Games and Economic Behavior 97 (2016) 19–31

Contents lists available at ScienceDirect

Games and Economic Behavior

www.elsevier.com/locate/geb

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Quantal response methods for equilibrium selection in 2×2 coordination games

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ARTICLE INFO

Article history:
Received 2 May 2013
Available online 21 March 2016

JEL classification:
C61
C73
D58

Keywords:
Quantal response equilibrium
Equilibrium selection
Logit equilibrium
Logarithmic game
Punishment

ABSTRACT

The notion of quantal response equilibrium (QRE), introduced by McKelvey and Palfrey (1995), has been widely used to explain experimental data. In this paper, we use quantal response equilibrium as a homotopy method for equilibrium selection, and study this in detail for 2×2 bimatrix coordination games. We show that the risk dominant equilibrium need not be selected. In the logarithmic game, the limiting QRE is the Nash equilibrium with the larger sum of square root payoffs. Finally, we apply the quantal response methods to the mini public goods game with punishment. A cooperative equilibrium can be selected if punishment is strong enough.

1. Introduction

Quantal response equilibrium (QRE) was introduced by McKelvey and Palfrey (1995) in the context of bounded rationality. In a QRE, players do not always choose best responses. Instead, they make decisions based on a probabilistic choice model and assume other players do so as well. A general interpretation of this model is that players observe randomly perturbed payoffs of strategies and choose optimally according to those noisy observations (McKelvey and Palfrey, 1995, 1998; Goeree et al., 2005; Turocy, 2005; Sandholm, 2010). For a given error structure, QRE is defined as a fixed point of this process.[1]

The most common specification of QRE is the logit equilibrium, where the random perturbations on the payoffs follow the extreme value distribution (Luce, 1959; McFadden, 1976; Blume, 1993, 1995; McKelvey and Palfrey, 1995, 1998; Anderson et al., 2004; Turocy, 2005; Hofbauer and Sandholm, 2002, 2007; Sandholm, 2010). The logit response function has one free parameter λ, whose inverse $\frac{1}{\lambda}$ has been interpreted as the temperature, or the intensity of noise. At $\lambda = 0$, players have no information about the game and each strategy is chosen with equal probability. As λ approaches infinity, players achieve full information about the game and choose best responses. McKelvey and Palfrey (1995) then defined an

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[1] The model is equivalent to an incomplete information game where the actual payoff is the sum of payoffs of some fixed game and independent random terms, and each player's private signal is his own payoffs. A QRE is a probability distribution of action profiles in a Bayesian Nash equilibrium (Ui, 2006). Ui (2002) also provided an evolutionary interpretation for QRE. In a population game, if a stochastic best response process satisfies the detailed balance condition, then the support of the stationary distribution converges to the set of quantal response equilibria as the population size goes to infinity.

http://dx.doi.org/10.1016/j.geb.2016.03.002

equilibrium selection from the set of Nash equilibria by "tracing" the branch of the logit equilibrium correspondence starting at the centroid of the strategy simplex (the only QRE when $\lambda = 0$) and following this branch to its terminus at $\lambda = +\infty$.[2] For almost all normal form games, this branch converges to a unique Nash equilibrium as λ goes to infinity. Later, McKelvey and Palfrey (1998) extended the original notion of QRE to extensive-form games (AQRE), and they found that the logit-AQRE also provides a unique selection from the set of sequential equilibria in generic extensive form games.

QRE allows every strategy to be played with non-zero probability, and therefore can be applied to explain data from laboratory experiments which Nash equilibrium analysis cannot. Recent studies include auctions (Anderson et al., 1998; Goeree et al., 2002), bargaining (Goeree and Holt, 2000; Yi, 2005), social dilemmas (Capra et al., 1999; Goeree and Holt, 2001), coordination games (Anderson et al., 2001) and games on networks (Choi et al., 2012). In these experiments, estimates of λ usually increased as the game progresses. This then provides empirical evidence of the equilibrium selection above. As players gain experience from repeated observations, they can be expected to make more precise estimates of the expected payoffs of different strategies.[3]

Although quantal response methods have been widely used in experimental studies, few papers investigated the QRE analytically. Goeree et al. (2005) (see also Haile et al., 2008) proposed a reduced form definition of QRE by restricting quantal response functions to satisfy several economically sensible axioms[4] and studied the properties of these equilibria. Recently, Tumennasan (2013) investigated the set of QRE as λ goes to infinity, and used QRE in this limit set to solve an implementation problem. In particular, to our knowledge, the only mathematical work on equilibrium selection using QRE is due to Turocy (2005). However, his claim that the QRE method selects the risk dominant equilibrium, requires stronger assumptions than he states, as we will show.

In this paper, we analyze equilibrium selection by quantal response methods in 2×2 bimatrix games. Section 2 defines logit equilibrium and reviews some basic results. In section 3 we study quantal response equilibrium selection for 2×2 bimatrix coordination games, present a characterization and a simple first order approximation if the mixed equilibrium is close to the centroid, and contrast it with risk-dominance. Section 4 provides higher order approximations for different types of quantal response functions. Section 5 discusses two equilibrium selection approaches that are closely related to the quantal response method, namely the logarithmic game (Harsanyi, 1973) and the centroid dominant equilibrium of the replicator equation (Zhang and Hofbauer, 2015). For the logarithmic game, we obtain a simple square root rule for equilibrium selection. Section 6 applies these results to the mini public goods game (PGG) with punishment. Section 7 suggests further developments.

2. The logit equilibrium

Let us start with a 2×2 bimatrix game, where A_1 and A_2 are the two pure strategies of player A, and B_1 and B_2 are the two pure strategies of player B. Let a_{ij} denote the payoff to player A using strategy A_i when it meets player B using strategy B_j, and denote the payoff to player B in this interaction by b_{ij}. The payoff matrix is then written as

$$\begin{array}{c c} & \begin{array}{cc} B_1 & B_2 \end{array} \\ \begin{array}{c} A_1 \\ A_2 \end{array} & \begin{pmatrix} a_{11}, b_{11} & a_{12}, b_{12} \\ a_{21}, b_{21} & a_{22}, b_{22} \end{pmatrix} \end{array}. \tag{1}$$

The bimatrix game is called a coordination game if pure strategy pairs (A_1, B_1) and (A_2, B_2) are strict Nash equilibria. That is, $a_{11} - a_{21} > 0$ and $a_{22} - a_{12} > 0$ for player A, and $b_{11} - b_{12} > 0$ and $b_{22} - b_{21} > 0$ for player B. In this case, the payoff matrix can be normalized as[5]

$$\begin{array}{c c} & \begin{array}{cc} B_1 & B_2 \end{array} \\ \begin{array}{c} A_1 \\ A_2 \end{array} & \begin{pmatrix} 1-q, c(1-p) & 0, 0 \\ 0, 0 & q, cp \end{pmatrix} \end{array}, \tag{2}$$

where

$$p = \frac{b_{22} - b_{21}}{b_{11} + b_{22} - b_{21} - b_{12}}, \quad q = \frac{a_{22} - a_{12}}{a_{11} + a_{22} - a_{21} - a_{12}}, \quad c = \frac{b_{11} + b_{22} - b_{21} - b_{12}}{a_{11} + a_{22} - a_{21} - a_{12}}, \tag{3}$$

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[2] As pointed out by Turocy (2005), the branch of the logit equilibria correspondence starting at the centroid may have turning points leading to intervals on which λ is decreasing while following the branch in the direction from the centroid at $\lambda = 0$ to the limiting Nash equilibrium.

