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Evolutionary Selection against Dominated Strategies*

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A class of evolutionary selection dynamics is defined, and the defining property, convex monotonicity, is shown to be sufficient and essentially necessary for the elimination of strictly dominated pure strategies. More precisely: (1) all strictly dominated strategies are eliminated along all interior solutions in all convex monotonic dynamics, and (2) for all selection dynamics where the pure-strategy growth rates are functions of their current payoffs, violation of convex monotonicity implies that there exist games with strictly dominated strategies that survive along a large set of interior solutions. The class of convex monotonic dynamics is shown to contain certain selection dynamics that arise in models of social evolution by way of imitation. *Journal of Economic Literature* Classification Number: C72.

1. INTRODUCTION

A basic rationality postulate in non-cooperative game theory is that players never use pure strategies that are strictly dominated. This postulate only requires that a player's payoffs indeed represent her preferences over outcomes. In particular, no knowledge of other players' preferences or behavior is required. A more stringent rationality postulate is that players never use pure strategies that are iteratively strictly dominated. In addition,

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this postulate requires that all players know each other's payoffs, that they know that they know each other's payoffs, etc. up to a finite level k of mutual knowledge, where k is the number of rounds required to halt the procedure of iterated elimination of strictly dominated pure strategies (see, e.g., Tan and Werlang [23]).

A fundamental question in evolutionary game theory thus is whether evolutionary selection processes do eliminate all strictly dominated pure strategies or even all iteratively strictly dominated pure strategies. If all iteratively strictly dominated strategies do vanish, this provides an evolutionary justification for the presumption that strategically interacting agents behave *as if* it were mutual knowledge that they are rational in the sense of never using strictly dominated strategies.[1] Clearly, this justification is more compelling the wider the class of evolutionary selection processes for which this result is valid.

So far, the result has been established for so-called aggregate monotonic selection dynamics in Samuelson and Zhang [21].[2] This is a class of continuous-time dynamics that contains the biological replicator dynamics. Samuelson and Zhang also show that all aggregate monotonic selection dynamics are closely related to the replicator dynamics: the differential equations for any dynamics in this class differ from the replicator equations only by a positive factor, a factor which may be player specific and population state dependent.

Here we generalize Samuelson's and Zhang's result to a considerably wider class of evolutionary selection dynamics which we call *convex monotonic*. We also show that this result is sharp within a wide class of selection dynamics where the growth rate of each pure strategy is determined by a function of its payoff. For all dynamics in this class which fail our condition there exist games in which strictly dominated strategies survive along (large sets of) solutions. The new class of dynamics is shown to contain certain (not aggregate monotonic) selection dynamics that arise in models of social evolution by way of imitation.

2. GAMES AND SELECTION DYNAMICS

Consider any finite n-player game in normal form, $G=(I, S, u)$, where $I=\{1, ..., n\}$ is the set of players, $S=\prod_{i \in I} S_i$ is the set of pure-strategy

[1] The "as if" approach to rationality has a long tradition in economics, with early advocates in Alchian [1] and Friedman [9]. See Weilbull [26] for a recent discussion of the "as if" approach applied to game theory.

[2] In contrast, the result is known not to be valid for the discrete-time version of the replicator dynamics, see Dekel and Scotchmer [7].

profiles, each player's pure-strategy set S_i being finite, $S_i = \{1, ..., m_i\}$, and $u: S \to \mathbb{R}^n$ is the combined payoff function. Let m denote the total number of pure strategies in the game, $m = m_1 + ... + m_n$.

For each player i, let Δ_i denote her set of mixed strategies,

$$\Delta_i = \left\{ x_i \in \mathbb{R}_+^{m_i} : \sum_{h \in S_i} x_{ih} = 1 \right\}. \tag{1}$$

We write $e_i^h \in \Delta$ for the mixed strategy for player i that assigns unit probability to her pure strategy $h \in S_i$. Geometrically, e_i^h is the hth vertex of the unit simplex Δ_i. A *face* of Δ_i is the convex hull of a subset of its vertices; the face spanned by $S_i' \subset S_i$ is denoted

$$\Delta_i(S_i') = \text{co}\{e_i^h : h \in S_i'\} = \{x_i \in \Delta_i : x_{ik} = 0 \ \forall k \notin S_i'\}. \tag{2}$$

Let Θ denote the polyhedron in $\mathbb{R}^m$ of mixed strategy profiles, $\Theta = \Theta(S) = \Pi_{i \in I} \Delta_i$. The *face* of Θ associated with any collection of pure strategy subsets $S_i' \subset S_i$ is accordingly defined by $\Theta(S') = \Pi_{i \in I} \Delta_i(S_i')$. As usual, statistically independent individual randomizations extend the domain of the payoff function u from the vertices of Θ to all of Θ, and this renders u polynomial. In fact, u is an n-linear function defined on the whole Euclidean space $\mathbb{R}^m$ where Θ is embedded. We write $u_i(y_i, x_{-i})$ for the payoff to player i when she plays $y_i \in \Delta_i$ and the others play according to the strategy profile $x \in \Theta$.

