theta t)$ and $\underline{v}(x,t)=\psi(x-\theta t)$ we have

$$\frac{\partial \underline{u}}{\partial t}-D\frac{\partial^2 \underline{u}}{\partial x^2}-f(\underline{u},\underline{v},v_1)=-\theta\dot{\phi}-D\ddot{\phi}-\phi(1-\phi)(\psi-v_1) \qquad (2.5)$$
$$=-\theta\dot{\phi}-\phi(1-\phi)(v_0-v_1)$$

since $\ddot{\phi}+\phi(1-\phi)(\psi-v_0)=0$. It is not hard to see that the right hand side of (2.5) is non-positive for small enough $|\theta|$. For choose z_0 so large that the linearization is approximately valid for $|z|>z_0$ and let $\dot{\phi}=\lambda_{\pm}\phi$ as $z\to\pm\infty$, respectively. Then the inequality holds if $|\theta|\max(|\lambda_+|,|\lambda_-|)<(v_0-v_1)/2$ and $|z|>z_0$. For $|z|\le z_0$, it is enough to choose

$$|\theta|\max_{|z|\le z_0}\dot{\phi}<(v_0-v_1)\min_{|z|\le z_0}\phi(1-\phi).$$

A similar argument holds for the equation for $\underline{v}$ and it follows that $(\phi(x-\theta t),\psi(x-\theta t))$ is a subsolution for $P(u_1,v_1)$. Furthermore, by the remarks above, when $t=0,\phi(x),\psi(x)$ lie below $u(x),v(x)$ respectively. Hence, by a standard theorem, see for example [1, Theorem 4.1], for $t\ge 0$,

$$u(x-ct)\ge\phi(x-ct),\quad v(x-ct)\ge\psi(x-ct).$$

But $\theta<0$, so for each fixed x, both $\phi(x-ct)$ and $\psi(x-ct)$ are increasing and tend to unity as $t\to\infty$. However, $c\ge 0$ and so u and v are non-increasing. This gives a contradiction.

A combination of these two theorems enables us to give the first main result concerning the sign of c.

COROLLARY 2.3.

(i) If $D=E$ and $\bar{u}+\bar{v}>1$ (respectively <1), then $c>0$ (respectively <0) and (0,0) (respectively (1,1)) is dominant.

(ii) If $(\bar{u},\bar{v})\in(\frac{1}{2},1)\times(\frac{1}{2},1)$ (i.e. the equilibrium point is in the top right-hand square of Fig 3.2) then $c>0$ for all D,E. That is, (0,0) is uniformly dominant. Similarly (1,1) is uniformly dominant if the equilibrium point lies in the bottom left-hand square.

3. DOMINANCE OF EQUILIBRIUM. ASYMPTOTIC ANALYSIS

The results of the previous section give some information about the sign of c and so the dominance of (0,0) and (1,1). However, their applicability is somewhat limited. It appears to be an extremely difficult task to obtain precise information of this nature for the whole range of diffusion rates and parameters in the reaction terms. We shall show that an asymptotic analysis can be carried out when one of the rate constants is large (or equivalently the corresponding diffusion rates is small) and this provides good insight into the general situation. It also suggests a plausible conjecture which provides an almost complete qualitative description for most parameters.

Consider then the situation when $D\to 0$ (or equivalently $A\to\infty$). Precise results may be obtained by the methods used in [12] with trivial modifications. An outline is as follows; the reader is referred to the original paper for further details and proofs.

The central idea, which is motivated by singular perturbation arguments, is that the limit of the projection of the orbit in the (u,v) plane follows the lines $u=0$ and $u=1$ (where $f=0$) with a 'jump' at $v=v^*$ joining these two lines, see Fig 3.1. The value of v^* is obtained as the solution to the equation

$$\int_0^{v^*} g(0,s,\bar{u})\mathrm{d}s + \int_{v^*}^{1} g(1,s,\bar{u})\mathrm{d}s = 0. \tag{3.1}$$

Having obtained an approximation to the projection of the orbit, it is straightforward to find the sign of c. The following theorem gives more information.

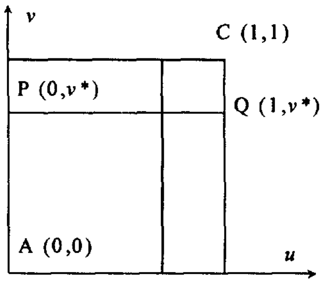

Fig 3.1 The projection of the orbit in the limit $D \to 0$ is the union of the sections AP, PQ, QC where v^* is determined by eqn (3.1).