[3] Some strong assumptions are needed in order to explain the experimental data by the quantal response equilibrium selection method. First, the initial λ is small enough such that there is a unique logit equilibrium. Second, as λ increases, the players always follow the same branch of the logit correspondence. Third, this branch does not bend backward, so that the branch can be followed without λ having to decrease.

[4] These restrictions are according to an earlier draft of Haile et al. (2008). Haile et al. (2008) pointed out that without further restrictions, QRE can be constructed to match any choice probabilities in any normal form game. Therefore, sensible empirical assumptions on the distributions of payoff perturbations are necessary.

[5] As pointed out by Goeree et al. (2005), (structural) quantal response functions involve only payoff differences.

and these parameters satisfy $0 < p, q < 1$, and $c > 0$. Besides the two strict pure Nash equilibria, the coordination game also has a mixed equilibrium (p, q), where at this equilibrium player A uses strategy A_1 with probability p and player B uses strategy B_1 with probability q.

In the framework of best responses, players choose the strategy with the highest payoff. Denote the probability of player A using strategy A_1 by x and the probability of player B using strategy B_1 by y. One can easily show that A_1 is the best response strategy of player A if and only if $y > q$ and B_1 is the best response strategy of player B if and only if $x > p$. As pointed out by Harsanyi and Selten (1988), best responders are more strongly attracted by the so-called *risk dominant equilibrium* when they are uncertain about the strategies of other players. A Nash equilibrium is called risk dominant if it has the largest Nash product, where the term Nash product refers to the product of the deviation losses of both players at a particular equilibrium. In the 2×2 bimatrix game (2), (A_1, B_1) is said to risk dominate (A_2, B_2) if the Nash products satisfy the inequality

$$c(1-q)(1-p) > cpq, \tag{4}$$

or equivalently

$$p + q < 1. \tag{5}$$

In a QRE, players are assumed to be boundedly rational and observe noisy realizations of the payoffs (McKelvey and Palfrey, 1995). In a 2×2 bimatrix game, player i will choose the first strategy if and only if

$$u_{i1} + \varepsilon_{i1} > u_{i2} + \varepsilon_{i2}, \tag{6}$$

where u_{ij} denotes the payoff of player i using strategy j and ε_{ij} is the payoff disturbance. The best response function becomes probabilistic rather than deterministic. Suppose that player i's noise vector, $\varepsilon_i = (\varepsilon_{i1}, \varepsilon_{i2})$, is distributed according to a joint distribution with density function $p_i(\varepsilon_i)$. Then player i adopts the first strategy with probability

$$\sigma_{i1}(u_{i1}, u_{i2}) = \int_{-\infty}^{\infty} \int_{-\infty}^{u_{i1}-u_{i2}+\varepsilon_{i1}} p_i(\varepsilon_i) d\varepsilon_{i2} d\varepsilon_{i1}, \tag{7}$$

where $\sigma_{ij}(u_{i1}, u_{i2})$ is called the (structural) quantal response function.[6]

The most common specification of QRE is the logit equilibrium. For any given $\lambda \geq 0$, the logit quantal response function is given by

$$\sigma_{i1}(u_{i1}, u_{i2}) = \frac{e^{\lambda u_{i1}}}{e^{\lambda u_{i1}} + e^{\lambda u_{i2}}} = \frac{1}{1 + e^{\lambda(u_{i2} - u_{i1})}}. \tag{8}$$

This arises from Eq. (7) if all noises follow the extreme value distribution with cumulative distribution function $\exp(-\exp(-\lambda\varepsilon - \gamma))$, where γ is Euler's constant. Therefore, if each player uses a logit quantal response function, the corresponding logit equilibria of game (2) are the solutions (x, y) of

$$\begin{aligned} x &= \frac{1}{1 + e^{\lambda(q-y)}}, \\ y &= \frac{1}{1 + e^{\lambda c(p-x)}}. \end{aligned} \tag{9}$$

Consider the logit equilibria (x, y) as a function of λ. $\lambda = 0$ means full noise and $\lambda = +\infty$ means no noise. It is obvious that for $\lambda = 0$, Eq. (9) has a unique solution $(\frac{1}{2}, \frac{1}{2})$. On the other hand, when $\lambda \to +\infty$, the set of logit equilibria approaches the set of all three Nash equilibria of the game.

As shown by McKelvey and Palfrey (1995), for almost all normal form games, the graph of the logit equilibrium correspondence contains a unique branch which starts for $\lambda = 0$ at the centroid of the strategy simplex and converges to a unique Nash equilibrium as λ goes to infinity. This then defines a selection from the set of Nash equilibria by "tracing" the graph of the logit equilibrium correspondence. The selected Nash equilibrium is called the *limiting logit equilibrium* of the game.[7]

[6] Eq. (7) was first introduced by McKelvey and Palfrey (1995) and they named it quantal response function. Later, Goeree et al. (2005) renamed it structural quantal response function since it is generated from an error structure.

[7] Besides the tracing procedure of Harsanyi and Selten (1988) this is one of the few methods that selects a unique equilibrium for almost all normal form games.

3. Equilibrium selection via QRE

In this section, we study equilibrium selection via QRE, where the quantal response function is defined in Eq. (7). Notice that $\sigma_{ij}(u_{i1}, u_{i2})$ involves only the payoff difference $u_i = u_{i1} - u_{i2}$. Hence we write $\sigma_i(u_i)$ for the probability of player i choosing the first strategy with payoff difference u_i. We assume that both players have the same quantal response function $\sigma = \sigma_1 = \sigma_2$. Following Goeree et al. (2005), we say that $\sigma : \mathbb{R} \to [0, 1]$ is a *regular quantal response function*, if it satisfies the following conditions

(a) σ is continuously differentiable with $\sigma'(u) > 0$ for all u,
(b) symmetry: $\sigma(u) + \sigma(-u) = 1$,
(c) $\lim_{u\to-\infty} \sigma(u) = 0$.

Properties (a)–(c) imply that σ is a cumulative distribution function of a symmetric distribution with positive density, where $\sigma(0) = \frac{1}{2}$, $\sigma'(u) = \sigma'(-u)$. For convenience, we continuously extend the domain of σ to $[-\infty, +\infty]$, where $\sigma(-\infty) = 0$ and $\sigma(+\infty) = 1$. The explicit formula of σ of course depends on the density function and we will investigate different types of noise in section 4.

Following the logit equilibrium Eq. (9), let us introduce a parameter $\lambda \geq 0$ representing the (inverse) level of noise and write the quantal response function at noise level λ by $\sigma(\lambda u)$. $\lambda = 0$ means full noise and $\lambda = +\infty$ means no noise. As pointed out by Goeree et al. (2005), there exists a QRE for any $\lambda \geq 0$.

For the 2×2 bimatrix game (2), QRE at noise level λ are the solutions of

$$x = \sigma(\lambda(y - q)),$$
$$y = \sigma(\lambda c(x - p)). \tag{10}$$

Let us now regard the solution of Eq. (10) as a 3-dimensional vector (x, y, λ), where (x, y) is the QRE at noise level λ. When $\lambda = 0$, Eq. (10) has a unique solution $(\frac{1}{2}, \frac{1}{2}, 0)$. When $\lambda = +\infty$, Eq. (10) has three solutions, $(0, 0, +\infty)$, $(1, 1, +\infty)$ and $(p, q, +\infty)$, which correspond to the three Nash equilibria of the coordination game. Similarly as the logit equilibrium, Eq. (10) induces a continuous path starting from the center point and leading to one of the Nash equilibria for almost all 2×2 bimatrix coordination games.[8] This then defines a unique equilibrium selection. For given σ, we call the selected Nash equilibrium the *limiting QRE* of the game (2). The following theorem provides sufficient conditions that hold for all quantal response functions.