A pure strategy $h \in S_i$ is *strictly dominated* if there is some (pure or mixed) strategy $y_i \in \Delta_i$ such that $u_i(y_i, x_{-i}) > u_i(e_i^h, x_{-i})$ for all $x \in \Theta$. A pure strategy is *iteratively strictly dominated* if it is strictly dominated in the original game G, or in the reduced game G' obtained by elimination from G of all strictly dominated pure strategies in G, or in the further reduced game G'' obtained by elimination from G' of all strictly dominated pure strategies in G', etc. The set S of pure-strategy profiles being finite, this procedure stops after a finite number of iterations (and the result is independent of the details of the elimination procedure, see, e.g., Fudenberg and Tirole [10]).

In evolutionary game theory one considers large populations of individuals who are randomly matched to play a given game. Following Taylor [24] we here imagine one population for each player position in a finite n-player game, where all individuals use pure strategies. Accordingly, a mixed-strategy profile $x \in \Theta$ is now interpreted as a population state, where each mixed strategy x_i represents the distribution of individuals in population $i \in I$ over the set of pure strategies S_i. Selection processes are thought to operate over time on the composition of behaviors—pure strategies—in each player population in the form of a *regular selection* *dynamics* on the polyhedron Θ. By this is meant a system of (autonomous, first-order) ordinary differential equations

$$\dot{x}_{ih} = x_{ih} g_{ih}(x), \tag{3}$$

where the function $g\colon X \to \mathbb{R}^m$ has open domain $X \supset \Theta$, is locally Lipschitz continuous, and satisfies the orthogonality condition ("$\cdot$" denotes the inner product)

$$x_i \cdot g_i(x) = 0 \qquad [\forall i \in I, x \in \Theta]. \tag{4}$$

This condition implies that the sum of population shares in each player population remains constantly equal to one. Any regular selection dynamics has a unique global solution $x(\cdot)\colon \mathbb{R} \to X$ through any initial state $x(0) \in \Theta$, and leaves Θ, as well as its interior $\text{int}(\Theta)$, and each of its faces, invariant.

Following Samuelson and Zhang [21] we call a regular selection dynamics (3) *monotonic* if it meets the following axiom:[3]

$$(\mathbf{M}) \quad u_i(e_i^h, x_{-i}) > u_i(e_i^k, x_{-i}) \quad \Leftrightarrow \quad g_{ih}(x) > g_{ik}(x).$$

They call a regular selection dynamics (3) *aggregate monotonic* if the growth-rate functions g_{ih} satisfy the more stringent axiom

$$(\mathbf{AM}) \quad u_i(y_i, x_{-i}) > u_i(z_i, x_{-i}) \quad \Leftrightarrow \quad y_i \cdot g_i(x) > z_i \cdot g_i(x).$$

Moreover, they show that the growth-rate functions associated with any aggregate monotonic selection dynamics can be written in the form

$$g_{ih}(x) = \lambda_i(x)(u_i(e_i^h, x_{-i}) - u_i(x)), \tag{5}$$

for some positive functions $\lambda_i\colon X \to \mathbb{R}$. *The standard replicator dynamics* (Taylor [24]) corresponds to the special case $\lambda_i(x) \equiv 1$ for all players i. The *payoff-adjusted replicator dynamics* (Maynard Smith [18], see also Hofbauer and Sigmund [17]), corresponds to the special case $\lambda_i(x) \equiv 1/u_i(x)$ for all players i (presuming all payoffs are positive).

Akin [2] shows that all strictly dominated pure strategies vanish along any interior solution trajectory to the (single-population) replicator dynamics in any (finite) symmetric two-player game. Samuelson and Zhang [21] establish that this conclusion is indeed valid for all iteratively strictly dominated pure strategies in any aggregate monotonic (two-population) selection dynamics in any (finite) two-player game. They also show that all pure strategies that are iteratively strictly dominated by other *pure*

[3] This property is called *relative monotonicity* in Nachbar [19] and *order compatibility* in Friedman [8].

strategies vanish in any monotone selection dynamics in such games (see also Nachbar [19]). Björnerstedt [3] shows by way of a counter-example that this is not generally true for pure strategies that are strictly dominated only by mixed strategies.

