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UNIFORM PERSISTENCE AND REPELLORS FOR MAPS

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ABSTRACT. We establish conditions for an isolated invariant set M of a map to be a repellor. The conditions are first formulated in terms of the stable set of M. They are then refined in two ways by considering (i) a Morse decomposition for M, and (ii) the invariantly connected components of the chain recurrent set of M. These results generalize and unify earlier persistence results.

1. INTRODUCTION

Let $\mathscr{X}$ be a metric space with metric d. A map $f:\mathscr{X} \to \mathscr{X}$ defines a discrete semi-dynamical system $T: \mathbf{Z}_+ \times \mathscr{X} \to \mathscr{X}$ by $T(n, x) = f^n(x)$, where $\mathbf{Z}_+$ denotes the set of non-negative integers and $f^n(x)$ denotes the nth iterate of x under f. Let $\mathscr{Y}$ be a subspace of $\mathscr{X}$. We say that f is *uniformly persistent* (with respect to $\mathscr{Y}$) if there exists $\eta > 0$ such that for all $x \in \mathscr{X}\backslash\mathscr{Y}$, $\liminf_{n\to\infty} d(f^n(x), \mathscr{Y}) > \eta$. In applications to ecological equations, $\mathscr{X}$ will be the set of all possible states of the system and $\mathscr{Y}$ the set of extinction states. In that case, uniform persistence captures the idea of non-extinction of the system.

The object of this paper is to obtain criteria for uniform persistence. These criteria are formulated as conditions imposed on the global attractor M of $\mathscr{Y}$. Hence, we first study uniform persistence on compact spaces X (§§2 and 3) and apply these results to the general problem in §4. In particular, the persistence results in Freedman and So [9,10] will be improved to include uniform persistence. The approach we use here is similar to the one used in Garay [11] and Hofbauer [14] for flows. It provides a more elegant approach to the persistence problem as it uses modern dynamical systems theory. Hence, this approach allows us to give simpler proofs for earlier persistence results based on average

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Lyapunov functions (cf. Hutson and Moran [17], Fonda [8], Hofbauer, Hutson and Jansen [15] and Hofbauer and Sigmund [16]).

2. Repellors

In this section and the next, the standing assumptions are:

(a) X is a compact metric space,

(b) $f: X \to X$ is a continuous map, and

(c) $M \subset X$ is compact invariant, that is, $f(M) = M$.

For notation and terminology concerning discrete semi-dynamical systems not explained here, we refer the reader to LaSalle [18], Bhatia and Hajek [2] and Hale [12].

M is said to be a *repellor* if there exists a neighborhood U of M such that for all $x \notin M$ there exists $n_0 = n_0(x) > 0$ satisfying $f^n(x) \notin U$ for all $n \geq n_0$. U is called a *repellor neighborhood* of M.

The following theorem is a special case of the Ura-Kimura theorem for maps (cf. Bhatia and Hajek [2, Chapter 9]).

Theorem 2.1. *M is a repellor if and only if*

(1) *M is isolated, that is, there exists a closed neighborhood U of M (U is called an isolating neighborhood of M) such that M is the largest invariant set in U, and*

(2) *$W^s(M) \subset M$, where $W^s(M) := \{x \in X: f^n(x) \to M \text{ as } n \to +\infty\}$ is the stable set of M.*

Proof. The $\Rightarrow$ part is obvious. To show the converse, let V be a compact isolating neighborhood of M. Then for each $x \notin M$, there exists arbitrarily large n such that $f^n(x) \notin V$. (Otherwise, the *ω-limit set of* $x, \Lambda^+(x) := \bigcap_{n=0}^{\infty} \mathrm{cl}(\bigcup_{k=n}^{\infty} f^k(x))$, will be an invariant subset of V. Since V is an isolating neighborhood of $M, \Lambda^+(x) \subset M$. This contradicts (2).) Let $B = \bigcup_{n=0}^{\infty} f^n(\mathrm{cl}(X \backslash V))$.

Claim. There exists $N \geq 0$ such that $B = \bigcup_{n=0}^{N} f^n(\mathrm{cl}(X \backslash V))$.