Theorem 1. *For all regular quantal response functions σ, the limiting QRE of game (2) is $(1, 1)$ if $p + q \leq 1$ and $cp + q < \frac{1}{2} + \frac{c}{2}$, and is $(0, 0)$ if $p + q \geq 1$ and $cp + q > \frac{1}{2} + \frac{c}{2}$.*

Proof. Outline: We divide the solution space of Eq. (10), i.e., $\{(x, y, \lambda) | x \in [0, 1], y \in [0, 1], \lambda \in [0, +\infty]\}$, into two parts $A_1 = \{(x, y, \lambda) | x + y < 1, \lambda \geq 0\}$ and $A_2 = \{(x, y, \lambda) | x + y > 1, \lambda \geq 0\}$ by the plane $x + y = 1$, and investigate the intersections of the QRE correspondence and the plane $x + y = 1$. It is evident that $(\frac{1}{2}, \frac{1}{2}, 0)$ is an intersection. If it is the only one, then for $\lambda > 0$, the QRE correspondence starting at the centroid will enter either region A_1 or region A_2 and cannot escape from it.[9] This implies that the limiting point of the QRE correspondence starting at the centroid must be a Nash equilibrium in that region (Zhang, 2013).[10]

We first study the case $c = 1$ and then the case $c \neq 1$.

(a) For $c = 1$, the intersections of the QRE correspondence and the plane $x + y = 1$ satisfy

$$\sigma(\lambda(y - q)) + \sigma(\lambda(x - p)) = 1. \tag{11}$$

Since σ is symmetric, Eq. (11) implies

$$\lambda(y - q + x - p) = 0. \tag{12}$$

Therefore, $(\frac{1}{2}, \frac{1}{2}, 0)$ is the only intersection if $p + q \neq 1$.

We next specify the region that the QRE correspondence enters. For $\lambda \to 0^+$, Eq. (10) can be approximated as

$$x = \frac{1}{2} + \lambda(y - q)\sigma'(0) + O(\lambda^2),$$
$$y = \frac{1}{2} + \lambda(x - p)\sigma'(0) + O(\lambda^2), \tag{13}$$

[8] This follows from the proof of Theorem 2 below. For general games see Zhang (2013).
[9] Theorem 2 in Zhang (2013) shows that the QRE is unique for small λ if σ is Lipschitz continuous.
[10] Theorem 6 in Zhang (2013) shows that the QRE correspondence includes a component that connects the centroid and a Nash equilibrium if σ is a regular quantal response function.

and therefore

$$\begin{aligned} x+y &= 1+\lambda(x+y-p-q)\sigma'(0)+O(\lambda^2) \\ &= 1+\lambda\left(\frac{1}{2}+O(\lambda)+\frac{1}{2}+O(\lambda)-p-q\right)\sigma'(0)+O(\lambda^2) \\ &= 1+\lambda(1-p-q)\sigma'(0)+O(\lambda^2). \end{aligned} \tag{14}$$

If $p+q<1$, then we have $x+y>1$. This implies that the QRE correspondence enters region $A_2=\{(x,y,\lambda)|x+y>1,\lambda\geq 0\}$ and cannot escape. Hence, the limiting QRE must be $(1,1)$ because it is the only Nash equilibrium in region A_2. On the other hand, if $p+q>1$, then the QRE correspondence enters region $A_1=\{(x,y,\lambda)|x+y<1,\lambda\geq 0\}$ and the limiting QRE is $(0,0)$.

(b) For $c\neq 1$, suppose that $c<1$. The intersections of the QRE correspondence and the plane $x+y=1$ satisfy

$$\sigma(\lambda(y-q))+\sigma(\lambda c(x-p))=1. \tag{15}$$

This implies

$$\lambda(y-q+c(x-p))=0. \tag{16}$$

Systems of Eq. (10) and Eq. (16) have two possible solutions, $(\frac{1}{2},\frac{1}{2},0)$ and $(\frac{1-cp-q}{1-c},\frac{cp+q-c}{1-c},\lambda_{pq})$, where λ_{pq} is the solution of

$$\frac{1-cp-q}{1-c}=\sigma\left(\lambda_{pq}c\frac{p+q-1}{1-c}\right). \tag{17}$$

If $(\frac{1-cp-q}{1-c},\frac{cp+q-c}{1-c},\lambda_{pq})$ is not in the region $\{(x,y,\lambda)|x\in[0,1],y\in[0,1],\lambda\in[0,+\infty]\}$, then $(\frac{1}{2},\frac{1}{2},0)$ is the only intersection of the QRE correspondence and the plane. This happens for $\frac{1-cp-q}{1-c}<0$ (i.e., $x<0$ or $y>1$), $\frac{cp+q-c}{1-c}<0$ (i.e., $x>1$ or $y<0$) or $(\frac{1-cp-q}{1-c}-\frac{1}{2})\frac{p+q-1}{1-c}\leq 0$ (i.e., $\lambda_{pq}<0$. Since $c\neq 1$, $\frac{1-cp-q}{1-c}-\frac{1}{2}$ and $\frac{p+q-1}{1-c}$ cannot equal to 0 at the same time). Since $c<1$, the above three inequalities can be simplified as

$$\begin{aligned} 1-cp-q&<0, \\ cp+q-c&<0, \\ \left(\frac{1}{2}+\frac{c}{2}-q-cp\right)(p+q-1)&\leq 0. \end{aligned} \tag{18}$$

Similarly as (a), for $\lambda\to 0^+$, Eq. (10) can be approximated as

$$\begin{aligned} x&=\frac{1}{2}+\lambda(y-q)\sigma'(0)+O(\lambda^2), \\ y&=\frac{1}{2}+\lambda c(x-p)\sigma'(0)+O(\lambda^2), \end{aligned} \tag{19}$$

and

$$\begin{aligned} x+y &= 1+\lambda(y+cx-q-cp)\sigma'(0)+O(\lambda^2) \\ &= 1+\lambda\left(\frac{1}{2}+O(\lambda)+\frac{c}{2}+O(\lambda)-q-cp\right)\sigma'(0)+O(\lambda^2) \\ &= 1+\lambda\left(\frac{1}{2}+\frac{c}{2}-q-cp\right)\sigma'(0)+O(\lambda^2). \end{aligned} \tag{20}$$

From Eq. (20), $x+y>1$ if $\frac{1}{2}+\frac{c}{2}>cp+q$ and $x+y<1$ if $\frac{1}{2}+\frac{c}{2}<cp+q$. Thus, $(1,1)$ is the limiting QRE if $cp+q<\frac{1}{2}+\frac{c}{2}$ and $p+q\leq 1$ (obtained from the third inequality in Eq. (18)) since in this case the QRE correspondence enters region $A_2=\{(x,y,\lambda)|x+y>1\}$ and $(1,1)$ is the only Nash equilibrium in this region. Similarly, $(0,0)$ is the limiting QRE if $cp+q>\frac{1}{2}+\frac{c}{2}$ and $p+q\geq 1$. □

For the 2×2 coordination game (2), $(1,1)$ is risk dominant if and only if $p+q<1$ (i.e., $(1,1)$ has the larger Nash product). When $c=1$, Theorem 1 shows that the limiting QRE is exactly the risk dominant equilibrium. However, when $p+q=1$ (i.e., $(1,1)$ and $(0,0)$ have equal Nash products and there is no risk dominant equilibrium), Theorem 1 shows that $(1,1)$ is the limiting QRE if and only if $cp+q<\frac{1}{2}+\frac{c}{2}$. This implies that not always the risk dominant equilibrium is selected in the case $c\neq 1$. This may be surprising since the QRE method is in its spirit very close to the "tracing procedure" of Harsanyi and Selten (1988) and Harsanyi (1975) which always leads to the risk dominant equilibrium. On the other hand, $(1,1)$ is the limiting QRE if $p<\frac{1}{2}$ and $q<\frac{1}{2}$. This implies that if a strategy is "risk dominant for both players", or