3. CONVEX MONOTONIC SELECTION DYNAMICS

The following axiom is a weakening of aggregate monotonicity (the mixed strategy z_i in (AM) is replaced by a pure strategy, and the equivalence is replaced by an implication):

$$\textbf{(CM)} \quad u_i(y_i, x_{-i}) > u_i(e_i^h, x_{-i}) \quad \Rightarrow \quad y_i \cdot g_i(x) > g_{ih}(x).$$

This property, which we call convex monotonicity, is below shown to be sufficient for the elimination of iteratively strictly dominated pure strategies. By inserting the current state for population i, we obtain the implication that strategies receiving worse-than-average payoffs must have negative growth rates: an application of (CM) to $y_i = x_i$ gives $u_i(x) > u_i(e_i^h, x_{-i}) \Rightarrow 0 = x_i \cdot g_i(x) > g_{ih}(x)$.[4]

The following special case motivates the name given to axiom (CM). Consider the class of regular selection dynamics in which all growth-rate functions g_{ih} are of the form

$$g_{ih}(x) = \lambda_i(x)\, f[u_i(e_i^h, x_{-i})] + \mu_i(x) \tag{6}$$

for some functions $f\colon \mathbb{R} \to \mathbb{R}$, $\lambda_i\colon X \to \mathbb{R}_{++}$ and $\mu_i\colon X \to \mathbb{R}$. By a *payoff functional* (PF) selection dynamics we mean a function $f\colon \mathbb{R} \to \mathbb{R}$, and for each game $G = (I, S, u)$ a pair of functions $\lambda_i\colon X \to \mathbb{R}_{++}$ and $\mu_i\colon X \to \mathbb{R}$, such that (3, 6) defines a regular selection dynamics for game G (note that the μ_i are determined by (4)).

As a special case, call such a dynamics *linear* if f is linear with positive slope. Both the standard and the payoff adjusted replicator dynamics are linear in this sense. Set $f(v) \equiv v$, $\lambda_i(x) \equiv 1$ and $\mu_i(x) \equiv -u_i(x)$ to obtain the standard replicator dynamics, and set $f(v) \equiv v$, $\lambda_i(x) \equiv 1/u_i(x)$ and $\mu_i(x) \equiv -1$ to obtain the payoff-adjusted replicator dynamics (in the latter case presuming $u_i(x) > 0$). Moreover, since all aggregate monotonic dynamics can be written in the form (5) these are linear PF dynamics (let $f(v) \equiv v$ and $\mu_i(x) \equiv -\lambda_i(x)\, u_i(x)$).

[4] Thus a face of Θ spanned by a product subset of pure strategies that is "closed under the better-reply correspondence" is asymptotically stable in any dynamics satisfying (CM), see Ritzberger and Weibull [20].

More generally, we call a payoff functional dynamics convex if f in (6) is convex and strictly increasing. Heuristically, nonlinear convex PF dynamics have individuals react *over-proportionally* to higher payoffs.

Convex payoff functional selection dynamics may also be interpreted in terms of risk aversion with respect to "fitness." For suppose replication of pure strategies occurs as in equation (3) with growth rate functions as in (6). The numbers $\varphi_{ih}(x)=f[u_i(e_i^h, x_{-i})]$ can then be interpreted as the relative fitness of pure strategy h in player population i when the overall population states is x: this is the relative rate at which pure strategy h is reproduced in population i. Let the functions φ_{ih} be given data. For a convex PF dynamics (3, 6) we may recover the associated utility function by simply inverting the strictly increasing function f: At any given population state x, $u_i(e_i^h, x_{-i})=f^{-1}[\varphi_{ih}(x)]$. Hence, utility is a strictly increasing and concave function of fitness. In this sense, it is as if individuals were (weakly) risk averse with respect to fitness. For instance, if f is exponential (as in (7) below), then utility is logarithmic in relative fitness, and the Arrow–Pratt measure of absolute risk aversion, here with respect to relative fitness, meets the usual (DARA) condition of decreasing absolute risk aversion.

We noted above that all aggregate monotonic selection dynamics are linear PF dynamics. Hence they are convex PF dynamics. The following proposition establishes that all convex PF dynamics, and no other PF dynamics, meet axiom (CM):

PROPOSITION. *A payoff functional selection dynamics satisfies axiom (CM) if and only if it is convex.*

Proof. For the first claim, suppose g is of the form (6), where f is convex and strictly increasing. Suppose $u_i(y_i, x_{-i})>u_i(e_i^h, x_{-i})$. Using Jensen's inequality:

$$
\begin{aligned}
y_i\cdot g_i(x)-e_i^h\cdot g_i(x) &= \lambda_i(x)\left(\sum_{k\in S_i} y_{ik} f[u_i(e_i^k, x_{-i})]-f[u_i(e_i^h, x_{-i})]\right)\\
&\geqslant \lambda_i(x)\left(f\left[\sum_{k\in S_i} y_{ik} u_i(e_i^k, x_{-i})\right]-f[u_i(e_i^h, x_{-i})]\right)\\
&= \lambda_i(x)(f[u_i(y_i, x_{-i})]-f[u(e_i^h, x_{-i})]).
\end{aligned}
$$

The last expression is positive since λ_i is positive and f strictly increasing, so (CM) is met.

