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Minmax via Replicator Dynamics

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Abstract I present a short proof of the minmax theorem using the replicator dynamics.

Keywords Minmax theorem · Zero-sum game · Replicator dynamics

1 Introduction

The minmax theorem of von Neumann [10] says that

$$\max_{x \in X} \min_{y \in Y} U(x, y) = \min_{y \in Y} \max_{x \in X} U(x, y)$$

where X, Y are the unit simplices in $\mathbb{R}^n$, $\mathbb{R}^m$ and $U : X \times Y \to \mathbb{R}$ is a continuous function, quasi-concave in x and quasi-convex in y. The proof was by induction on the number of variables, see also [7]. An important special case is where U is a bilinear function $U(x, y) = x \cdot Ay$, with A an $n \times m$ matrix.

The idea to use dynamics for proving the minmax theorem (and computing the equilibria) goes back to Brown [1,2]: for symmetric zero-sum games, i.e., $A = -A^{\mathsf{T}}$, he proved together with von Neumann [2] that the solutions of a certain differential equation converge to the set of equilibria. In [1], he showed that the (continuous time) fictitious play process approaches the set of equilibria in any finite zero-sum game, which implies the minmax theorem for any $n \times m$ matrix A. Brown's fictitious play process [1] is now often framed as best response dynamics and can be used to prove the minmax theorem for more general payoff functions U, which are continuous and concave/convex, see [6]. For the original version [10] for continuous quasi-concave/quasi-convex functions, a dynamic proof is still missing. Another proof based on differential inclusions can be found in [8].

In the present note, I give a short proof of the minmax theorem in the matrix case, based on the replicator dynamics.

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2 Replicator Dynamics

The *replicator dynamics* [5] for an $n \times m$ bimatrix game (A, B) is given by

$$\dot{x}_i = x_i \left(e^i \cdot Ay - x \cdot Ay \right) \qquad i = 1, \ldots, n$$
$$\dot{y}_j = y_j \left(e^j \cdot Bx - y \cdot Bx \right) \qquad j = 1, \ldots, m$$

Here x_i denotes the frequency of strategy i of player 1, hence $x = (x_1, \ldots, x_n)$ is in the probability simplex $\Delta_n = \{x \in [0,1]^n : \sum x_i = 1\}$, y_j is the frequency of strategy j of player 2, $y = (y_1, \ldots, y_m) \in \Delta_m$, and e^i denotes the ith unit vector.

Besides its original derivation from evolution and natural selection, there are at least two economic motivations based on imitation and on reinforcement learning.

For a zero-sum game $B = -A^\mathsf{T}$, in the interior of $\Delta_n \times \Delta_m$ we obtain

$$\dot{x}_i / x_i = e^i \cdot Ay - x \cdot Ay \quad i = 1, \ldots, n \tag{1}$$
$$\dot{y}_j / y_j = -x \cdot Ae^j + x \cdot Ay \quad j = 1, \ldots, m \tag{2}$$

Now add these equations

$$\frac{\dot{x}_i}{x_i} + \frac{\dot{y}_j}{y_j} = e^i \cdot Ay - x \cdot Ae^j \quad \forall i, j$$

and integrate

$$\frac{\log x_i(T) - \log x_i(0) + \log y_j(T) - \log y_j(0)}{T} = e^i \cdot A\bar{y}(T) - \bar{x}(T) \cdot Ae^j$$

where

$$\bar{x}(T) = \frac{1}{T} \int_0^T x(t) \mathrm{d}t, \quad \bar{y}(T) = \frac{1}{T} \int_0^T y(t) \mathrm{d}t$$

denote time averages of the solutions of (1, 2). Now consider limit points, i.e., choose a sequence $T_k \to \infty$ s.t. $\bar{x}(T_k) \to \bar{x}$, $\bar{y}(T_k) \to \bar{y}$. Since $\log x_i(T) \le 0$, we obtain $0 \ge e^i \cdot A\bar{y} - \bar{x} \cdot Ae^j \quad \forall i, j$ or

$$e^i \cdot A\bar{y} \le \bar{x} \cdot Ae^j \quad \forall i, j. \tag{3}$$

Multiplying by x_i and y_j and summing over i and j, we obtain

$$\max_{x \in \Delta_n} x \cdot A\bar{y} \le \min_{y \in \Delta_m} \bar{x} \cdot Ay \tag{4}$$

and

$$\min_y \max_x x \cdot Ay \le \max_x x \cdot A\bar{y} \le \min_y \bar{x} \cdot Ay \le \max_x \min_y x \cdot Ay,$$

and together with the obvious inequality, we obtain

$$\min_{y \in \Delta_m} \max_{x \in \Delta_n} x \cdot Ay = \max_{x \in \Delta_n} \min_{y \in \Delta_m} x \cdot Ay. \tag{5}$$

Additionally, (3) or (4) also imply

$$x \cdot A\bar{y} \le \bar{x} \cdot A\bar{y} \le \bar{x} \cdot Ay \quad \forall x, y \tag{6}$$

so $(\bar{x}, \bar{y})$ is a pair of optimal strategies for the zero–sum game. (In particular, this shows the existence of equilibria.)

Furthermore, if we integrate (1, 2) directly, then we obtain $0 \geq e^i \cdot A\bar{y} - \bar{a}$ and $0 \geq -\bar{x} \cdot Ae^j + \bar{a}$, with $\bar{a} = \lim_{T\to\infty} \frac{1}{T}\int_0^T x(t)Ay(t)\mathrm{d}t$, and hence

$$e^i \cdot A\bar{y} \leq \bar{a} \leq \bar{x} \cdot Ae^j \quad \forall i, j. \tag{7}$$

Comparing with (6), we get $\bar{a} = \bar{x} \cdot A\bar{y}$.

Summarizing, besides minmax theorem (5), we have shown:

Theorem *Every limit point $(\bar{x}, \bar{y})$ of the time averages $(\bar{x}(T), \bar{y}(T))$ of positive solutions $(x(t), y(t))$ of the replicator dynamics is a pair of optimal strategies of the zero-sum game. And the time averages of the payoffs*

$$\frac{1}{T}\int_0^T \sum_{i,j} a_{ij} x_i(t) y_j(t) dt$$

converge to the value $\bar{x} \cdot A\bar{y}$ of the game, as $T \to \infty$.

3 Remarks

1. If $\log x_i(T)$ and $\log y_j(T)$ are bounded functions of T (i.e., the solution stays at a positive distance from the boundary of Δ_n and Δ_m), then we have equality in (3) for all i, j, and the existence of a fully mixed equilibrium follows. The converse holds as well, see [3,5,9]: If $(p, q) > 0$ is an equilibrium of the zero-sum game, then the relative entropy

$$H(x, y) = -\sum_i p_i \log \frac{x_i}{p_i} - \sum_j q_j \log \frac{y_j}{q_j} \geq 0$$

   or Kullback–Leibler divergence is a constant of motion for (1, 2): $\dot{H} = 0$. The replicator dynamics is even a Hamiltonian system w.r.t. a suitable symplectic or Poisson structure [3], and hence, on each level set of H, by Poincaré's recurrence theorem, almost every solution is recurrent. The behavior of the solutions might be chaotic, but by the above theorem, their time averages approach the set of equilibria.
2. For nonzero-sum games, a similar argument shows that the time averages

$$\frac{1}{T}\int_0^T x_i(t) y_j(t)\mathrm{d}t$$

   (i.e., how often does player 1 use strategy i against strategy j of player 2 in a given period) converge (as $T \to \infty$) to the set of *exact coarse correlated equilibria*, see [4]. This holds also for N player normal form games.

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