Indeed for all $x \in \mathrm{cl}(X \backslash V)$, there exists $n(x) \geq 1$ such that $f^{n(x)}(x) \in X \backslash V$. By compactness of $\mathrm{cl}(X \backslash V)$, there exists $N \geq 0$ such that for all $x \in \mathrm{cl}(X \backslash V)$, there exists $n, 1 \leq n \leq N$ such that $f^n(x) \in X \backslash V$, since $X \backslash V$ is open. This proves the claim.

Thus B is compact, *positively invariant* (that is, $f(B) \subset B$) and disjoint from M and $U = X \backslash B$ is a repellor neighborhood for M. □

Remark. The set $A = \bigcap_{n=0}^{\infty} f^n(B)$ is an *attractor*, that is, there is a neighborhood U of A such that the *ω-limit set* of U, $\Lambda^+(U) := \bigcap_{n=0}^{\infty} \mathrm{cl}(\bigcup_{k=n}^{\infty} f^k(U))$, is A. It is called the *dual attractor* to the repellor M. Moreover, $W^s(A) = X \backslash M$.

Corollary 2.2 (Fonda [8]). *Let $X\backslash M$ be positively invariant. Then M is a repellor if there exists a continuous function $P: X \to \mathbf{R}^+$ satisfying the conditions:*

(1) *$P(x) = 0$ for $x \in M$, and*
(2) *there exists a neighborhood U of M such that $\forall x \in U\backslash M \, \exists n > 0$ such that $P(f^n(x)) > P(x)$.*

Proof. Without loss of generality, we can assume U is closed. Suppose $\exists x \notin M$ with $\mathrm{cl}(\gamma^+(x)) \subset U, \gamma^+(x) := \{x, f(x), f^2(x), \dots\}$; then $\exists y \in \mathrm{cl}(\gamma^+(x))$ such that $P(y) \geq P(z)$ for all $z \in \mathrm{cl}(\gamma^+(x))$. Hence, $y \notin M$ and $P(y) \geq P(f^n(y))$ for all $n \geq 0$. This contradicts (2). Therefore, U is an isolating neighborhood of M and $\forall x \notin M$, $\Lambda^+(x) \not\subset M$. Hence, by Theorem 2.1, M is a repellor. □

Corollary 2.3 (Hutson and Moran [17]). *Let $X\backslash M$ be positively invariant. Then M is a repellor if there exists a continuous function $P: X \to \mathbf{R}^+$ such that*

(1) *$P(x) = 0$ if and only if $x \in M$, and*
(2) *for all $x \in M$, $\sup\{\prod_{k=0}^{n-1} \psi(f^k(x)): n \geq 1\} > 1$, where $\psi: X \to \mathbf{R}^+$ is a continuous function with $P(f(x)) \geq \psi(x)P(x)$.*

Proof. Let $U_n = \{x \in X: \psi(x) \cdots \psi(f^{n-1}(x)) > 1\}$. Then U_n is open and $U := \bigcup_{n=1}^{\infty} U_n \supset M$. Therefore P satisfies the conditions in Corollary 2.2 and M is a repellor. □

Remark. This shows that M is a repellor if one can find a function P as above such that $\psi > 1$ on $\bigcup_{x \in M} \Lambda^+(x)$.

Let Γ be a compact invariant set and let $x, y \in \Gamma$. An *ϵ-chain* from x to y is a sequence of points $x_0 = x$, $x_1, \dots, x_n = y$ in Γ with $d(f(x_i), x_{i+1}) < \epsilon$ for $i = 0, \dots, n-1$. We say that x is *chained* to y if for all $\epsilon > 0$, there exists an ϵ-chain from x to y. A compact, invariant subset Λ of Γ is *chain transitive* if any two points x and y in Λ are chained in Λ. If x is chained to itself (in Γ), then x is said to be a *chain recurrent* point and we write $x \in \mathscr{R}(\Gamma)$. Now, $x \notin \mathscr{R}(\Gamma)$ iff there exists an open set $U \subset \Gamma$ such that $f(\mathrm{cl}(U)) \subset U$, $x \notin U$ and $f(x) \in U$, or equivalently, iff there exists an attractor $A \subset \Gamma$ such that $x \in W^s(A)\backslash A$ (see Block and Franke [4, Theorem A]). Hence, $\mathscr{R}(\Gamma) = \bigcap(A \cup A^*)$, the intersection being taken over all attractor-repellor pairs (A, A^*) of Γ.

