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A q-analog of the Lagrange expansion

By

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1. The classical Lagrange inversion theorem gives an explicit formula for the coefficients in the expansion

$$f(z) = f(0) + \sum_{n \geq 1} \frac{c_n}{n!} \frac{z^n}{\varphi(z)^n} \tag{1.1}$$

where $f(z)$ and $\varphi(z)$ are (formal) power series with $\varphi(0) \neq 0$:

$$c_n = \left(\frac{\partial}{\partial z}\right)^{n-1} (f'(z)\,\varphi^n(z))\Big|_{z=0} \tag{1.2}$$

A second less often used formula is

$$c_n = \left(\frac{\partial}{\partial z}\right)^{n} \left(f(z)\,\varphi^n(z)\left(1 - z\frac{\varphi'(z)}{\varphi(z)}\right)\right)\Big|_{z=0}. \tag{1.3}$$

Simple proofs of this formula have been given based on Rota's umbral calculus [17, 18], see e.g. [8, 17] or [12, 15] for the higher dimensional generalization of Good.

To obtain a q-analog of (1.1) one has to replace $\varphi(z)^n$ by any q-analog $\varphi_n(z)$ of the n-th power of $\varphi(z)$. After Carlitz's striking example [2] $\varphi_n(z) = (1 - z)(1 - qz)\ldots(1 - q^{n-1}z)$ it seemed natural to consider $\varphi_n(z) = \varphi(z)\,\varphi(qz)\ldots\varphi(q^{n-1}z)$. Andrews [1] gave an analog of (1.2) in this case: he represented c_n by an $(n + 1) \times (n + 1)$ determinant. Garsia and Joni [7, 9] found the analog of the second version (1.3) and have developed a far reaching theory including a nice q-analog of the umbral calculus. They have also yielded the connection to Gessels paper [10], who had found a q-analog of Lagrange inversion from a different point of view. Garsia [7] has also given interesting applications to partition identities. But for the original purpose – to give such explicite and calculable formulas for the coefficients c_n as in the classical case – only Carlitz's special case seems to be practicable.

Reformulating some q-Abel identities of Jackson [14], Cigler [4] found an equally simple formula as (1.2) for $\varphi_n(z) = e(a[n]z)$. He gave a unified approach to this and Carlitz's special case and proposed how the general q-analog of (1.2) should look like. Based on his ideas we will state this formula, giving the appropriate concept of n-th power. As an application we obtain some q-analogs of Abel's identity. Finally, recent generalizations due to Krattenthaler [16] are mentioned.

A fairly good example for comparing the different nature of Garsia's q-analog to that proposed in this paper may be found in [6], where two versions of q-Catalan numbers are studied.

2. The main idea is to interprete a formal power series $f(z) = \sum a_k z^k$ as a linear operator $f(D) = \sum a_k D^k$ on the space of polynomials (over some field of characteristic zero), where D is the q-differentiation operator. (We adopt the notation of [3, 4].) If L denotes the linear functional $Lp = p(0)$ then $a_k \cdot [k]! = Lf(D)x^k$.

In order to calculate the coefficients in an expansion of the form

$$f(z) = \sum_{n=1}^{\infty} \frac{c_n}{[n]!} \frac{z^n}{\varphi_n(z)} \quad \text{with} \quad \varphi_n(0) \neq 0 \tag{2.1}$$

we consider the polynomial sequence $(p_n)_{n \geq 0}$ with $\deg p_n = n$, whose dual functionals are given by $LD^n/\varphi_n(D)$:

$$L \frac{D^n}{\varphi_n(D)} p_m(x) = [n]!\, \delta_{nm}. \tag{2.2}$$

The relevance of those polynomials for (2.1) becomes clear from

$$Lf(D)p_m(x) = \sum \frac{c_n}{[n]!} L \frac{D^n}{\varphi_n(D)} p_m(x) = c_m. \tag{2.3}$$

Now if $\varphi_n(D) = \varphi(D)^n$ (and $q = 1$) then (p_n) is the unique sequence of binomial type with deltaoperator $\frac{D}{\varphi(D)} p_n = n p_{n-1}$ (see [4, 17]). Steffensens formula $p_n = x\varphi(D)^n x^{n-1}$ together with (2.3) gives

$$c_n = Lf(D)x\varphi(D)^n x^{n-1} = Lf'(D)\varphi(D)^n x^{n-1} = LD^{n-1}f'(x)\varphi(x)^n$$

since $f(D)x - xf(D) = f'(D)$ implies $Lf(D)x = Lf'(D)$ and $Lf(D)g(x) = Lg(D)f(x)$ holds. But this is just Lagrange's formula (1.2).

