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A SHORT PROOF OF THE LAGRANGE–GOOD FORMULA

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A new proof of Good's generalization to several variables of the Lagrange inversion formula is given, which is mainly based on methods used in Rota's theory of polynomial sequences of binomial type.

1. Introduction

Since Good's paper [5] appeared in 1960, a lot of other works have been published on this theme. Good's proof of his theorem being analytical, the later authors considered Lagrange inversion within the theory of formal power series. The first attempt for a purely combinatorial proof was made by Chottin [1], who treated a special case of the two dimensional formula. Tutte gave an extensive development of this subject in [11] for an arbitrary number of variables.

While studying Rota's theory of polynomial sequences of binomial type (see [8, 9, 10]), Cigler [2], Garsia and Joni [3] realized its intimate connection with Lagrange inversion (in the one dimensional case). Generalizing Rota's theory to higher dimensions they obtained an analogous setting for the Lagrange-Good formula (see [2, 4, 6, 7]).

This paper is mainly based on these works, it follows the same ideas and gives only a simplified version of the more technical part of their proofs. Readers interested in the connection to Rota's theory which will not be treated here are referred to the above cited papers.

Remark. We will use the standard multi-index notation, i.e. for $n=(n_1,\dots,n_s)\in \mathbf{N}^s$ ($\mathbf{N}$ is the set of nonnegative integers) we mean by $n!=n_1!\cdots n_s!$, $(n)_k=(n_1)_{k_1}\cdots(n_s)_{k_s}$ (the lower factorials) and $e_i=(\delta_{ij})_{1\le j\le s}$ shall denote the ith basic vector. For $x=(x_1,\dots,x_s)$ we mean $x^n=x_1^{n_1}\cdots x_s^{n_s}$, etc.

2. The Lagrange-Good formula

Let $\varphi_i(z_1,\dots,z_s)$, $i=1,2,\dots,s$ be formal power series (over a field K of characteristic zero) with $\varphi_i(0)\neq 0$. Consider the formal power series w_i defined by

$$w_i(z)=\frac{z_i}{\varphi_i(z)},\qquad i=1,\dots,s.$$

If one expands any formal power series $f(z_1, \ldots, z_s)$ into a series in the variables $w = (w_1, \ldots, w_s)$,

$$f(z(w)) = \sum \frac{c_n}{n!} w^n,$$

then the coefficients c_n are given by the formula

$$c_n = \left[f\varphi^n \det\left(\delta_{ij} - w_i \frac{\partial \varphi_i}{\partial z_j}\right)\right]^{(n)}(0)$$
$$= \frac{\partial^{n_1+n_2+\cdots+n_s}}{\partial^{n_1}_{z_1} \cdots \partial^{n_s}_{z_s}} \left[f(z)\varphi_1^{n_1}(z) \cdots \varphi_s^{n_s}(z) \det\left(\delta_{ij} - w_i \frac{\partial \varphi_i}{\partial z_j}\right)\right]_{z=0}.$$

The intrinsic idea of the proof is the following:
Every formal power series

$$f(z) = \sum a_n z^n = \sum a_{n_1 \cdots n_s} z_1^{n_1} \cdots z_s^{n_s}$$

defines a linear operator

$$f(D) = \sum a_n D^n = \sum a_{n_1 \cdots n_s} D_1^{n_1} \cdots D_s^{n_s}$$

on the space $P = K[x_1, \ldots, x_s]$ of all polynomials in s variables over the field K, where D_i denotes the partial derivation with respect to x_i.

The mapping $f(z) \mapsto f(D)$ gives an imbedding of the algebra $K[[z_1, \ldots, z_s]]$ of formal power series into the algebra $L(P)$ of K-linear operators on P. This mapping is one-to-one, for $Q = \sum a_n/n! D^n$ implies $Qx^n |_{x=0} = a_n$.

Let Q_i be those operators, which correspond to the formal power series $w_i(z)$. We will in the following construct a sequence of polynomials $(p_n)_{n \in N^s}$ with the properties

(i) $p_n(0) = \delta_{on}$,

(ii) $Q_i p_n = n_i p_{n-e_i} \quad (1 \leq i \leq s, n \in N^s)$.

Observing that (ii) implies $Q^k p_n = (n)_k p_{n-k}$ for $n, k \in N^s$ one obtains the coefficient of a formal power series in $w = (w_1, \ldots, w_s)$ by applying the corresponding operator $\sum c_k/k! Q^k$ to the polynomials p_n:

$$f(D) p_n |_{x=0} = \left(\sum \frac{c_k}{k!} Q^k\right) p_n |_{x=0} = \sum \frac{c_k}{k!} (n)_k p_{n-k}(0) = c_n.$$

On the other hand we have

$$\left[f\varphi^n \det\left(\delta_{ij} - w_i \frac{\partial \varphi_i}{\partial z_j}\right)\right]^{(n)}(0) = f(D)\varphi^n(D) \det\left(\delta_{ij} - w_i(D) \frac{\partial \varphi_i}{\partial z_j}(D)\right) x^n \Big|_{x=0}$$

Computing these two terms we see that the only thing we have to show is that the polynomials defined by

$$p_n = \varphi^n(D)\det\left(\delta_{ij} - w_i(D)\frac{\partial\varphi_i}{\partial z_j}(D)\right)x^n$$

satisfy the above conditions (i) and (ii).