For the second claim, consider any PF dynamics (3, 6) that meets (CM). It follows from (CM) that f is necessarily strictly increasing. Suppose f is not convex. Then there are $b, c\in\mathbb{R}$ such that $f((b+c)/2)>\frac{1}{2}[f(b)+f(c)]$.

By continuity of f there are $a<((b+c)/2)$ such that $f(a)>\frac{1}{2}[f(b)+f(c)]$. Let G be a game where player i has three pure strategies, $h=1, 2, 3$, that earn payoffs a, b, c, respectively, against some strategy profile $x\in\Theta$. Let $y_i\in\Delta_i$ be the mixed strategy that assigns probability $\frac{1}{2}$ to pure strategies 2 and 3. Then

$$u_i(y_i, x_{-i})=\tfrac{1}{2}(b+c)>a=u_i(e_i^1, x_{-i}).$$

However,

$$y_i\cdot g_i(x)=\lambda_i(x)\,\frac{f(b)+f(c)}{2}<\lambda_i(x)\,f(a)=e_i^1\cdot g_i(x),$$

in violation of (CM). *End of Proof.*

Remark. It is easily verified that axiom (CM) is satisfied by any selection dynamics (3) with growth rate functions in the more general functional form $g_{ih}(x)=F_i[u_i(e_i^h, x_{-i}), x]$ for $F_i\colon \mathbb{R}\times X\to\mathbb{R}$ convex and strictly increasing in its first argument.

A number of researchers have recently worked with models of social evolution by way of imitation, see, e.g., Cabrales [6], Weibull [25], Björnerstedt and Weibull [5], Weibull [25], Schlag [22], Gale *et al.* [11], and Weibull [27]. Björnerstedt and Weibull [5] consider a few classes of payoff functional selection dynamics derived from models of adaptation by way of imitation. They imagine that each individual in the interacting populations every now and then reviews her pure strategy choice in the light of noisy empirical information about current payoffs to alternative pure strategies.

First, suppose that the review rate is constantly equal to one for all individuals, but each individual imitates an individual in her own player population, randomly drawn with a higher probability for currently more successful individuals. Then one obtains a payoff functional selection dynamics with $f(v)\equiv s(v)$, where $s(v)$ is the probability "weight" factor given to an individual who earns payoff v.[5] A convex PF dynamics arises

[5] Let the review rate of all individuals be identically equal to one, and let the probability that a reviewing individual in population i will select pure strategy h be proportional to $x_{ih}s[u_i(e_i^h, x_{-i})]$ for some strictly increasing and positive function s. In terms of expected values, this results in

$$g_{ih}(x)=\frac{s[u_i(e_i^h, x_{-i})]}{\sum_k x_{ik}s[u_i(e_i^k, x_{-i})]}-1,$$

see Eq. (7) in Björnerstedt and Weibull [5], Eqs. (4.37) and (5.32) in Weibull [27].

if s is strictly increasing and convex. For example, by setting $s(v)=\exp(\sigma v)$ for some $\sigma>0$, one obtains:[6]

$$\dot{x}_{ih}=x_{ih}\left(\frac{\exp[\sigma u_i(e_i^h, x_{-i})]}{\sum_{k\in S_i} x_{ik}\exp[\sigma u_i(e_i^k, x_{-i})]}-1\right). \tag{7}$$

For small σ this dynamics approaches the standard replicator dynamics slowed down by the factor σ.[7] For large σ, the dynamics approaches, at interior population states, the *best-reply dynamics* which assigns (equal) negative growth rates (-1) to all non-best replies.[8]

Secondly, suppose instead that the review rates are decreasing in the individual's current payoff, and assume now that each reviewing individual imitates "the first man in the street," i.e., an individual in her own player population who is randomly drawn according to a uniform probability distribution over this population. (This corresponds to $s(v)\equiv 1$ above.) Then one obtains a payoff functional selection dynamics with $f(v)\equiv -r(v)$, where $r(v)$ is the relative review rate of an individual earning payoff v.[9] A concave PF dynamics arises if r is strictly decreasing and convex. For instance, if $r(v)=\exp(-\sigma v)$ for some $\sigma>0$, then

$$\dot{x}_{ih}=x_{ih}\left(1-\frac{\exp[-\sigma u_i(e_i^h, x_{-i})]}{\sum_{k\in S_i} x_{ik}\exp[-\sigma u_i(e_i^k, x_{-i})]}\right). \tag{8}$$

This dynamics constitutes a "concave dual" to the dynamics (7). For small σ, (8) performs approximately like (7); it approaches the standard

[6] See Eq. (9) in Björnerstedt and Weibull [5], Eq. (4) in Weibull [26], and Example 4.5 in Weibull [27].