THEOREM 3.1. With v^* given by (3.1),

$$c(\bar{v}-v^*) > 0 \qquad (\bar{v} \neq v^*)$$

for small enough D. Furthermore, if $\bar{v} \neq v^*$ then, for fixed $\bar{u}, \bar{v}$ there are constants $K_1, K_2 > 0$ such that

$$K_1 \leq |c|/\sqrt{D} \leq K_2.$$

A simple calculation based upon eqn (3.1) shows that the change of sign from c positive to negative occurs (in the limit $D \to 0$) when

$$\bar{u} = 2\bar{v}^3 - 3\bar{v}^2 + 1. \tag{3.2}$$

There is a corresponding result as $E \to 0$.

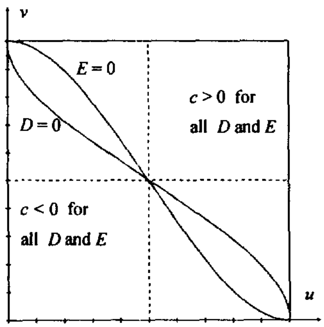

Fig 3.2 The Petal diagram. If the equilibrium point $(\bar{u},\bar{v})$ lies to the right of the curve labelled $D=0$ then c is positive for small D. In the top right-hand square c is positive for all D, E and so (0,0) is uniformly dominant.

Fig 3.2 is deduced from Corollary 2.3 and Theorem 3.1. The aim is to combine in a single figure all that is known about the effect of the position of the equilibrium point $(\bar{u},\bar{v})$ and the diffusion coefficients D, E (or equivalently the rate constants $A=d/D, B=d/E$) on the sign of c. An explanation of the figure is as follows. For $(\bar{u},\bar{v})$ to the right of the curve labelled $D=0$ (respectively left of $E=0$) the wave speed c is positive for small D (respectively negative for small E). An interesting conclusion concerning uniform dominance is that certainly this does not hold inside the petals. In these regions the sign of c (and so the dominance of the equilibria) may be changed by varying the ratio D/E. On the other hand, it has been shown that (0,0) is uniformly dominant if $(\bar{u},\bar{v})$ lies in the top-right square. It is tempting to assume montonicity of the family of curves $c=0$ in D/E. This leads to the following

CONJECTURE. The wave speed, c, is positive for all D,E (or (0,0) is uniformly dominant) if $(\bar{u},\bar{v})$ lies to the right of the petals (that is, to the right of the curves labelled $D=0$ and $E=0$). A similar result applies for $(\bar{u},\bar{v})$ to the left of the petals.

Although we are unable to prove this conjecture, it is supported by numerical calculations. It is an interesting theoretical challenge to supply a proof. We further speculate that for fixed D/E, the $c=0$ curve which gives the position of $\bar{u},\bar{v}$ connects (0,1) to (1,0) passing through (½,½) and lies within the petals. They may be thought of as providing 'veins' for the petal. This is partially confirmed by the analysis in the next section.

Theorems 2.2 and 3.1 are sufficiently strong to be able to yield the following remarkable result which contrasts with the results in [7] concering the best-response dynamic.

THEOREM 3.2. The dominance relationship obtained from the replicator-dynamic (1.3) is, in general, different from the risk-dominance relation for (1.2). Also the new dominance relation is not transitive for coordination games: There are 3×3 coordination games with three strategy pairs (a_i,b_i) $(i=1,2,3)$ such that (a_1,b_1) dominates (a_2,b_2), which dominates (a_3,b_3), which in turn dominates (a_1,b_1).

PROOF. From Theorem 3.1, when (a_1+a_2) is much larger than (b_1+b_2), i.e. D is small, the dominance relation for (1.3) is $v^*>\bar{v}$, in the sense that if this inequality is satisfied then strategy 1 is dominant. But this inequality implies that

$$\frac{a_2^2(3a_1+a_2)}{a_1^2(a_1+3a_2)}<\frac{b_1}{b_2},$$

which is clearly different from the risk-dominance criterion, $a_1b_1>a_2b_2$. Also, if we choose

$$a_1=K, a_2=2K, a_3=6K, \quad b_1=32, b_2=11, b_3=2, \quad (K \text{ large})$$

then it is easily verified that the 3 pure strategies have the required cyclical property.