Proof of (ii):

$$Q_i p_n = \frac{D_i}{\varphi_i(D)}\varphi^n(D)\det(\cdot)x^n = \varphi^{n-e_i}(D)\det(\cdot)D_i x^n$$
$$= \varphi^{n-e_i}(D)\det(\cdot)n_i x^{n-e_i} = n_i p_{n-e_i}.$$

Proof of (i): $p_n(0) = \delta_{0n}$ means that the coefficient of D^n in the expansion of $\varphi^n(D)\det(\cdot)$, that is

$$\left[\varphi^n(z)\det\left(\delta_{ij} - w_i\frac{\partial\varphi_i}{\partial z_j}\right)\right]^{(n)}(0) = \delta_{0n} \tag{1}$$

This is clear for $n = 0$.

From the definition of $w_i(z) = z_i/\varphi_i(z)$ follows

$$\left[\varphi^n(z)\det\left(\delta_{ij} - w_i(z)\frac{\partial\varphi_i}{\partial z_j}\right)\right]^{(n)}(0)$$
$$= [\det(\varphi_i^{n_i}\delta_{ij} - z_i\varphi_i^{n_i-1}\varphi_{i,j})]^{(n)}(0) = \sum_\sigma \operatorname{sgn}\sigma P(\sigma) \tag{2}$$

with

$$P(\sigma) = \left\{\prod_{1\le i\le s}(\varphi_i^{n_i}\delta_{i\sigma(i)} - z_i\varphi_i^{n_i-1}\varphi_{i,\sigma(i)})\right\}^{(n)}(0)$$

and $\varphi_{i,j} = \partial\varphi_i/\partial z_j$ for short.

If some $n_i = 0$, then the corresponding factor in $P(\sigma)$ has the form $(\delta_{i\sigma(i)} - z_i\varphi_i^{-1}\varphi_{i,\sigma(i)})$. Not taking the derivative with respect to z_i, for $z = 0$ we get $P(\sigma) = 0$, if $\sigma(i) \neq i$. That means that the sum (2) reduces to permutations of the set $W = \{i : n_i \neq 0\}$. Using a generalized version of the Leibniz product rule

$$\left(\prod_{1\le i\le p} f_i\right)^{(n)} = \sum_{\sum_{1\le i\le p}k_i = n}\frac{n!}{k_1!\cdots k_p!}\prod_{1\le i\le p} f_i^{(k_i)},$$

$P(\sigma)$ takes the form (non indexed $\sum$ and $\prod$ will run over all $i \in W$)

$$P(\sigma) = \sum_{\sum k_i = n}\frac{n!}{\prod k_i!}\prod(\varphi_i^{n_i}\delta_{i\sigma(i)} - z_i\varphi_i^{n_i-1}\varphi_{i,\sigma(i)})^{(k_i)}(0)$$
$$= \sum_{\sum k_i = n}\frac{n!}{\prod k_i!}\prod\left\{(\varphi_i^{n_i})^{(k_i)}(0)\delta_{i\sigma(i)} - z_i(\varphi_i^{n_i-1}\varphi_{i,\sigma(i)})^{(k_i)}\Big|_{z=0}\right.$$
$$\left. - \binom{k_i}{e_i}(\varphi_i^{n_i-1}\varphi_{i,\sigma(i)})^{(k_i-e_i)}(0)\right\}.$$

The term with z_i vanishes for $z = 0$.

$$\binom{k_i}{e_i} = k_{ii} = i\text{th coordinate of the multi index } k_i.$$

Observing

$$\varphi_i^{n_i-1}\varphi_{i,\sigma(i)} = \frac{1}{n_i}(\varphi_i^{n_i})_{,\sigma(i)}$$

and using the abbreviation $\varphi_i^{n_i} = f_i$, we get

$$P(\sigma) = \sum_{\sum k_i = n} \frac{n!}{\prod k_i!} \prod \left\{ f_i^{(k_i)}\delta_{i\sigma(i)} - \frac{k_{ii}}{n_i} f_i^{(k_i - e_i + e_{\sigma(i)})} \right\}.$$

Substituting the indices k_i by $l_i = k_i - e_i + e_{\sigma(i)}$ gives in view of $\sum l_i = n$

$$P(\sigma) = \sum_{\sum k_i = n} \frac{n!}{\prod k_i!} \prod \left(f_i^{(l_i)}\sigma_{i\sigma(i)} - \frac{k_{ii}}{n_i} f_i^{(l_i)} \right)$$

$$= \sum_{\sum l_i = n} \frac{n!}{\prod l_i!} \prod f_i^{(l_i)} \left(\sigma_{i\sigma(i)} - \frac{l_{i\sigma(i)}}{n_i} \right),$$

for $(1/k_i!)k_{ii} = (1/l_i!)l_{i\sigma(i)}$.
Notice that the delicate cases $l_{ii} = -1$ and $l_{i\sigma(i)} = 0$ do not disturb the calculation.

In this last representation of $P(\sigma)$ the only term still depending on the special permutation σ is $\prod(\delta_{i\sigma(i)} - (l_{i\sigma(i)}/n_i))$. Summing (2) over all permutations of the set W, we get just the determinant of the matrix $(\delta_{ij} - (l_{ij}/n_i))_{i,j\in W}$, which is zero, for its rows are linearly dependent:

$$\sum_{i\in W} n_i\left(\delta_{ij} - \frac{l_{ij}}{n_i}\right) = n_j - \sum l_{ij} = \left(n - \sum l_i\right)_j = 0.$$

Thus (1) is established for $n > 0$ and the proof is completed.

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