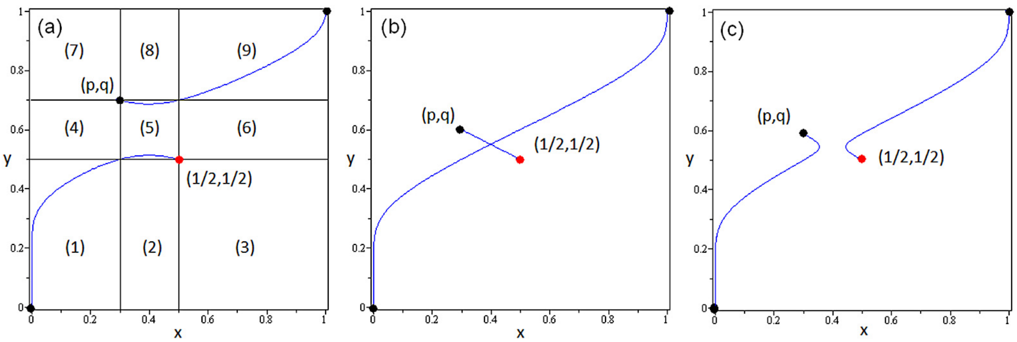

Fig. 1. The logit equilibrium correspondence for game (2). Black points are NE and red points are $(\frac{1}{2}, \frac{1}{2})$. Parameters are taken as $p = 0.3$, $c = 0.25$, $q = 0.7$ in (a), $q = 0.6005$ in (b), and $q = 0.59$ in (c). For almost all 2×2 coordination games, the graph of Eq. (14) consists of two branches, where one passes through the mixed equilibrium (p, q) and the other passes through the centroid $(\frac{1}{2}, \frac{1}{2})$. In the critical case 1(b), $(\frac{1}{2}, \frac{1}{2})$ is connected to all three Nash equilibria. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

$\frac{1}{2}$-dominant (Morris et al., 1995), then it will be selected by the QRE methods independently of c. Turocy (2005) made a mistake in his proof of Theorem 7 by assuming that risk dominance implies $\frac{1}{2}$-dominance.

As an extension of Theorem 1, we look at the two limit cases $c \to 0$ and $c \to +\infty$. If $c = 0$, Eq. (17) has a solution if and only if $q = \frac{1}{2}$. Hence, $(\frac{1}{2}, \frac{1}{2}, 0)$ is the unique intersection of the QRE correspondence and the plane $p + q = 1$ if $q \neq \frac{1}{2}$. From Eq. (20), $(1, 1)$ (resp. $(0, 0)$) is the limiting QRE if and only if $q < \frac{1}{2}$ $(q > \frac{1}{2})$, which is independent of p. This implies that the influence of player B on the equilibrium selection is negligible if his payoffs are much smaller than those of player A. Similarly, if $c \to +\infty$, $(1, 1)$ (resp. $(0, 0)$) is the limiting QRE if and only if $p < \frac{1}{2}$ $(p > \frac{1}{2})$, which is independent of q. This shows clearly that the equilibrium selection depends crucially on c, in contrast to Nash products.

In order to derive a criterion for the limiting QRE, we introduce a new concept. For given c, define the *separatrix* as the curve (or set) in the (p, q) plane separating the two regions where the limiting QRE are $(0, 0)$ and $(1, 1)$, respectively. Theorem 1 says that the separatrix is $p + q = 1$ if $c = 1$, and lies (non-strictly) between lines $p + q = 1$ and $cp + q = \frac{1}{2} + \frac{c}{2}$ if $c \neq 1$. Furthermore, $(1, 1)$ is selected for (p, q) below the separatrix and $(0, 0)$ is selected for (p, q) above the separatrix (see Fig. 2). In the next theorem, we provide a first order approximation of the separatrix, which is again independent of the quantal response function.

Theorem 2. *Let σ be a C^2 regular quantal response function. For p and q close to $\frac{1}{2}$, the separatrix is a curve whose tangent line at $p = q = \frac{1}{2}$ is given by $q = \frac{1}{2} - c^{1/2}(p - \frac{1}{2})$. That is, for p and q sufficiently close to $\frac{1}{2}$, the limiting QRE is $(1, 1)$ if $q < \frac{1}{2} - c^{1/2}(p - \frac{1}{2})$, and is $(0, 0)$ if $q > \frac{1}{2} - c^{1/2}(p - \frac{1}{2})$.*

Proof. Outline: The proof of this theorem consists of three steps. In step (i), we project the QRE correspondence onto the (x, y) plane. In step (ii), we investigate the topological structure of the projection of the QRE correspondence. Generically, it consists of two (disjoint) branches, where one passes through the mixed equilibrium (p, q) and the other passes through the centroid $(\frac{1}{2}, \frac{1}{2})$. The limiting QRE is $(1, 1)$ (or $(0, 0)$) if and only if $(1, 1)$ (or $(0, 0)$) and the centroid $(\frac{1}{2}, \frac{1}{2})$ are on the same branch. In step (iii), we derive an expression for the separatrix.

(i) Define $f = \sigma^{-1} : [0, 1] \to [-\infty, +\infty]$. From the properties of σ, f is monotonically increasing in $[0, 1]$, $f(0) = -\infty$, $f(\frac{1}{2}) = 0$, $f(1) = +\infty$, $f(\frac{1}{2} - z) = -f(\frac{1}{2} + z)$ for $0 < z < \frac{1}{2}$, and $f''(\frac{1}{2}) = 0$. For the logit quantal response function or logit distribution $\sigma(u) = (1 + e^{-u})^{-1}$ we have $f(x) = \log \frac{x}{1-x}$. Then, Eq. (10) can be written as

$$\begin{aligned} f(x) &= \lambda(y - q), \\ f(y) &= \lambda c(x - p), \end{aligned} \tag{21}$$

and the projection of Eq. (10) on the (x, y) plane is

$$(y - q) f(y) = c(x - p) f(x). \tag{22}$$

Note that solutions of Eq. (22) contain all solutions of Eq. (10) (or Eq. (21)), including those with $\lambda < 0$. In the next step, we investigate the topological structure of the graph of Eq. (22) instead of Eq. (10).

(ii) Eq. (22) has four interior solutions, $(\frac{1}{2}, \frac{1}{2})$, $(p, \frac{1}{2})$, $(\frac{1}{2}, q)$ and (p, q), and four boundary solutions $(0, 0)$, $(1, 0)$, $(0, 1)$ and $(1, 1)$. For convenience, we divide the (x, y) plane into nine regions by four lines $x = \frac{1}{2}$, $x = p$, $y = \frac{1}{2}$ and $y = q$, and study the graph of Eq. (22) in each region. Without loss of generality, we assume $p < \frac{1}{2}$ and $q > \frac{1}{2}$. By analyzing the signs of Eq. (21) and Eq. (22), it is easy to verify that there is no solution in regions (2), (4), (6) and (8). Furthermore, $\lambda > 0$ in regions (1) and (9), and $\lambda < 0$ in regions (3) and (7). (See Fig. 1(a).)