Cigler [4] now suggested to determine for $q \neq 1$ those sequences (φ_n) for which the p_n's are again given by $p_n(x) = x\varphi_n(D)\, x^{n-1}$ as he had shown for the two special cases of Carlitz and Jackson. This is done through

Lemma 1. *Consider the following statements for a sequence $(\varphi_n)_{n \geq 1}$ of formal power series with $\varphi_n(0) \neq 0$. ($f'(z)$ here means the q-derivative of $f(z)$):*

(1) $\qquad \dfrac{\varphi_n'(z)}{[n]\varphi_n(z)}$ *is a formal power series independent of n;*

(2) $\qquad \dfrac{D^k}{\varphi_k(D)} x\varphi_n(D)x^{n-1} = [n]_k x \dfrac{\varphi_n(D)}{\varphi_k(D)} x^{n-k-1},$ *where* $[n]_k = \dfrac{[n]!}{[n-k]!}$;

(3) $\qquad L\dfrac{D^k}{\varphi_k(D)} x\varphi_n(D)x^{n-1} = [n]!\,\delta_{nk}.$

Then (1) $\Rightarrow$ (2) $\Rightarrow$ (3).

Proof. We will use the following operator relations (see [3], eq. (5) and (6)) where $\varepsilon p(x) = p(qx)$:

$$\begin{aligned} f(D)x - xf(qD) &= f'(D) \\ f(D)x - xf(D) &= \varepsilon f'(D) \qquad (2.4)\\ f(D)\varepsilon &= \varepsilon f(qD), \end{aligned}$$

to deduce an equivalent formulation of (2):

$$D^k x \varphi_n(D) x^{n-1} = [n]_k \varphi_k(D) x \frac{1}{\varphi_k(D)} \varphi_n(D) x^{n-k-1}$$

$$(q^k x D^k + [k] D^{k-1}) \varphi_n(D) x^{n-1} = [n]_k \left(x + \varepsilon \frac{\varphi_k'(D)}{\varphi_k(D)}\right) \varphi_n(D) x^{n-k-1}$$

$$q^k [n-1]_k x \varphi_n(D) x^{n-k-1} + [k][n-1]_{k-1} \varphi_n(D) x^{n-k} =$$

$$= [n]_k x \varphi_n(D) x^{n-k-1} + [n]_k \varepsilon \frac{\varphi_k'(D)}{\varphi_k(D)} \varphi_n(D) x^{n-k-1}.$$

Hence $[n]_k \varepsilon \frac{\varphi_k'(D)}{\varphi_k(D)} \varphi_n(D) x^{n-k-1} = [k][n-1]_{k-1} \varphi_n(D) x^{n-k} -$

$$- [n-1]_{k-1} x \varphi_n(D) x^{n-k-1}([n] - q^k[n-k]) = [k][n-1]_{k-1} \{\varphi_n(D)x - x\varphi_n(D)\} x^{n-k-1}$$

$$= [k][n-1]_{k-1} \varepsilon \varphi_n'(D) x^{n-k-1}.$$

Therefore (2) is equivalent (for $k \leq n$) to

$$\varepsilon \left\{\frac{\varphi_n'(D)}{[n]\varphi_n(D)} - \frac{\varphi_k'(D)}{[k]\varphi_k(D)}\right\} x^{n-k-1} = 0 \qquad (2.5)$$

which is an obvious consequence of (1). The other implication (2) $\Rightarrow$ (3) is clear.

Remarks. (2) is the straightforward generalization of [4], (14). It is easy to see from (2.5) that for $q \neq 0$ even (1) $\Leftrightarrow$ (2) holds, but the converse of (1) $\Rightarrow$ (3) is false for every q.

Of course for $q = 1$ (1) together with appropriate initial conditions, like $\varphi_n(0) = 1$, reduces to $\varphi_n(z) = \varphi_1(z)^n$.