The equivalence classes $C(x) = \{y \in \Gamma: x$ is chained to y and y is chained to $x\}$ are called the *basic sets* of Γ. $C(x)$ are chain transitive and they are the invariantly connected components of $\mathscr{R}(\Gamma)$ (see Conley [7] for flows and Akin [1] for maps). A compact invariant set is *invariantly connected* if it is not the disjoint union of two non-empty compact invariant sets.

Corollary 2.4 (Butler-McGehee lemma for maps (cf. Freedman and So [10])). *Let M be an isolated invariant set and let $x \in X$ be such that $\Lambda^+(x) \cap M \neq \varnothing$ and $\Lambda^+(x) \not\subset M$. Then $\Lambda^+(x) \cap (W^s(M)\backslash M) \neq \varnothing$.*

Proof. Let $X_1 = \Lambda^+(x), M_1 = \Lambda^+(x) \cap M$, $f_1 = f|_{X_1}$ and restrict ourselves to the dynamics of f_1 on X_1 for the moment. Suppose $\Lambda^+(x) \cap (W^s(M)\backslash M) = \emptyset$. Then M_1 is isolated invariant, $X_1 \backslash M_1$ is positively invariant and $W^s(M_1) \subset M_1$. By Theorem 2.1, M_1 is a (non-empty) repellor. Since X_1 is chain transitive (see Bowen [3]), it cannot have a non-trivial attractor-repellor pair. Thus, $M_1 = X_1$ and this contradicts $\Lambda^+(x) \not\subset M$. □

Remark. There is a characterization for attractors analogous to Theorem 2.1. It can be used to show that, under the hypotheses of Corollary 2.4, there exists a negative orbit in $\Lambda^+(x)\backslash M$ whose α-limit is contained in M.

3. Morse decompositions, basic sets and repellors

Let M be a compact, invariant set. A finite collection $\{M_1, \dots, M_n\}$ of compact invariant subsets of M is called a *Morse decomposition* if the M_i are pairwise disjoint and for each $x \in M \backslash \bigcup_{i=1}^n M_i$ there is an i with $\Lambda^+(x) \subset M_i$ and for any negative orbit x_- through x there is a $j > i$ with $\Lambda^-(x_-) \subset M_j$, where $\Lambda^-(x_-)$ is the α-limit set of x_- (cf. Mischaikow and Franzosa [19]).

Remark. It was shown in Freedman and So [10] that an acyclic covering, as introduced in Butler, Freedman and Waltman [5] and Butler and Waltman [6], is a Morse decomposition. The proof is based on the Butler-McGehee Lemma.

The following theorem is a refinement of Theorem 2.1 if a Morse decomposition of M is given. It improves Theorem 3.3 of Freedman and So [10] as it yields uniform instead of strong persistence. Its proof shows that the existence of an acyclic covering for M is equivalent to M being isolated. The point is that one should choose an acyclic covering of M so that the assumptions can be easily checked.

Theorem 3.1. *Suppose $\{M_1, \dots, M_n\}$ is a Morse decomposition of M such that each M_i is isolated in X. Then M is a repellor if and only if $W^s(M_i) \subset M$ for each i.*

Proof. The $\Rightarrow$ part is obvious. For the converse, we have to show that conditions (1) and (2) in Theorem 2.1 are satisfied.

First we will show (2). Let $x \notin M$. Suppose $\Lambda^+(x) \subset M$. Then $\Lambda^+(x) \subset \mathscr{R}(M)$, the chain recurrent set of M. Since $\mathscr{R}(M) \subset \bigcup_{i=1}^n M_i$ and $\Lambda^+(x)$ is invariantly connected, there exists i such that $\Lambda^+(x) \subset M_i$, contradicting $W^s(M_i) \subset M$.