[7] The orbits approach those of the standard replicator dynamics as $\sigma\to 0$, but the speed of adjustment goes down toward zero. In the limit all population states are stationary.

[8] The limit of the right-hand side in (7) is a discontinuous vector field that does not admit solutions in general. On the other hand, limits of solutions of (7), as $\sigma\to+\infty$, are solutions of the multi-valued and upper hemi-continuous best reply dynamics $\dot{x}=BR(x)-x$, where $BR(x)$ denotes the set of (mixed) best replies to x. This is a differential inclusion, and its solutions are in general not uniquely determined by the initial state. See Hofbauer [15] for a rigorous treatment of this dynamics, and see Gaundersdorfer and Hofbauer [13] for a comparison of its asymptotic behavior with that of the replicator and other selection dynamics. It is easily seen that this best-reply dynamics eliminates all (iteratively) strictly dominated strategies.

[9] Let $r[u_i(e_i^h, x_{-i})]/\sum_k x_{ik} r[u_i(e_i^k, x_{-i})]$ be the review rate of a h-strategist in player population i, for r positive and decreasing, and let x_{ik} be the probability that a reviewing individual will select pure strategy k. In terms of expected values, this results in

$$g_{ih}(x)=1-r[u_i(e_i^h, x_{-i})]\Big/\sum_k x_{ik} r[u_i(e_i^k, x_{-i})],$$

see Eq. (4) in Björnerstedt and Weibull [5], and Eqs. (4.28) and (5.24) in Weibull [27].

replicator dynamics slowed down by the factor σ. For large σ, however, (8) approaches, at interior population states, the *worst-reply dynamics* which assigns (equal) positive growth rates $(+1)$ to all non-worst replies.[10]

4. ELIMINATION OF DOMINATED STRATEGIES

Suppose player i has a pure strategy $h \in S_i$ that is strictly dominated by some mixed strategy $y_i \in \Delta_i$, and consider the function $P: int(\Theta) \to \mathbb{R}_{++}$ defined by $P(x) = x_{ih} \prod_{k \in S_i} x_{ik}^{-y_{ik}}$. Evaluated along any interior solution trajectory $x(\cdot): \mathbb{R} \to \Theta$ to a regular selection dynamics (3):

$$\dot{P}(x) = \sum_{k \in S_i} \frac{\partial P(x)}{\partial x_{ik}} \dot{x}_{ik} = P(x)(e_i^h - y_i) \cdot g_i(x). \tag{9}$$

In particular, under (CM) we have $\dot{P}(x) < 0$ for all $x \in int(\Theta)$. Then $P(x)$ decreases strictly along any interior solution. In fact, since Θ is compact and g_i continuous, there is, by (CM), some $\delta > 0$ such that $(e_i^h - y_i) \cdot g_i(x) < -\delta$ for all $x \in \Theta$. Thus, $\dot{P}(x) < -\delta P(x)$ and hence $x_{ih}(t)$ decreases exponentially to zero from any interior initial state

$$x_{ih}(t) = P(x(t)) \prod_{k \in S_i} x_{ik}(t)^{y_{ik}} \leqslant P(x(t)) < \theta \exp(-\delta t) \tag{10}$$

for some $\theta > 0$. Strictly dominated pure strategies are indeed eliminated in this class of dynamics!

A repetition of this argument leads to the conclusion that all iteratively strictly dominated pure strategies vanish along all interior solutions. Since axiom (CM) is much weaker than axiom (AM), this considerably generalizes the result in Samuelson and Zhang ([21], Theorem 2) that all iteratively strictly dominated pure strategies get wiped out in all aggregate monotonic selection dynamics.

THEOREM 1. *If a pure strategy $h \in S_i$ is iteratively strictly dominated and $x(0) \in int(\Theta)$, then $x_{ih}(t)_{t \to +\infty} \to 0$ under any regular selection dynamics* (3) *satisfying* (*CM*).

Proof. Fix $x(0) \in int(\Theta)$. It has already been established that for each player position $i \in I$ and strictly dominated pure strategy $h \in S_i$ there exists some δ_{ih}, $\theta_{ih} > 0$ such that $x_{ih}(t) < \theta_{ih} \exp(-\delta_{ih} t)$ for all $t > 0$. Let $S' \subset S$ be the subset of pure strategy profiles that are not strictly dominated in the game. Let $\delta = \min\{\delta_{ih}: i \in I,\ h \in S_i \backslash S'_i\}$ and $\theta = \max\{\theta_{ih}: i \in I,\ h \in S_i \backslash S'_i\}$.