If we recall (2.3), an immediate consequence of Lemma 1 is

Theorem 1. *If $(\varphi_n)_{n \geq 1}$ is a sequence of formal power series $\varphi_n(0) \neq 0$ and $\varphi_n'(z)/[n]\varphi_n(z)$ independent of n, then the coefficients in the expansion*

$$f(z) = f(0) + \sum_{n \geq 1} \frac{c_n}{[n]!} \frac{z^n}{\varphi_n(z)} \quad \textit{are given by} \quad c_n = LD^{n-1} f'(z) \varphi_n(z).$$

Corollary 1. *If $(\varphi_n)_{n \geq 1}$ satisfies condition (1) of Lemma 1 and $p_n(x) = x\varphi_n(D)x^{n-1}$, $p_0(x) \equiv 1$ are the corresponding polynomials, then*

$$e(xz) = \sum_{n \geq 0} \frac{p_n(x)}{[n]!} \frac{z^n}{\varphi_n(z)}.$$

Proof. According to Theorem 1 the coefficients in the expansion of $e(az)$ are given by

$$LD^{n-1}e(az)a\cdot\varphi_n(z)=a\,Le(aD)\,\varphi_n(D)\,z^{n-1}=p_n(a)$$

since $Le(aD)$ is just the evaluation functional at $x=a$.

Examples. Set $\varphi_n(z)=\dfrac{e(-a[n]z+bz)}{e(z)}$.

Here $e(z)=\sum_{n=0}^{\infty}z^n/[n]!$ is the q-exponential function which satisfies $e'(z)=e(z)$ or $e(qz)=(1+(q-1)z)\,e(z)$. The quotient rule for q-differentiation then implies

$$\begin{aligned}\varphi_n'(z)&=\frac{e(-a[n]z+bz)}{e(qbz)}(-a[n]+b)-\frac{b}{e(qbz)}e(-a[n]z+bz)\\&=-[n]\,\varphi_n(z)\frac{a}{1+(q-1)bz}\end{aligned}$$

and (1) is fulfilled. As special cases we obtain the two cases studied by Cigler:

$$b=0 \qquad \varphi_n(z)=e(-a[n]z)$$

$$b=\frac{1}{1-q},\ a=1 \qquad \varphi_n(z)=\frac{e\left(\dfrac{q^n}{1-q}z\right)}{e\left(\dfrac{z}{1-q}\right)}=(1-z)(1-qz)\ldots(1-q^{n-1}z).$$

The latter gives exactly Carlitz's expansion, as mentioned in the introduction.

For $f(D)=\dfrac{e(bD)}{e(-yD)}$ Theorem 1 gives (see also [4], (17) for $b=0$)

$$\begin{aligned}\frac{1}{e(-yD)}=\sum_{k=0}^{\infty}&\frac{(y+b)(qy-a[k]+b)\ldots(q^{k-1}y-a[k]+b)}{[k]!}\\&\cdot\frac{D^k}{e(-a[k]D+bD)}.\end{aligned}$$

Applying these operators to x^n we get a generalization of Jackson's result [14], (2):

(2.6)
$$\begin{aligned}&(x+y)(x+qy)\ldots(x+q^{n-1}y)\\&=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}(y+b)(qy-a[k]+b)\ldots(q^{k-1}y-a[k]+b)\\&\quad\cdot(x-b+a[k])\ldots(x+q^{n-k-1}(-b+a[k])).\end{aligned}$$

For $q=1$ this reduces to Abel's famous formula

(2.7)
$$(x+y)^n=\sum_{k=0}^{n}\binom{n}{k}y(y-ka)^{k-1}(x+ak)^{n-k}.$$

Corollary 1 gives in this special case (for $b = 0$, see [4], (15))

$$e(yD) = \sum_{k=0}^{\infty} \frac{B_k^{(a)}(y)}{[k]!} \frac{D^k}{e(-a[k]D)}$$

where $B_n^{(a)}(x) = x\,e(-a[n]D)\,x^{n-1}$ is the corresponding q-analog of the Abel polynomial $x(x-an)^{n-1}$, studied extensively by Cigler. This implies a second q-analog of Abel's equation (2.7)

$$H_n(x, y) = \sum_{k=0}^{n} \left[{n \atop k}\right] B_k^{(a)}(y)(x + a[k]) \dots (x + a[k]\,q^{n-k-1})$$

where

$$H_n(x, a) = e(aD)\,x^n = \sum \left[{n \atop k}\right] a^{n-k} x^k = (x + a\varepsilon)^n\, 1$$

is another q-analog of $(x+a)^n$.