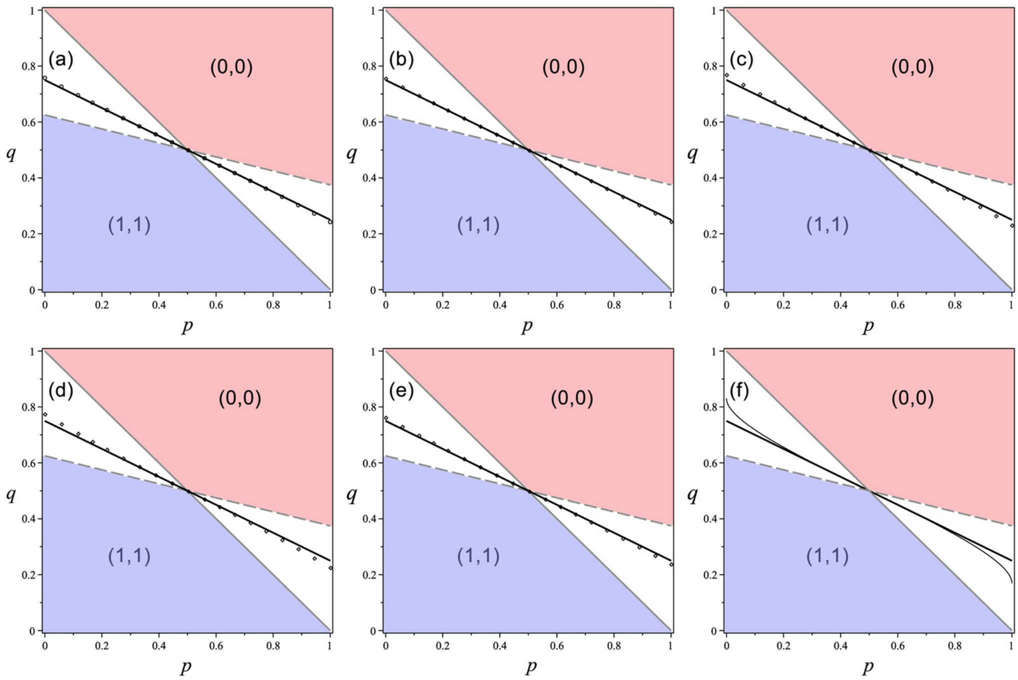

Fig. 2. Separatrix in the (p, q) plane, and its tangent line at $p = q = \frac{1}{2}$, for $c = \frac{1}{4}$. Panels (a)–(f) are respectively the logit equilibrium, the probit equilibrium, the Cauchy noise, the exponential noise, the uniform noise and the logarithmic game. As shown in Theorem 1, the limiting QRE in the blue (= southwest) regions and pink (= northeast) regions are $(1, 1)$ and $(0, 0)$, respectively. Black dots in (a)–(e) denote numerically computed points on the separatrices. In (f) we show the precise form of the separatrix using Eq. (44). For all six types of quantal response functions, the separatrix lies between the blue and the pink region. The Nash equilibrium $(1, 1)$ is selected for (p, q) in the region below the separatrix and $(0, 0)$ is selected for (p, q) in the region above the separatrix. The slope of the tangent line is $-\frac{1}{2}$ (independent of σ). The linear approximation works well for all six types of quantal response functions. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Define

$$S(x, y) = (y - q) f(y) - c(x - p) f(x). \tag{23}$$

The derivatives of $S(x, y)$ satisfy

$$\begin{aligned} S_x &= -c(x - p) f'(x) - c f(x), \\ S_{xx} &= -c(2 f'(x) + (x - p) f''(x)), \\ S_y &= (y - q) f'(y) + f(y), \\ S_{yy} &= 2 f'(y) + (y - q) f''(y), \end{aligned} \tag{24}$$

where $S_x < 0$ if $x > \frac{1}{2}$, $S_x > 0$ if $x < p$, $S_y > 0$ if $y > q$ and $S_y < 0$ if $y < \frac{1}{2}$. Furthermore, for p and q close to $\frac{1}{2}$, $S_{xx} < 0$ if $p < x < \frac{1}{2}$ and $S_{yy} > 0$ if $\frac{1}{2} < y < q$. Applying the implicit function theorem, in region (1), $S(x, y) = 0$ is an increasing curve from $(0, 0)$ to $(p, \frac{1}{2})$ since $S_x > 0$ and $S_y < 0$; in region (3), it is a decreasing curve from $(\frac{1}{2}, \frac{1}{2})$ to $(1, 0)$ since $S_x < 0$ and $S_y < 0$; in region (7), it is a decreasing curve from $(0, 1)$ to (p, q) since $S_x > 0$ and $S_y > 0$; and in region (9), it is an increasing curve from $(\frac{1}{2}, q)$ to $(1, 1)$ since $S_x < 0$ and $S_y > 0$ (see Fig. 1(a)).

On the other hand, in region (5), we have $S_{xx} < 0$ and $S_{yy} > 0$. Therefore, the solution of $S_x(x) = 0$, denoted by x^*, is the (unique local and hence global) maximizer of $x \mapsto S(x, y)$, and the solution of $S_y(y) = 0$, denoted by y^*, is the minimizer of $y \mapsto S(x, y)$. This implies that for given $\hat{y}$, $S(x, \hat{y}) = 0$ has (a) two solutions $(\hat{x_1}, \hat{y})$ and $(\hat{x_2}, \hat{y})$ if $S(x^*, \hat{y}) > 0$, (b) one solution $(x^*, \hat{y})$ if $S(x^*, \hat{y}) = 0$ and (c) no solution if $S(x^*, \hat{y}) < 0$. Similarly, for given $\hat{x}$, $S(\hat{x}, y) = 0$ has (d) two solutions $(\hat{x}, \hat{y_1})$ and $(\hat{x}, \hat{y_2})$ if $S(\hat{x}, y^*) < 0$, (e) one solution $(\hat{x}, y^*)$ if $S(\hat{x}, y^*) = 0$ and (f) no solution if $S(\hat{x}, y^*) > 0$. Thus, the graph of $S(x, y) = 0$ in region (5) consists of two disjoint curves that are separated by the line $x = x^*$ if $S(x^*, y^*) > 0$ (see Fig. 1(c)) and separated by the line $y = y^*$ if $S(x^*, y^*) < 0$ (see Fig. 1(a)).

In sum, the graph of $S(x, y) = 0$ generically consists of two disjoint branches, and the Nash equilibrium (p, q) and the centroid $(\frac{1}{2}, \frac{1}{2})$ are always on different branches. $(\frac{1}{2}, \frac{1}{2})$ is on the curve passing through $(0, 0)$ if $S(x^*, y^*) > 0$ and $x^* > \frac{1}{2}$ or $S(x^*, y^*) < 0$ and $y^* > \frac{1}{2}$ (see Fig. 1(a)), and $(\frac{1}{2}, \frac{1}{2})$ is on the curve passing through $(1, 1)$ if $S(x^*, y^*) > 0$ and $x^* < \frac{1}{2}$ or $S(x^*, y^*) < 0$ and $y^* < \frac{1}{2}$ (see Fig. 1(c)). In the critical case $S(x^*, y^*) = 0$, i.e., 0 is a critical value of S, the two branches intersect at (x^*, y^*) and tracing the QRE correspondence beginning at $(\frac{1}{2}, \frac{1}{2})$ could reach all three Nash equilibria (see Fig. 1(b)).

