Dyn Games Appl (2012) 2:195–205
DOI 10.1007/s13235-012-0040-0

Perturbations of Set-Valued Dynamical Systems, with Applications to Game Theory

Michel Benaïm · Josef Hofbauer · Sylvain Sorin

Published online: 25 February 2012
© Springer Science+Business Media, LLC 2012

Abstract We present upper-semicontinuity results for attractors and the chain-recurrent set of differential inclusions, in particular w.r.t. discretizations, and applications to game dynamics.

Keywords Differential inclusion · Attractor · Chain-recurrence · Discretization · Game dynamics

1 Set-Valued Dynamical Systems: Notations and Basic Properties

Let F be an upper semi-continuous set-valued map from $\mathbb{R}^k$ to itself with compact convex values and X be a given compact convex subset of $\mathbb{R}^k$.

Σ denotes the set of solutions of the differential inclusion

$$\dot{\mathbf{x}} \in F(\mathbf{x}) \tag{1}$$

for which X is forward invariant. More precisely, $\mathbf{x} \in \Sigma$ iff $\mathbf{x}$ is an absolutely continuous map from an interval $I_{\mathbf{x}}$ to $\mathbb{R}^k$, where $[0, +\infty) \subset I_{\mathbf{x}}$, that satisfies (1) a.e., such that $\mathbf{x}(t) \in X$ for some $t \in I_{\mathbf{x}}$ and for any $s, t \in I_{\mathbf{x}}$, $s < t$ and $\mathbf{x}(s) \in X$ implies $\mathbf{x}(t) \in X$.

M. Benaïm
Institut de Mathematiques, Université de Neuchâtel, Rue Emile-Argand 11, Suisse, Neuchâtel, Switzerland
e-mail: michel.benaim@unine.ch

J. Hofbauer (✉)
Fakultät für Mathematik, Universität Wien, Nordbergstrasse 15, 1090 Vienna, Austria
e-mail: josef.hofbauer@univie.ac.at

S. Sorin
Combinatoire et Optimisation, IMJ, CNRS UMR 7586, Faculté de Mathématiques,
Université P. et M. Curie—Paris 6, Tour 15-16, 1 étage, 4 place Jussieu, 75005 Paris, France
e-mail: sorin@math.jussieu.fr

S. Sorin
Laboratoire d'Econométrie, Ecole Polytechnique, Palaiseau, France

Elements of Σ are called trajectories.

$\mathbf{x}$ is a complete trajectory in X if $I_{\mathbf{x}} = \mathbb{R}$ and for all $t \in \mathbb{R}$: $\mathbf{x}(t) \in X$.

We assume that for each $x \in X$ there exists a trajectory $\mathbf{x} \in \Sigma$ with $\mathbf{x}(0) = x$.

Note that Σ is compact in the topology of uniform convergence on compact time intervals.

The *set-valued semiflow* Φ associated to the differential inclusion (1) is defined on $[0, +\infty) \times X$ by:

$$\Phi_t(x) = \{\mathbf{x}(t); \mathbf{x} \in \Sigma,\ \mathbf{x}(0) = x\}$$

$(x, t) \mapsto \Phi_t(x)$ is a closed set-valued map with compact values, see [1, Chap. 2, Sect. 2].

For $T \times M \subset [0, +\infty) \times X$ we define

$$\Phi_T(M) = \bigcup_{t \in T, x \in M} \Phi_t(x).$$

For $\delta > 0$, let F^δ be a set-valued map from $\mathbb{R}^k$ to itself satisfying

$$\mathrm{Graph}(F^\delta) \subset N^\delta(\mathrm{Graph}(F))$$

where $N^\delta(V)$ stands for the closed δ neighborhood of V.

Σ^δ is the set of solutions of $\dot{\mathbf{x}} \in F^\delta(\mathbf{x})$ for which X is forward invariant.

Finally Φ^δ denotes the set-valued flow associated to Σ^δ.

$M \subset X$ is *invariant* if for every $x \in M$ there exists a complete trajectory $\mathbf{x}$ contained in M with $\mathbf{x}(0) = x$. See [6, Sect. 3.2] for further discussion.