[10] The worst-reply dynamics was introduced in (a 1993 version of) Björnerstedt [3], see Section 7 below for a discussion.

The sets I and S being finite, δ, $\theta > 0$, and $x_{ih}(t) < \theta \exp(-\delta t)$ for all $i \in I$, $h \notin S'_i$ and $t > 0$.

For any $\varepsilon > 0$ there is a finite time T after which $x(t)$ stays within distance ε from the face $\Theta(S')$. In the reduced game G' defined by the pure-strategy subsets S'_i, let $S''_i \subset S'_i$ be the subset of pure strategies (for each $i \in I$) that are not strictly dominated in G'. For each $i \in I$ and $h \in S'_i \backslash S''_i$ let $y^h_i \in \Delta_i$ strictly dominate h in G'. By continuity of g, compactness of $\Theta(S') \subset \Theta$ and finiteness of S_i there exists some ε', $\delta' > 0$ such that $(y^h_i - e^h_i)\, g_i(x) > \delta'$ for all $i \in I$, $h \notin S'_i$ and $x \in \Theta$ within distance ε' of $\Theta(S')$. After some finite time T', $x(t)$ stays within this distance ε' from $\Theta(S')$, and by the above argument for exponential decay, $x_{ih}(t) < \theta' \exp(-\delta' t)$ for all $i \in I$, $h \in S'_i \backslash S''_i$ and all $t > T'$. Consequently, all pure strategies in the subset $(S_i \backslash S'_i) \cup (S'_i \backslash S''_i)$ decay at least at the exponential rate $\delta'' = \min\{\delta, \delta'\} > 0$.

A finite repetition of this argument, by way of iterated elimination of strictly dominated pure strategies, leads to the conclusion that there exists some finite time T'' and $\delta'' > 0$ such that $x_{ih}(t) < \theta'' \exp(-\delta'' t)$ for all player positions $i \in I$, iteratively strictly dominated strategies $h \in S_i$, and times $t > T''$. *End of proof.*

5. SINGLE-POPULATION DYNAMICS

In this section we focus on the standard set-up for evolutionary game theory: a single population of individuals randomly matched to play a symmetric and finite two-player game. For this purpose, let the common set of pure strategies available to each of the two players be denoted $S = S_1 = \{1, ..., m\}$, write Δ for the associated unit simplex of mixed strategies, and let $\tilde{u}(x, y)$ be the payoff to mixed strategy $x \in \Delta$ when used against mixed strategy $y \in \Delta$.

A population state is now a vector $x \in \Delta$, where x_h, for each pure strategy $h \in S$, is the population share of individuals using pure strategy h. Accordingly, a regular selection dynamics is a system of ordinary differential equations

$$\dot{x}_h = x_h \tilde{g}_h(x) \qquad [\forall h \in S] \tag{11}$$

where $\tilde{g}: X \to \mathbb{R}^m$ has open domain $X \supset \Delta$, is locally Lipschitz continuous, and satisfies the orthogonality condition $x \cdot \tilde{g}(x) = 0$ for all $x \in \Delta$.

Axiom (CM) becomes

$$(\text{CM}') \quad \tilde{u}(y, x) > \tilde{u}(e^h, x) \quad \Rightarrow \quad y \cdot \tilde{g}(x) > \tilde{g}_h(x).$$

Payoff functional (PF) selection dynamics are defined as in the multi-population setting: these are single-population dynamics (11) with growth rate functions of the form

$$\tilde{g}_h(x) = \lambda(x)\, f[\tilde{u}(e^h, x)] + \mu(x), \tag{12}$$

for some functions λ, μ and f, where f is the same for all games but λ and μ may depend on the game in question. Convex PF dynamics constitute the subclass where f is convex and strictly increasing. The single-population replicator dynamics is the special case $\lambda(x) \equiv 1$, $\mu(x) = -u(x, x)$ and $f(v) \equiv v$.

The same argument as that for Theorem 1 establishes

COROLLARY. *If a pure strategy $h \in S$ is iteratively strictly dominated in a symmetric two-player game, and $x(0) \in int(\Delta)$, then $x_h(t)_{t \to +\infty} \to 0$ under any single-population dynamics* (11) *satisfying* (CM′). *A payoff functional dynamics* (11) *satisfies* (CM′) *if and only if it is convex.*

In contrast to the multi-population setting, all aggregate monotonic single-population dynamics have the same orbits as the single-population replicator dynamics. They only differ in the velocity with which the solutions move along the replicator orbits (reflected by the positive factor $\lambda(x)$). In contrast, convex monotonic single-population dynamics may have orbits which are quite distinct from those of the replicator dynamics. Examples for which this applies are given by the single-population dynamics version of (7) (see Fig. 4.9 in Weibull [27]).