(iii) From step (ii), for given c, (p, q) is on the separatrix if and only if $S(x^*, y^*) = 0$. That is, the system of three equations

$$\begin{aligned} S(x, y) &= (y-q)f(y) - c(x-p)f(x) = 0, \\ S_x(x, y) &= f(x) + (x-p)f'(x) = 0, \\ S_y(x, y) &= f(y) + (y-q)f'(y) = 0 \end{aligned} \tag{25}$$

has a solution, (x^*, y^*). The second and the third equation in Eq. (25) can be written as

$$p = x + \frac{f(x)}{f'(x)}, \qquad q = y + \frac{f(y)}{f'(y)}. \tag{26}$$

Define

$$F(x) = x + \frac{f(x)}{f'(x)}. \tag{27}$$

Since $f(\frac{1}{2}) = 0$ and $f''(\frac{1}{2}) = 0$, we obtain $F(\frac{1}{2}) = \frac{1}{2}$ and

$$F'\left(\frac{1}{2}\right) = 2 - \frac{f(\frac{1}{2})f''(\frac{1}{2})}{f'(\frac{1}{2})^2} = 2. \tag{28}$$

Therefore, $F(x)$ is locally increasing for x close to $\frac{1}{2}$, and $F^{-1}(p)$ is well defined for p close to $\frac{1}{2}$, where $F^{-1}(\frac{1}{2}) = (F^{-1})'(\frac{1}{2}) = \frac{1}{2}$. Hence, the first equation in Eq. (25) can be represented as

$$(F^{-1}(q) - q)f(F^{-1}(q)) = c(F^{-1}(p) - p)f(F^{-1}(p)). \tag{29}$$

This is an implicit equation for the separatrix.

From $F^{-1}(\frac{1}{2}) = \frac{1}{2}$, $f(F^{-1}(\frac{1}{2})) = 0$ and $(F^{-1})'(\frac{1}{2}) = \frac{1}{2}$, we obtain for p close to $\frac{1}{2}$,

$$\begin{aligned} F^{-1}(p) - p &= (F^{-1}(p) - p)'\left(p - \frac{1}{2}\right) + O\left(p - \frac{1}{2}\right)^2 \sim -\frac{1}{2}\left(p - \frac{1}{2}\right), \\ f(F^{-1}(p)) &= (f(F^{-1}(p))'\left(p - \frac{1}{2}\right) + O\left(p - \frac{1}{2}\right)^2 \sim \frac{1}{2}f'\left(\frac{1}{2}\right)\left(p - \frac{1}{2}\right). \end{aligned} \tag{30}$$

Therefore, Eq. (29) can be approximated as

$$-\frac{1}{4}f'\left(\frac{1}{2}\right)\left(q - \frac{1}{2}\right)^2 = -\frac{c}{4}f'\left(\frac{1}{2}\right)\left(p - \frac{1}{2}\right)^2. \tag{31}$$

Thus, the separatrix is to first order given by

$$q - \frac{1}{2} = -c^{1/2}\left(p - \frac{1}{2}\right), \tag{32}$$

where the minus before $c^{1/2}$ is determined by Theorem 1. □

One may notice that the limit set of QRE as $\lambda \to +\infty$ includes three Nash equilibria, but Theorems 1 and 2 only referred the two pure Nash equilibria. Part (ii) in the proof of Theorem 2 shows that there is no path connecting $(\frac{1}{2}, \frac{1}{2})$ and (p, q) if $S(x^*, y^*) \neq 0$ (see Figs. 1(a) and 1(c)), and $(\frac{1}{2}, \frac{1}{2})$ is connected to all three Nash equilibria if $S(x^*, y^*) = 0$ (see Fig. 1(b)). Thus, the mixed equilibrium (p, q) cannot be selected as the (unique) limiting QRE.

For the case $c = 1$, we obtain from Theorem 1 and (the proof of) Theorem 2 the following corollary.

Corollary 1. *When $c = 1$, the separatrix is the line $p + q = 1$. That is, the limiting QRE is $(1, 1)$ if $p + q < 1$, and is $(0, 0)$ if $p + q > 1$.*

Proof. The second part of this corollary is directly from Theorem 1. Hence the separatrix is contained in the line $p + q = 1$. We only need to show that every point (p, q) on the line segment $p + q = 1$ is on the separatrix.

From the proof of Theorem 2, (p, q, c) is on the separatrix if and only if Eq. (25) has a solution (x^*, y^*). Suppose that the solution has the form $x^* + y^* = 1$. (It follows from eq. (12) that this must hold.) From the properties of $f(x)$, we have $f(x^*) = -f(y^*)$ and $f'(x^*) = f'(y^*)$. Thus, the first equation of Eq. (25) always holds, and the third equation of Eq. (25)

holds if and only if the second equation holds. Notice that $f'(x) > 0$, $f(0) = -\infty$ and $f(1) = +\infty$, the second equation of Eq. (25), $f(x) + (x - p)f'(x) = 0$, must have a solution x^* in $(0, 1)$. This implies that if $p + q = 1$ and $c = 1$, the QRE correspondence has a bifurcation point (x^*, y^*) with $y^* = 1 - x^*$, and tracing the correspondence beginning at the centroid could reach all three Nash equilibria (see Fig. 1(b) as an example). □

4. Different types of quantal response functions

Theorem 2 provides a linear approximation of the separatrix, in which the first order term is independent of the quantal response function but depends on the payoff matrix only. In this section, we derive the higher order terms of Eq. (29) and calculate the separatrix for five types of quantal response functions.

For convenience, define $H(p) = 2(\frac{(p-F^{-1}(p))f(F^{-1}(p))}{f'(1/2)})^{1/2}$. Eq. (29) could be then simplified as

$$H(q)^2 = cH(p)^2. \tag{33}$$

From the proof of Theorem 2, the separatrix is given by

$$H(q) = -c^{-1/2}H(p). \tag{34}$$

Eqs. (29)–(32) imply that the first order term of $H(q)$ is $q - \frac{1}{2}$. This implies that $H(q)$ is locally increasing for q close to $\frac{1}{2}$. Thus, Eq. (34) could be written as

$$q = H^{-1}(-c^{-1/2}H(p)). \tag{35}$$

We next derive the 6th order Taylor series of Eq. (35) for p and q close to $\frac{1}{2}$. Let $\Delta p = p - \frac{1}{2}$ and $\Delta q = q - \frac{1}{2}$. Following Eq. (27), the Taylor expansion of $F^{-1}(p)$ at $\frac{1}{2}$ is

$$F^{-1}(p) = \frac{1}{2} + \frac{\Delta p}{2} + \frac{\Delta p^3}{48}\frac{f^{(3)}}{f^{(1)}} + \frac{\Delta p^5}{1920}\frac{f^{(5)}}{f^{(1)}} + O(\Delta p^7), \tag{36}$$

where $f^{(i)}$ denotes the i-th order derivative of f at $\frac{1}{2}$ (from the symmetry of σ, $f^{(2)} = f^{(4)} = f^{(6)} = 0$). Applying Eq. (36), the Taylor series of $H(p)$ at $\frac{1}{2}$ is

$$\begin{aligned} H(p) &= \Delta p + \frac{\Delta p^3}{48}\frac{f^{(3)}}{f^{(1)}} + \Delta p^5\left(\frac{f^{(5)}}{3840 f^{(1)}} + \frac{(f^{(3)})^2}{1536(f^{(1)})^2}\right) + O(\Delta p^7), \\ &= \Delta p - \frac{\Delta p^3}{48}\frac{\sigma^{(3)}}{(\sigma^{(1)})^3} - \Delta p^5\left(\frac{\sigma^{(5)}}{3840(\sigma^{(1)})^5} - \frac{5(\sigma^{(3)})^2}{1536(\sigma^{(1)})^6}\right) + O(\Delta p^7), \end{aligned} \tag{37}$$

where $\sigma^{(i)}$ denotes the i-th order derivative of σ at 0. Thus, from Eq. (35), the 6th order Taylor series of the separatrix is given by

$$\begin{aligned} \Delta q = &-c^{1/2}\Delta p - \frac{\sigma^{(3)}}{48(\sigma^{(1)})^3}(-c^{1/2} + c^{3/2})\Delta p^3 \\ &- \left(\frac{(c+1)\sigma^{(5)}}{3840(\sigma^{(1)})^5} - \frac{(3c+5)(\sigma^{(3)})^2}{1536(\sigma^{(1)})^6}\right)(-c^{1/2} + c^{3/2})\Delta p^5 + O(\Delta p^7). \end{aligned} \tag{38}$$

Eq. (38) shows that higher order terms include derivatives of σ, and the coefficients of the higher order terms are small.