For $M \subset X$, the ω-limit set is defined by

$$\omega_\Phi(M) = \bigcap_{t \geq 0} \overline{\Phi_{[t, +\infty)}(M)}.$$

Similarly the limit set of a trajectory $\mathbf{x}$ is

$$L(\mathbf{x}) = \bigcap_{t \geq 0} \overline{\mathbf{x}([t, +\infty))}.$$

$A \subset X$ is an *attractor* if it is compact, invariant and there exists a neighborhood U of A in X (for the induced topology on X) such that for any $\varepsilon > 0$ there is a time $\tau(\varepsilon) \geq 0$ with

$$\Phi_{[\tau(\varepsilon), +\infty)}(U) \subset N^\varepsilon(A).$$

Such a U is called a fundamental neighborhood of A.

An equivalent characterization is that A is *asymptotically stable*, that is, invariant, Lyapunov stable and attractive (for all x in a neighborhood U, $\omega_\Phi(x) \subset A$) or even for any trajectory $\mathbf{x}$ with $\mathbf{x}(0) \in U$, $L(\mathbf{x}) \subset A$, see [6, Corollary 3.18].

Given an attractor A, its *basin of attraction* is given by

$$\begin{aligned} B(A) &= \{x \in X : \omega_\Phi(x) \subset A\} \\ &= \{x \in X : \text{ for any trajectory } \mathbf{x} \text{ with } \mathbf{x}(0) = x, L(\mathbf{x}) \subset A\}. \end{aligned}$$

Note that $B(A)$ is the union of all fundamental neighborhoods of A, and hence open.

The *global attractor* of Φ is $\omega_\Phi(X)$. It is the maximal compact invariant set in X, and is the union of all complete trajectories in X.

2 Continuation of Attractors

The following result is well known for smooth dynamical systems, see e.g. [40, Proposition 8.1] and [32, Lemma 7.7].

An extension to the set-valued framework can be found in [35]. Here we provide a short proof based on properties of attractors proved in [6].

2.1 Upper-Semicontinuity of an Attractor

Theorem 2.1 *Let U be a compact fundamental neighborhood of an attractor A for Φ. Then for $\varepsilon > 0$ small enough, there is $\delta_0 > 0$ such that for all $0 \le \delta \le \delta_0$, there exists a unique attractor A^δ for Φ^δ with $A^\delta \subset N^\varepsilon(A)$ and such that U is a fundamental neighborhood of A^δ.*

Proof Choose ε small enough so that $N^{2\varepsilon}(A) \subset U$ and note that for all t large enough, $\Phi_t(U) \subset \mathrm{int} N^\varepsilon(A)$. Fix such a t. We then use the following approximation property from [1, Chap. 2, Sect. 2]: *For any $\varepsilon > 0$, there exists $\delta > 0$ such that, for any $Y \subset X$, $\Phi_t^\delta(Y) \subset N^\varepsilon(\Phi_t(Y))$.* Then for δ_0 small enough and all $0 \le \delta \le \delta_0$, $\Phi_t^\delta(U) \subset \mathrm{int} N^\varepsilon(A)$ as well. Hence, as in Proposition 3.19 of [6], $A^\delta = \omega_{\Phi^\delta}(U) \subset N^\varepsilon(A)$ is an attractor for Φ^δ with fundamental neighborhood U. □

2.2 Application: Perturbations of the Best Response Dynamics

Consider the best response dynamics [22, 36] for an N person game (here X is a product of simplices)

$$\dot{x}^n \in \mathrm{BR}^n\left(x^{-n}\right) - x^n, \quad n \in N \tag{2}$$

or a symmetric 2 person game with finite strategy set I played within one population (here $X = \Delta(I)$ is a probability simplex)

$$\dot{x} \in \mathrm{BR}(x) - x. \tag{3}$$

Introduce any single valued perturbation of this dynamics,

$$\dot{x} = b_\varepsilon(x) - x, \tag{4}$$

such that $\mathrm{Graph}(b_\varepsilon) \subset N^\varepsilon(\mathrm{Graph}(\mathrm{BR}))$. The most prominent such smooth perturbation is the logit map [8, 16]:

$$b_\varepsilon^L(x) = L\big(U(x)/\varepsilon\big) \tag{5}$$

where $U(x) \in \mathbb{R}^n$ is the payoff vector against profile x and

$$L_i(U) = \frac{e^{U_i}}{\sum_k e^{U_k}}, \quad i \in I. \tag{6}$$

Maybe the first such perturbation is due to Nash [38],

$$b_\varepsilon^N(x)_i = \frac{c_i(x)}{\sum_k c_k(x)} \quad \text{with } c_i(x) = \Big[U_i(x) - \max_k U_k(x) + \varepsilon\Big]_+, \ i \in I \tag{7}$$

where $u_+ = \max(u, 0)$ denotes the positive part. Only strategies which lose at most ε compared to the maximal payoff are used in this approximate best response. Other examples are the logarithmic games of Harsanyi [20], the quantal response [37], and more general smoothings based on deterministic or stochastic payoff perturbations [5, 7, 11, 15, 17, 23, 25].