6. SURVIVAL OF DOMINATED STRATEGIES

We now turn to converse results. For this purpose it is sufficient to consider single-population dynamics (see remark at the end of this section). More specifically, we will show that Theorem 1 is sharp for single-population payoff functional selection dynamics. If f is not throughout convex, then there are symmetric two-player games with strictly dominated strategies surviving along interior solutions to the associated single-population dynamics. We establish this by a slight modification of a game given in Dekel and Scotchmer [7].

This is a ROCK–SCISSORS–PAPER game, augmented by a fourth strategy, called DUMB, which is strictly dominated. The payoff matrix is given by

$$A = \begin{bmatrix} a & c & b & \gamma \\ b & a & c & \gamma \\ c & b & a & \gamma \\ a+\beta & a+\beta & a+\beta & 0 \end{bmatrix}, \tag{13}$$

where $c < a < b$, $0 < \beta < b - a$, and $\gamma > 0$. The pure strategies $h \in H = \{1, 2, 3\}$ form a cycle of best replies. For a single-population selection dynamics (11) this implies that the (relative) boundary Γ_1 of the face $\Phi = \Delta(H)$ forms a *heteroclinic cycle*: Γ_1 is an invariant set that consists of three rest points e^h, for $h \in H$, which are saddle points in any monotonic selection dynamics, and three connecting orbits. (Clearly Γ_1 is unstable in the e^4 direction since $\beta > 0$.) In particular, DUMB can invade a monomorphic population consisting of only h-strategists, for each of the pure strategies $h = 1, 2, 3$. Hence, on the boundary of Δ there are three more rest points (corresponding to symmetric Nash equilibria of each of the associated 2×2 "subgames") for any monotonic selection dynamics (11): $z^1 = (\gamma/(\beta+\gamma), 0, 0, \beta/(\beta+\gamma))$, $z^2 = (0, \gamma/(\beta+\gamma), 0, \beta/(\beta+\gamma))$ and $z^3 = (0, 0, \gamma/(\beta+\gamma), \beta/(\beta+\gamma))$.

Note that $z^h \in \Delta$ attracts all orbits on the (relative) interior of the boundary face of Δ where $x_{h+1} = 0$, for any monotonic selection dynamics. Hence, there is another heteroclinic cycle Γ_2 connecting these three rest points. The connecting orbits are now curves in these two-dimensional subfaces, invariant under the flow of (11), namely the unstable manifolds of the saddles z^h (see Fig. 1).

Let $p = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0)$ be the Nash equilibrium point of the RSP subgame. The strategy p is in Nash equilibrium with itself in the full game, iff $(a+b+c)/3 \geqslant a+\beta$. Moreover, p strictly dominates pure strategy 4 (=DUMB) iff

$$\frac{a+b+c}{3} > a+\beta. \tag{14}$$

For the replicator dynamics, and, more generally all those meeting axiom (CM′), this implies that $x_4(t) \to 0$ along all interior solutions.

The inequality (14) holds only if $a < (b+c)/2$. The latter inequality implies that the Nash equilibrium strategy p is globally stable in the replicator dynamics: Every solution that has all pure strategies $h \in H$ initially present, will converge to p. For other (regular) monotonic selection dynamics this need not be true. In particular, it may happen that p is not globally stable within the face Φ, since the boundary cycle Γ_1 may be attracting on that face. Then orbits close to Γ_1 will spiral away from p. Near Γ_1, x_4 will increase most of the time and the orbits will converge to the heteroclinic cycle Γ_2 formed by the z^h. The dominated pure strategy 4 will not be eliminated along such orbits.

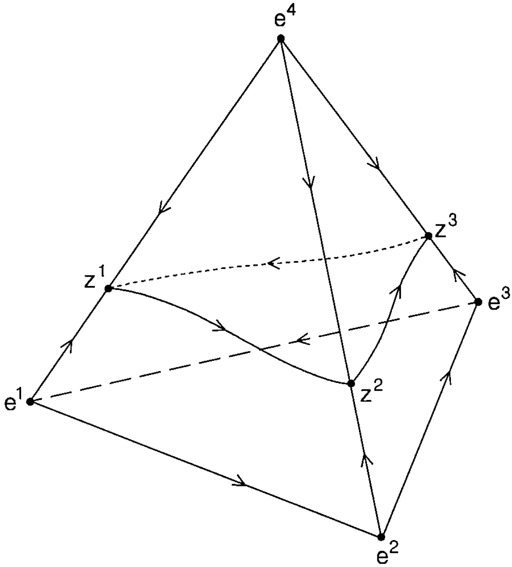

FIG. 1. The rest points z^h ($h=1, 2, 3$) and the connecting orbits together form the heteroclinic cycle Γ_2. This invariant set is an attractor under the assumptions of Theorem 2, leading to the survival of the strictly dominated fourth pure strategy.