We then apply Eq. (38) to calculate the separatrix for five different types of payoff disturbances, which are respectively the extreme value distribution (i.e., the logit equilibrium), the normal distribution (i.e., the probit equilibrium) (Palfrey and Prisbrey, 1997; Staudigl, 2012), the Cauchy distribution, the exponential distribution and the uniform distribution (Gale et al., 1995). In general, the separatrix does not have an explicit formula. We provide both the Taylor series and numerical simulations and show that the limiting QRE is affected little by the noise structure but mainly decided by the payoff matrix (see Fig. 2(a)–(e) and Online appendix). This suggests that Eq. (32) is a good approximation for the equilibrium selection by QRE methods.

5. Two related approaches

In this section, we discuss two equilibrium selection approaches, namely the logarithmic game (Harsanyi, 1973) and the centroid dominant equilibrium of the replicator equation (Zhang and Hofbauer, 2015). Both of them are closely related to the quantal response method since their selection criterion takes the same form (34). Furthermore, we obtain explicit expressions for H.

5.1. Logarithmic game

In the logarithmic game introduced by Harsanyi (1973), a player's utility has the form $\frac{\lambda}{\lambda+1}U + \frac{1}{\lambda+1}L$ with $\lambda \geq 0$, where U depends on payoff matrix and L depends on the player's own strategy. For the game (2), these utility functions are given by

$$\hat{u}_1(x) = \frac{\lambda}{\lambda+1}(xy(1-q) + (1-x)(1-y)q) + \frac{1}{\lambda+1}\log(x(1-x)),$$
$$\hat{u}_2(y) = \frac{\lambda}{\lambda+1}(yxc(1-p) + (1-y)(1-x)cp) + \frac{1}{\lambda+1}\log(y(1-y)), \tag{39}$$

where $\hat{u}_i(x)$ is the utility of player i using strategy x.

Suppose that individuals choose strategies maximizing their utilities. The corresponding logarithmic equilibrium is then the solution of

$$\frac{\partial \hat{u}_i(x)}{\partial x} = \frac{\lambda}{\lambda+1}(y-q) + \frac{1}{\lambda+1}\frac{1-2x}{x(1-x)} = 0,$$
$$\frac{\partial \hat{u}_2(y)}{\partial y} = \frac{\lambda}{\lambda+1}c(x-p) + \frac{1}{\lambda+1}\frac{1-2y}{y(1-y)} = 0. \tag{40}$$

These equations are of the form (21) and (10), with

$$f(x) = \frac{2x-1}{x(1-x)}$$

and the 'logarithmic' quantal response function

$$\sigma(u) = \frac{1}{2} + \frac{u}{4(1+\sqrt{1+u^2/4})}. \tag{41}$$

From Eqs. (25)–(29), we get

$$H(p) = \sqrt{\frac{1}{2} - \sqrt{\frac{1}{4} - (p-\frac{1}{2})^2}}, \tag{42}$$

and the separatrix has an explicit formula

$$q = \frac{1}{2} - c^{1/2}\sqrt{\frac{1}{2} - \sqrt{\frac{1}{4} - (p-\frac{1}{2})^2}}\sqrt{1 - c(\frac{1}{2} - \sqrt{\frac{1}{4} - (p-\frac{1}{2})^2})}. \tag{43}$$

Interestingly, Eq. (43) is equivalent to the following simpler expression

$$\sqrt{1-q} + \sqrt{c(1-p)} = \sqrt{q} + \sqrt{cp}, \tag{44}$$

which means that the equilibrium with the larger sum of square root payoffs will be selected (see Fig. 2(f)). Since the equilibrium selection seems to be affected little by the quantal response function (see Fig. 2), Eq. (44) then provides a simple way to estimate the limiting QRE from the payoff matrix. The Nash equilibrium (A_1, B_1) is selected by the QRE methods if it has larger sum of square root payoff differences, i.e.,

$$\sqrt{a_{11} - a_{21}} + \sqrt{b_{11} - b_{12}} > \sqrt{a_{22} - a_{12}} + \sqrt{b_{22} - b_{21}}. \tag{45}$$

5.2. Replicator equation

In this subsection, we discuss the relation between the centroid dominant equilibrium of the replicator equation (Zhang and Hofbauer, 2015) and the limiting QRE. The replicator equation is a well known dynamic approach in evolutionary game theory. It is closely related to evolution and learning (Hofbauer and Sigmund, 1998). Following the notations in previous sections, the replicator dynamics for game (2) is given by

$$\frac{dx}{dt} = x(1-x)(y-q),$$
$$\frac{dy}{dt} = cy(1-y)(x-p). \tag{46}$$

Recently, Zhang and Hofbauer (2015) considered the following equilibrium selection method: a Nash equilibrium is called *centroid dominant* if the trajectory of (46) starting at the centroid of the strategy simplex converges to it[11] as $t \to +\infty$. Since the orbits of (46) are explicitly known, see Hofbauer and Sigmund (1998), one can derive that the equilibrium $A_1B_1 = (1,1)$ (resp. $A_2B_2 = (0,0)$) is the centroid dominant equilibrium if

$$\log\left(2q^q(1-q)^{(1-q)}\right) > (<) c \log\left(2p^p(1-p)^{(1-p)}\right). \tag{47}$$

Since Eq. (47) has the form of Eq. (33) with $H(p) = (\log(2p^p(1-p)^{(1-p)}))^{\frac{1}{2}}$, the centroid dominant equilibrium is indeed a special case of the limiting QRE. Here we have an explicit formula for H, but we do not have an explicit formula for σ. In particular, it is not surprising that the slope of the separatrix Eq. (47) at $(\frac{1}{2}, \frac{1}{2})$ is $\sqrt{c}$ (Corollary 2(b) in Zhang and Hofbauer, 2015). Also the shape of the separatrix is similar to those in Fig. 2, see Fig. 2 in Zhang and Hofbauer (2015).

6. Application

Let us now look at *degenerate coordination games* with normalized payoff matrix (2) given in the limit $p \to 1$ by

$$\begin{array}{c c} & \begin{array}{cc} B_1 & B_2 \end{array} \\ \begin{array}{c} A_1 \\ A_2 \end{array} & \begin{pmatrix} 1-q, 0 & 0, 0 \\ 0, 0 & q, c \end{pmatrix} \end{array}. \tag{48}$$

Such games have a unique strict equilibrium (A_2, B_2) and a component of equilibria. The Nash product criterion always selects the strict equilibrium in this case. However, the limiting QRE can be from the component, if $q < \frac{1}{2}$ and c is sufficiently small. For example, Eq. (44) leads to the condition

$$\sqrt{1-q} > \sqrt{q} + \sqrt{c}. \tag{49}$$

As an application, we study the *mini public goods game* (PGG) with punishment (Sigmund et al., 2001; Sigmund, 2010). The only strict Nash equilibrium in this game is: *do not contribute to the public pool and do not punish free riders*. However, a huge number of empirical research indicated that punishment can curb free-riding (see a recent overview by Guala (2012)). By applying the results in section 3, we find that a cooperative equilibrium is selected by the QRE method if punishment is strong enough. For intermediate punishment, cooperation could also dominate the population when λ is not so large even if the limiting QRE is defection.