The above example includes also a q-analog of $(1+z)^a$. For $b = \dfrac{1}{q-1}$ let us define

$$s_a(z) = e\left(az + \frac{z}{q-1}\right) \Big/ e\left(\frac{z}{q-1}\right) = \sum_{k \geq 0} \frac{(a)_k}{[k]!} z^k \tag{2.8}$$

with $(a)_k = a(a-[1]) \dots (a-[n-1])$.

Since $s'_a(z) = \dfrac{a}{1+z} s_a(z)$, $\varphi_n(z) = s_{a[n]}(z)$ is the n-th power of $s_a(z)$ in the sense of Lemma 1.

Obviously $s_a(z)$ is a q-analog of $(1+z)^a$; its properties have been studied in [5]. We mention only the useful identity

$$\frac{s_a(z)}{s_b(z)} = \sum_{n=0}^{\infty} \frac{(a-b)(a-qb-[1]) \dots (a-q^{n-1}b-[n-1])}{[n]!} z^n$$

which follows immediately from (2.8) and the q-binomial theorem [3], (12).

As we derived q-Abel identities, one may now in a similar way obtain q-analogs of the Gould-Rothe identity

$$\binom{x+y}{n} = \sum_{k=0}^{n} \binom{y+ka}{k} \frac{y}{y+ka} \binom{x-ak}{n-k}$$

and of the Gould polynomials $\dfrac{x}{x+ak}\dbinom{x+ak}{k}$ (see [18], ch. 12) which are the difference analogs of the Abel polynomials.

In fact the resulting q-identity can be obtained directly from (2.6) replacing x by $x + \dfrac{1}{q-1}$, y by $y - \dfrac{1}{q-1}$ and $b = \dfrac{1}{q-1}$.

3. In the following we derive a second q-analog of the Lagrange expansion, which is based on

Lemma 2. *Consider the following conditions for a sequence $(\psi_n)_{n\geq 1}$ of formal power series with $\psi_n(0) \neq 0$, where $q \neq 0$:*

(1)′ $$q^n \frac{\psi_n'(z)}{[n]\psi_n(qz)} \text{ is independent of } n;$$

(2)′ $$\frac{D^k}{\psi_k(D)} x\psi_n(qD)x^{n-1} = [n]_k x \frac{\psi_n(qD)}{\psi_k(qD)} x^{n-k-1};$$

(3)′ $$L \frac{D^k}{\psi_k(D)} x\psi_n(qD)x^{n-1} = [n]!\,\delta_{nk}.$$

Then (1)′ ⇔ (2)′ ⇒ (3)′.

Proof. Using again relations (2.4) we see that (2)′ is equivalent to

$$D^k x\psi_n(qD)x^{n-1} = [n]_k \psi_k(D)x \frac{1}{\psi_k(qD)} \psi_n(qD)x^{n-k-1}$$

$$(xD^k + \varepsilon[k]D^{k-1})\psi_n(qD)x^{n-1} = [n]_k\left(x + \frac{\psi_k'(D)}{\psi_k(qD)}\right)\psi_n(qD)x^{n-k-1}.$$

Hence $$[n]_k \psi_k'(D)\frac{\psi_n(qD)}{\psi_k(qD)} x^{n-k-1} = \varepsilon[k][n-1]_{k-1}\psi_n(qD)x^{n-k}$$
$$- ([n]_k - [n-1]_k)x\psi_n(qD)x^{n-k-1}$$
$$= [n-1]_{k-1}[k]q^{n-k}(\psi_n(D)x - x\psi_n(qD))x^{n-k-1}$$
$$= [n-1]_{k-1}[k]q^{n-k}\psi_n'(D)x^{n-k-1}.$$

So (2)′ can be written as

$$\left\{\frac{q^k}{[k]}\frac{\psi_k'(D)}{\psi_k(qD)} - \frac{q^n}{[n]}\frac{\psi_n'(D)}{\psi_n(qD)}\right\} x^{n-k-1} = 0.$$

But this is obviously equivalent to (1)′.