Formally:

THEOREM 2. *Consider a regular single-population dynamics* (11) *where* $\tilde{g}$ *is of the form* (12). *If* f *is not convex, then there exists a symmetric two-player game with payoff matrix as in* (13) *such that the dominated pure strategy* 4 *survives along an open set of interior solutions of* (11).

Proof. As shown in the proof of the lemma: If f is not convex, there exist $a, b, c \in \mathbb{R}$ such that $2a < b + c$ and $2f(a) > f(b) + f(c)$. Consider now the RSPD game (13) with these a, b, c. As is easily seen, and was shown in Gaunersdorfer and Hofbauer ([13], Section 4), the eigenvalues of the vector field (11) at a vertex e^h (for $h \in H$) are given by $\rho = \lambda(e^h)[f(b) - f(a)] > 0$ and $-\tau = \lambda(e^h)[f(c) - f(a)] < 0$. Now $2f(a) > f(b) + f(c)$ implies $\tau > \rho$, i.e., the "incoming speed" is larger than the "outgoing speed," which means that Γ_1 is attracting within the face Φ, according to the stability criterion for heteroclinic cycles in Hofbauer [14], Hofbauer and Sigmund ([17], Sect. 22.1, in particular Exercise 6), and Gaunersdorfer [12].

Now choose $\beta > 0$ small enough to satisfy (14), and such that the z^h, the rest points of Γ_2, are close to the e^h, the rest points of Γ_1. Since the inequality guaranteeing stability for Γ_1 within the face Φ is strict it holds also for Γ_2: The "outgoing speed" is smaller than the "ingoing speed" at the rest points z^i. Since Γ_2 (unlike Γ_1) is asymptotically stable within the boundary of Δ this establishes the (local) asymptotic stability of Γ_2 in the full space Δ, for small $\beta > 0$.[11] The dominated pure strategy 4 thus survives for an open set $U \supset \Gamma_2$ of interior initial states. *End of proof.*

Remark. Theorem 2 also shows that two-population payoff functional dynamics (3, 6), with a non-convex function *f*, do not eliminate all strictly dominated strategies in all games. Just consider (13) as the payoff matrix of a symmetric bi-matrix game. The restriction of the associated two-population dynamics to the invariant diagonal of the state space $\Theta = \Delta^2$ coincides with the one-population dynamics studied above, and hence we obtain interior two-population solutions along which strategy DUMB survives.

7. CONCLUDING REMARKS

Theorem 1 identifies a class of evolutionary selection dynamics that select against all iteratively strictly dominated pure strategies in all (finite *n*-player) games. Our proof is an extension of Akin's [2] proof that strictly dominated strategies are eliminated in the single-population replicator dynamics for symmetric two-player games.

Theorem 2 provides a complementary class of evolutionary selection dynamics under which strictly dominated strategies do survive for some games. Björnerstedt [3], see also Björnerstedt *et al.* [4], presents a different, but related, class of evolutionary selection dynamics with the same property. He imagines that individuals every now and then review their strategy choice by way of a (possibly noisy) payoff comparison with all other strategies. Such a reviewing individual changes strategy if and only if her current strategy is observed to yield the worst payoff of all pure strategies. In this case, she imitates a (uniformly) randomly drawn individual. Björnerstedt gives a nice geometric proof that the strictly dominated pure strategy in a version of the Dekel–Scotchmer [7] game studied above survives the resulting "abandon the worst reply" dynamics for a large set of initial states. His argument is robust against small perturbations of the dynamics, so the result applies also to the monotonic concave dynamics (8) for large σ. In contrast, our proof of Theorem 2 is based on the stability criterion in Hofbauer [14] for heteroclinic cycles, and is not directly applicable to the worst-reply dynamics. On the other

[11] Compare with Theorem 22.1(b) in Hofbauer and Sigmund [17].

hand, the technique behind Theorem 2 is more powerful since it allows to obtain general and, in conjunction with Theorem 1 (and its corollary), sharp results.

In a parallel study, Hofbauer [16] shows (among other things) that strictly dominated strategies can survive under a class of selection dynamics based on models of social evolution by way of imitation introduced in Weibull [25, 27]. In these imitation processes individuals every now and then make a binary and noisy comparison with the strategy used by another, randomly selected, individual. The reviewing individual switches to the sampled strategy iff its observed payoff is higher than her current observed payoff. The replicator dynamics, which corresponds to an affine cumulative probability distribution function for the observational errors (over the range of payoffs in the game), is essentially the only imitation dynamics in that class that eliminates strictly dominated strategies in all games.

In sum, all evolutionary dynamics in the class of convex monotonic (CM) selection dynamics that we have introduced here lend support to the rationalistic principle of elimination of iteratively strictly dominated strategies, and it appears that this is the only class of evolutionary dynamics for which this is true.

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