6.1. Public goods game with punishment

Following Sigmund et al. (2001), we consider a two players PGG, where both can send a benefit b to their coplayer at a cost of a. In the second stage, they are offered the opportunity to punish their coplayer by imposing a fine. The fine amounts to a loss β to the punished player, and it entails a cost α to the punisher. Let us label with C (cooperator) those players who cooperate by sending a benefit and with D (defector) those who do not. Let P denotes those who punish defectors and N those who do not. The payoff matrix is then written as

$$\begin{array}{c c} & \begin{array}{cc} P & N \end{array} \\ \begin{array}{c} C \\ D \end{array} & \begin{pmatrix} -a, b & -a, b \\ -\beta, -\alpha & 0, 0 \end{pmatrix} \end{array}. \tag{50}$$

This mini PGG with punishment is equivalent to the mini ultimatum game (Gale et al., 1995) or the Prisoner's Dilemma game with punishment (Sigmund et al., 2001). The game has infinitely many Nash equilibria, one strict Nash equilibrium $(0,0)$ and non-isolated Nash equilibria $(1, \hat{y})$, where $\frac{a}{\beta} \le \hat{y} \le 1$. However, all the cooperative equilibria are weakly dominated by the second-order free-riding, $(1,0)$. Therefore, defecting and refusing to punish is the only strict Nash equilibrium (see Fig. 3(a)).

6.2. Quantal response analysis

Game (50) is a limit case of a coordination game. From Eq. (10), the limit set of QRE consists of three equilibria only, one asocial equilibrium $(0,0)$ and two cooperative equilibria $(1, \frac{1}{2})$ and $(1, q)$.[12] Similarly as Theorem 2, we can easily prove that the graph of Eq. (22) consists of two branches for almost all parameters, where the Nash equilibrium $(1, q)$ and the centroid $(\frac{1}{2}, \frac{1}{2})$ are always on different branches (see Fig. 3(a)). By using the linear approximation Eq. (32), the limiting QRE is $(1, \frac{1}{2})$, i.e., cooperate and punish defectors with probability $\frac{1}{2}$, if

[11] One could use any other dynamics besides the replicator dynamics for a similar equilibrium selection method. For the best response dynamics, this selects the risk-dominant equilibrium. For other game dynamics, an expression for the separatrix is not known.

[12] See Tumennasan (2013) for the relation between non-strict Nash equilibria and the limit set of QRE.

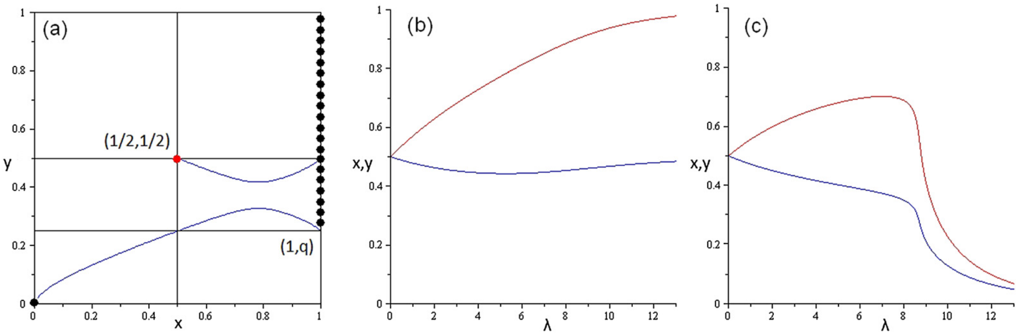

Fig. 3. Logit equilibrium correspondence for the mini public goods game with punishment. $a = \alpha = 1$, $b = 2$, $\beta = 5$ in (a) and (b), and $\beta = 4$ in (c). Black points are NE and red point is $(\frac{1}{2}, \frac{1}{2})$. Red curves and blue curves in (b) and (c) denote the frequencies of cooperation and punishment, respectively. (a) The graph of Eq. (14) consists of two branches, where one passes through the Nash equilibrium $(1, q)$ and the other passes through the centroid $(\frac{1}{2}, \frac{1}{2})$. (b) The limiting QRE is the cooperative equilibrium $(1, \frac{1}{2})$. (c) The limiting QRE is the asocial equilibrium $(0, 0)$ but cooperation can dominate the population for small value of λ. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

$$2\beta > 4a + \alpha + \sqrt{8a\alpha + \alpha^2}. \tag{51}$$

Hence, quantal response methods choose the cooperative equilibrium if the punishment is strong enough. For instance, if the cost of cooperation equals to the cost of punishment, i.e., $a = \alpha$, Eq. (51) can be simplified as $\beta > 4\alpha$. The limiting QRE is the cooperative equilibrium if the fine-to-fee ratio (also called the effectiveness of punishment) is greater than four (see Fig. 3(b)). On the other hand, if the cost of punishment is much larger than the cost of cooperation, i.e., $\alpha >> a$, Eq. (51) reduces to $\beta > \alpha$. In this case, punishment can be selected for lower effectiveness.

In general, cooperation is called "dominant" in the population if more than half of all players contribute in the first stage. Denote the frequencies of C and P by x and y, respectively. From part (b) in the proof of Theorem 2, the maximum value of x on the branch of QRE correspondence starting at the centroid is larger than $\frac{1}{2}$ if and only if $q < \frac{1}{2}$. This implies that if $\beta > 2a$, cooperation could dominate the population for some values of λ even if the QRE correspondence eventually converges to the asocial equilibrium. For instance, if $a = \alpha = 1$ and $c = 4$, the limiting QRE is $(0, 0)$ but the proportion of cooperators can reach 70 percent when $6 < \lambda < 8$ (see Fig. 3(c)).

7. Further developments

This paper considers 2×2 bimatrix coordination games only, and a natural development would be to calculate the limiting QRE for 3×3 symmetric/bimatrix coordination games. Zhang (2013) provided sufficient conditions for the limiting QRE in $n \times n$ games. In 3×3 symmetric/bimatrix coordination games, a strategy is selected by the quantal response method if it is $\frac{1}{2}$-dominant (Morris et al., 1995) and is the unique best response to the centroid of the strategy simplex. But the $\frac{1}{2}$-dominant equilibrium need not be the limiting QRE if it is not the best response to the centroid. Recently, Hommes and Ochea (2012) investigated the logit equilibria correspondence in 3×3 symmetric games. They showed that even in pure coordination games, the correspondence can display a fold bifurcation and the number of logit equilibria may change non-monotonically in λ. Thus, equilibrium selection in 3×3 games becomes an even more delicate issue and we cannot expect a simple formula to decide the limiting QRE.

Acknowledgments

This paper is a part of Boyu Zhang's Ph.D. thesis, written under the guidance of Josef Hofbauer and Karl Sigmund. We are grateful to Mathias Staudigl for helpful discussions, and a referee for his very careful reading. We also thank for the support of the NSFC (project number 11301032) of China and "the Fundamental Research Funds for the Central Universities" of China. WWTF (project number MA09-017) of Austria supported the presentation of the paper at the 4th World Congress of the Game Theory Society.

Appendix A. Supplementary material

Supplementary material related to this article can be found online at http://dx.doi.org/10.1016/j.geb.2016.03.002.

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