To show (1), assume M is not isolated. Then for every n, there exists $x_n \notin M$ with $\gamma^+(x_n) \subset \{x: d(x, M) < 1/n\}$, where $\gamma^+(x_n)$ is the positive orbit through x_n. The sets $\Omega_n = \Lambda^+(x_n)$ are compact, invariant and chain transitive. By (2), we know $\Omega_n \not\subset M$. A subsequence of $\{\Omega_n\}$, which we again denote by $\{\Omega_n\}$, converges in the Hausdorff metric to a compact set $\Omega \subset M$. Since $f(\Omega_n)$ converges to $f(\Omega)$, Ω is invariant. By the argument in Garay [11, Proof of Theorem 2], Ω is also chain transitive. Therefore

$\Omega \subset \mathscr{R}(M) \subset \bigcup_{i=1}^{n} M_i$. Since Ω is invariantly connected, $\Omega \subset M_i$ for some i. Hence any neighborhood of M_i must contain some Ω_n for large n and M_i is not isolated. □

The proof of Theorem 3.1 actually shows:

Theorem 3.2. *M is a repellor if and only if each basic set Λ of M satisfies*

(1) *Λ is M-isolated, that is, there is a neighborhood U of Λ in X such that if a complete orbit $\gamma(x) \subset U$ then $\gamma(x) \subset M$, and*
(2) *$W^s(\Lambda) \subset M$.*

4. Uniform persistence

Let $\mathscr{X}$ be a metric space with metric d, $f:\mathscr{X} \to \mathscr{X}$ be a continuous map and $\mathscr{Y} \subset \mathscr{X}$ is closed with $f(\mathscr{X}\backslash\mathscr{Y}) \subset \mathscr{X}\backslash\mathscr{Y}$.

We assume that $\mathscr{X}$ has a *global attractor* X, that is, X is the maximal compact invariant subset of $\mathscr{X}$ and $d(f^n(x), X) \to 0$ as $n \to \infty$, for all $x \in \mathscr{X}$. This assumption is satisfied in many applications including maps, time-periodic ordinary differential equations, retarded delay equations as well as semi-linear parabolic equations (cf. Hale [12] and Hale and Waltman [13]).

Note that $\mathscr{Y}$ is in general not a positively invariant set. Let M be the maximal compact invariant set in $\mathscr{Y}$. Then $M \subset X$.

Theorem 4.1. *f is uniformly persistent (w.r.t. $\mathscr{Y}$) if and only if*

(1) *M is isolated in X, and*
(2) *$W^s(M) \subset \mathscr{Y}$.*

Proof. The $\Rightarrow$ part is again obvious. For the converse, we first observe that Theorem 2.1 as applied to $f|_X$ (and the remark following it) gives us an attractor A (for $f|_X$) dual to M. Since A is non-empty and compact, $\mathscr{Y}$ is closed and $A \cap \mathscr{Y} = \varnothing$, we have $\eta := d(A, \mathscr{Y}) > 0$. Now consider $x \in \mathscr{X}\backslash\mathscr{Y}$. Then $\Lambda^+(x)$ (which exists by the above assumptions) is contained in X and $\Lambda^+(x) \not\subset M$, by (2). Since $\Lambda^+(x)$ is chain transitive, $\Lambda^+(x) \subset A$. Hence, $\liminf_{n\to\infty} d(f^n(x), \mathscr{Y}) \geq \eta$. □

In the same way we get the following modifications of Theorems 3.1 and 3.2.

Theorem 4.2. *Suppose $\{M_1, \ldots, M_n\}$ is a Morse decomposition of M (under $f|_M$) such that each M_i is isolated in X. Then f is uniformly persistent if and only if $W^s(M_i) \subset \mathscr{Y}$ for each i.*

Theorem 4.3. *f is uniformly persistent if and only if each basic set Λ of $f|_M$ satisfies*

(1) *Λ is M-isolated, that is, there is a neighborhood U of Λ in X such that if a complete orbit $\gamma(x) \subset U$ then $\gamma(x) \subset M$, and*
(2) *$W^s(\Lambda) \subset \mathscr{Y}$.*

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