Suppose that we write

$$a_1=A\bar{v}, a_2=A(1-\bar{v}), b_1=B\bar{u}, b_2=B(1-\bar{u}),$$

so that A, B are the rate constants as in (2.1), and consider $\bar{u},\bar{v}$ as fixed. If A, B are chosen randomly from the same distribution, then the expected criterion for strategy 1 to dominate is $\bar{u}+\bar{v}>1$, which (since $\bar{u}=b_1/(b_1+b_2)$ and $\bar{v}=a_1/(a_1+a_2)$) is the same as $a_1b_1>a_2b_2$. Thus the risk-dominance criterion can be thought of as an average condition over all rate constants. Also, if $a_1>a_2$ and $b_1>b_2$ then the equilibrium point $(\bar{u},\bar{v})$ lies in the top-right region of Fig. 3.2 and so strategy 1 dominates regardless of the rate constants.

4. A PERTURBATION ANALYSIS

In the special case $D=E$ and $\bar{u}=\bar{v}=½$, the travelling-wave solution is known explicitly. It has $c=0$ and the orbit lies along $u=v$. The aim is to show that for a small perturbation from this case, the wave speed may be found explicitly. The analysis is formal, but we believe convincing. Also the results are strongly confirmed by numerical calculations. These results are of particular interest as they partially confirm a conjecture for an important class of economic games, the 'bargaining' game. These implications are discussed at the end of the section.

In this section it is the form (1.3) of the equations which is the most appropriate. Write

$$a_1=1+(\alpha+\beta)\varepsilon+(\gamma+\delta)\varepsilon^2,\quad a_2=1+(\alpha-\beta)\varepsilon+(\gamma-\delta)\varepsilon^2,$$
$$b_1=1-(\alpha+\beta)\varepsilon+(\gamma+\delta)\varepsilon^2,\quad b_2=1-(\alpha-\beta)\varepsilon+(\gamma-\delta)\varepsilon^2,$$

where α, β, γ, δ are fixed and ε is a small parameter. Write the solution of the travelling-wave equation (2.3) as $u(z,\varepsilon), v(z,\varepsilon)$. It is easy to see that the special form assumed for the constants implies that

$$u(z,-\varepsilon)=v(z,\varepsilon)\ \text{ and }\ c(-\varepsilon)=c(\varepsilon).$$

This suggests that we should look for solutions of the form

$$u=u_0+\varepsilon u_1+\varepsilon^2 u_2+O(\varepsilon^3),$$
$$v=u_0-\varepsilon u_1+\varepsilon^2 u_2+O(\varepsilon^3),$$
$$c=\varepsilon^2\theta+O(\varepsilon^4),$$

where $u_0=1/(1+e^{-z})$. Thus $u_i(\pm\infty)=\dot{u}_i(\pm\infty)=0$ for $i=1,2$. Substitution of these forms into (2.3) then yields, on setting the coefficients of ε and ε^2 to zero,

$$\ddot{u}_1-u_1(2u_0^2-2u_0+1)=u_0(1-u_0)(\alpha+\beta-2\alpha u_0), \tag{4.1}$$

$$\ddot{u}_2-u_2(6u_0^2-6u_0+1)=-\theta\dot{u}_0+u_0(1-u_0)(\gamma+\delta-2\gamma u_0)+ u_1[2\alpha u_0^2+(\alpha+\beta)(1-2u_0)]+u_1^2(1-2u_0). \tag{4.2}$$

Introduce the new independent variable $T=1/(1+e^{-z})$, whose range is (0,1), and use a dash for differentiation with respect to T. Eqn (4.1) becomes

$$T^2(1-T)^2u_1''+T(1-T)(1-2T)u_1'-(2T^2-2T+1)u_1=T(1-T)(\alpha+\beta-2\alpha T). \tag{4.3}$$

The solution to this equation satisfying the conditions $u_1(0)=0=u_1(1)$ is

$$u_1=\frac{(\alpha+\beta)}{6}T\left[\frac{(3-2T)}{1-T}\ln T+1\right]-\frac{(\alpha-\beta)}{6}(1-T)\left[\frac{(1+2T)}{T}\ln(1-T)+1\right].$$