Theorem 2. *If $(\psi_n)_{n\geq 1}$ is a sequence of f.p.s. with $\psi_n(0) \neq 0$ and $q^n\psi_n'(z)/[n]\psi_n(qz)$ independent of n and $q \neq 0$, then the coefficients in the expansion*

$$f(z) = f(0) + \sum_{n\geq 1} \frac{d_n}{[n]!}\frac{z^n}{\psi_n(z)}$$

are given by

$$d_n = LD^{n-1}f'(z)\psi_n(qz).$$

Corollary 2. *Under the same assumptions*

$$e(xz) = \sum_{n\geq 0} \frac{q_n(x)}{[n]!}\frac{z^n}{\psi_n(z)} \quad \textit{holds with} \quad q_n(x) = x\psi_n(qD)x^{n-1}.$$

Remark. Conditions (1) and (1)′ are closely related: Substituting q by q^{-1} in a sequence (φ_n) which satisfies (1), one obtains a sequence satisfying (1)′. Hence the two Theorems are trivially equivalent. But there does not seem to be an easy relation between the corresponding polynomials p_n and q_n (if $q \neq 1$).

Examples. $\psi_n(z) = \dfrac{e(bz)}{e\left(a\dfrac{[n]}{q^n}z + bz\right)}$ satisfy (1)′.

Setting $b = \dfrac{1}{1-q}$, $a = 1$ we get $\psi_n(z) = \left(1 - \dfrac{z}{q}\right)\dots\left(1 - \dfrac{z}{q^n}\right)$. The resulting expansion theorem was obtained in [4], (9). For $b = 0$ the corresponding polynomials $q_n(x) = x\psi_n(qD)x^{n-1}$ have the form

$$\begin{aligned} q_n(x) &= x\,e\left(a\frac{[n]}{q^{n-1}}D\right)^{-1} x^{n-1} = x\left(x - a\frac{[n]}{q}\right)\left(x - a\frac{[n]}{q^2}\right)\dots\left(x - a\frac{[n]}{q^{n-1}}\right)\\ &= q^{-\binom{n}{2}}\, b_n^{(a)}(x), \end{aligned}$$

where the $b_n^{(a)}(x)$ are another q-analog of the Abel polynomials occurring in [4], (19).

Corollary 2 gives another result of Jackson [14], (5)

$$e(yD) = \sum_{k=0}^{\infty} \frac{1}{[k]!}\, y\left(y - a\frac{[k]}{q}\right)\dots\left(y - a\frac{[k]}{q^{k-1}}\right) D^k e\left(a\frac{[k]}{q^k}D\right).$$

These operators applied to x^n, we obtain a third q-analog of (2.7):

$$H_n(x, y) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} y\left(y - a\frac{[k]}{q}\right)\dots\left(y - a\frac{[k]}{q^{k-1}}\right) H_{n-k}\left(x, a\frac{[k]}{q^k}\right).$$

4. The polynomials $p_n(x) = x\varphi_n(D)x^{n-1}$ and $q_n(x) = x\psi_n(qD)x^{n-1}$ may be considered as q-analogs of the celebrated polynomials of binomial type which enjoy a great number of beautiful properties and satisfy nice identities. The question arises which of them may be extended to our q-analogs in a simple way. It seems that those are rather few. Among them are what Rota [18] called the "closed forms":

Proposition. *Suppose* (φ_n) *and* (ψ_n) *satisfy condition* (1) *resp.* (1)′ *and let*

$$\tilde{\varphi}(D) = -D\frac{\varphi_n'(D)}{[n]\varphi_n(D)} \quad \textit{and} \quad \tilde{\psi}(D) = -D\frac{q^n\psi_n'(D)}{[n]\psi_n(qD)}. \tag{4.1}$$

Then the following formulas hold:

(i) $p_n(x) = x\varphi_n(D)x^{n-1} = (1 + \varepsilon\tilde{\varphi}(D))\varphi_n(D)x^n = q^{-n}\varepsilon(1 + \tilde{\varphi}(D))\varphi_n(D)x^n$,

$p_n(x) = x\left(x - \varepsilon[n]\dfrac{\tilde{\varphi}(D)}{D}\right)^{n-1} 1$ (*if further* $\varphi_n(0) = 1$);

(i)′ $q_n(x) = x\psi_n(qD)x^{n-1} = (1 + \tilde{\psi}(D))\psi_n(qD)x^n$;

(ii) $[n]p_n(x) = xD(1+\varepsilon\tilde{\varphi}(D))^{-1}p_n(x)$,
$[n]p_n(qx) = xD(1+\tilde{\varphi}(D))^{-1}p_n(x)$;

(ii)′ $[n]q_n(x) = xD(1+\tilde{\psi}(D))^{-1}q_n(x)$.