Eqn (4.2) now becomes

$$\frac{d}{dT}\left[T(1-T)\frac{du_2}{dT}\right]-\frac{6T^2-6T+1}{T(1-T)}u_2=\frac{h(T)}{T(1-T)} \tag{4.4}$$

where

$$h(T)=-\theta T(1-T)+T(1-T)(\gamma+\delta-2\gamma T)+u_1[2\alpha T^2+(\alpha+\beta)(1-2T)]+u_1(1-2T).$$

Now $T(1-T)$ is, loosely speaking, an eigenfunction (corresponding to an eigenvalue of zero) of the formal linear operator appearing on the left-hand side of (4.4), that is, a solution of the corresponding homogeneous equation. We multiply (4.4) by $T(1-T)$, integrate by parts using the boundary conditions at 0 and 1, and find that the left-hand side vanishes. Therefore

$$\int_0^1 h(T)\,dT=0.$$

A somewhat tedious calculation, based upon the variation of constants formula, shows that this condition is also sufficient for the existence of u_2 satisfying the boundary conditions. The integral condition above is tedious to calculate by hand, but with the help of MAPLE it is readily found that

$$\theta = \delta - \tfrac{8}{9}\alpha\beta.$$

These results have close relevance to bargaining games, a subclass of coordination games which are of considerable interest in economics, see [6, p.13]. In these games, the rules are imposed by an external mechanism. In outline, the idea is that competitors can divide a fixed amount (of money) in any way that can be agreed. If they fail to agree, the money is lost. In our case, there are just two strategy pairs, (a_1, b_1) and (a_2, b_2) where $a_1 + b_1 = a_2 + b_2$ $(=2$, say). The first player may choose a_1 or a_2 and the second one b_1 or b_2. If player $\mathcal{A}$ chooses a_1 and player $\mathcal{B}$ b_1, then each receives the value of their bid. But if the second one plays b_2, then neither receives anything. The strategy pair $(1,1)$ has an obvious appeal as being the fair split but all strategies (with $a_i + b_i = 2$) are Nash equilibria. This is because any unilateral deviation yields a lower pay-off. [6] suggests extra axioms in order that just one solution shall be preferred. Not surprisingly, it is the fair split which is the one preferred. Here, in contrast, it is the direction of the travelling wave which, we argue, gives a natural criterion for the best strategy. With $\gamma = 0 = \delta$, the above analysis shows that, if $\varepsilon_1, \varepsilon_2$ are small then

$$a_1 = 1 + \varepsilon_1, \quad a_2 = 1 + \varepsilon_2, \quad b_1 = 1 - \varepsilon_1, \quad b_2 = 1 - \varepsilon_2 \quad \Rightarrow c \approx \tfrac{2}{9}(\varepsilon_2^2 - \varepsilon_1^2).$$

Thus it is the strategy with the more-equal split which is the winner. In particular, the 'fair' division of $(1,1)$ is unbeatable. These conclusions have been verified numerically and have been shown to apply (in so far as the sign of c is concerned) to any pair of strategies (with $a_1 + b_1 = a_2 + b_2$). A proof of this result is still lacking.

5. CONCLUDING REMARKS

The importance of reaction-diffusion equations, and in particular travelling-waves, has long been recognised in the area of mathematical biology, see for example [13, 14, 15]. Game theory was originally developed to solve economic problems, see [16], but its relevance to biology was demonstrated brilliantly by [17, 18]. Even though the basic notion of Maynard Smith, the evolutionarily stable strategy, involved invasion and so necessarily implied a spatial variation, it was some time before this was explicitly included into models [9]. Only very recently has the combination of game theory and travelling waves been applied to economics [7]. The purpose of this latest synthesis is to overcome one of the gravest weaknesses of the classic economic game theory, viz. determining which, of often very many, Nash equilibria is the one which is 'best'. The idea is that it is the direction of the travelling wave which determines the best strategy. The strength of this idea is its simplicity, no new axioms or assumptions have to be made (except, of course, that the players of the game have a spatial distribution). Its weakness is that the form of the reaction terms may be in dispute. [7] concentrated upon the best response dynamic and here the choice has been the replicator dynamic from evolutionary game theory. As we have seen in Section 3, this makes a difference. The theory presented here is equally applicable to both dynamics. Also, we have chosen to emphasise the economic (rather than the biological) aspects because of the novelty of this application. The bargaining game is but one of several economic games which can be analyzed in the manner indicated here and work is in progress which will be reported elsewhere.

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