Proof. We again use the relations (2.4):

$$p_n(x) = x\varphi_n(D)x^{n-1} = \varphi_n(D)x^n - \varepsilon\varphi_n'(D)x^{n-1} = (1+\varepsilon\tilde{\varphi}(D))\varphi_n(D)x^n$$

$$p_n(x) = x\varphi_n(D)x^{n-1} = x\left[\varphi_n(D)x\frac{1}{\varphi_n(D)}\right]^{n-1}1 = x\left[x+\varepsilon\frac{\varphi_n'(D)}{\varphi_n(D)}\right]^{n-1}1$$
$$= x(x-\varepsilon[n]\tilde{\varphi}(D)/D)^{n-1}1$$

$$[n]p_n(x) = [n]x\varphi_n(D)x^{n-1} = xD\varphi_n(D)x^n = xD(1+\varepsilon\tilde{\varphi}(D))^{-1}p_n(x).$$

The other relations follow in a similar way. □

As a consequence of (i) and (i)′ we obtain a q-analog of the second form (1.3) of the Lagrange formula:

Theorem 3. *Under the assumptions of Theorem* 1 *and* 2, *and if* $q \neq 0$, *the following additional formulas hold for the coefficients* c_n *resp.* d_n:

$$c_n = q^{-n}LD^nf(qx)\varphi_n(x)(1+\tilde{\varphi}(x)),$$
$$d_n = LD^nf(x)\psi_n(qx)(1+\tilde{\psi}(x))$$

where $\tilde{\varphi}$ *and* $\tilde{\psi}$ *are given by* (4.1).

5. Concluding remarks. Since the first version of this paper was presented for publication a lot of new and remarkable results have been obtained: First Ch. Krattenthaler has found a very short and elegant proof of Theorems 1–3 avoiding operators and the polynomials p_n. He further combined Theorem 1 and 2 in a very surprising way to obtain the following more general q-Lagrange inversion theorem (see [16]):

Theorem 4. *Let* φ_n, ψ_n *be as in Theorem* 1 *and* 2 *and* $\tilde{\varphi}$, $\tilde{\psi}$ *be given by* (4.1). *Then the coefficients in the expansion*

$$f(z) = f(0) + \sum_{n\geq 1}\frac{a_n}{[n]!}\frac{z^n}{\varphi_n(z)\psi_n(z)}$$

are given by

$$a_n = LD^{n-1}f'(z)\varphi_n(z)\psi_n(qz) = LD^nf(z)\varphi_n\left(\frac{z}{q}\right)\psi_n(qz)\left(1+\tilde{\varphi}\left(\frac{z}{q}\right)+\tilde{\psi}(z)\right).$$

The simplest but also most important special case of Theorem 4 is obtained from Carlitz's example

$$\varphi_n(z) = (1-az)(1-qaz)\dots(1-q^{n-1}az) \quad \text{and}$$
$$\psi_n(z) = \left(1-\frac{z}{q}\right)\dots\left(1-\frac{z}{q^n}\right), \quad \text{where } \tilde{\varphi}(z) = \frac{az}{1-az} \quad \text{and} \quad \tilde{\psi}(z) = \frac{z}{1-z}.$$

The resulting inversion formula is closely related to the q-analogs of the Jacobi-polynomials. It may be used to derive q-analogs of inverse relations of Legendre and Čebyshev type (see [16]) and also to study a new kind of q-Catalan numbers [6].

This important special case has been obtained independently by Gessel and Stanton [11], using basic hypergeometric series. They more generally gave q-analogs of expansions (1.1) with $\varphi(z) = (1 - z)^b$ which do not seem (besides $b = 1, 2, \frac{1}{2}$) to be covered by Theorem 4. But see [13] for a unified approach to both formulas, where Lagrange inversion is treated as an eigenvalue problem.

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