Let A be an attractor for (3). Theorem 2.1 implies that any such perturbed dynamics has an attractor A^ε contained in a neighborhood of A, for ε small enough. We remark that for 'nice' smoothings such as the logit dynamics or those based on deterministic or stochastic payoff perturbations considered in [5, 17, 23, 25], and 'nice' classes of games, such as 2-person zero-sum games [29], or 'stable games' [26], more is known: For the unperturbed dynamics (3), the global attractor A is given by the convex set of Nash equilibria. The smoothed dynamics (4) has, for each small $\varepsilon > 0$, a unique 'perturbed equilibrium' $\hat{x}_\varepsilon$, which is the global attractor $A_\varepsilon = \{\hat{x}_\varepsilon\}$. In this case Theorem 2.1 simply says that $\hat{x}_\varepsilon$ tends to the set of Nash equilibria as $\varepsilon \to 0$. However, Theorem 2.1 applies to all other perturbations (4) as well, such as Nash's (7), etc., for which similar global convergence results to a singleton are not known or even wrong.

Even for the logit dynamics, there are surprises. Consider the following symmetric 4×4 game, with payoff matrix

$$U = \begin{pmatrix} 0 & 0 & -1 & 2 \\ 2 & 0 & 0 & -1 \\ -1 & 2 & 0 & 0 \\ 0 & -1 & 2 & 0 \end{pmatrix}$$

or the 5×5 anti-coordination game in [34]. In these games the global attractor for the BR dynamics is the barycenter of the simplex, which is the unique equilibrium E of the game. This follows from the general stability result in [22, Theorem 5.1] or [24, Theorem 2.2]. Indeed, the Lyapunov function $V(x) = \max_i (Ux)_i$ satisfies $\dot{V}(x) \le -V(x)$ for $x \neq E$, and E is even reached in finite time along every solution of (3). Because of symmetry, E is also the unique equilibrium of the logit dynamics (4), (5). However, E is unstable for each small $\varepsilon > 0$ in (4), (5), as can be easily seen by linearization, and computing the eigenvalues of the resulting circulant matrix. There seems to be an attracting limit cycle around E. By Theorem 2.1, the global attractor A_ε is close to E for all small ε.

Consider a (single population) paper-rock-scissors game with payoff matrix

$$U = \begin{pmatrix} 0 & -b & a \\ a & 0 & -b \\ -b & a & 0 \end{pmatrix} \quad \text{with } b > a > 0. \tag{8}$$

Here the unique equilibrium E is repelling and the best response dynamics (3) has a closed orbit (a 'Shapley polygon') as attractor, see Fig. 1. This can be shown with the Lya-

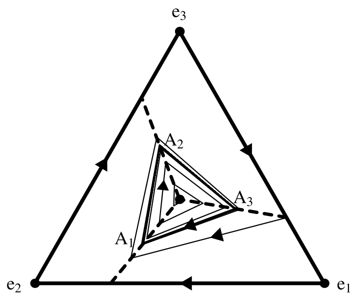

Fig. 1 Best response dynamics for a Paper–Rock–Scissors game with $a < b$

punov function $V(x) = \max_i (Ux)_i$, which satisfies $\dot{V} = -V$, so that $V(x) \to 0$, see [19, 22, 28] for details. Theorem 2.1 implies that (4) has an attractor A_ε nearby. Because X is two dimensional, the Poincaré–Bendixson theorem [21] shows that A_ε contains a closed orbit of (4). Typically, A_ε will consist of a single attracting closed orbit.

Consider a game where one of the players has a strictly dominated strategy or a strategy that is never a best reply. The global attractor of (3) is then contained in the face of X that places zero weight on this strategy. Iterating this argument, the global attractor is contained in the set of rationalizable strategies. Theorem 2.1 implies then that the global attractor A_ε of a perturbed dynamics (4) is contained in a small neighborhood of the set of rationalizable strategies. A local version of this result is the following. Every CURB set (this is a face $Y \subset X$ such that $\mathrm{BR}(Y) = Y$, see [2]) is an attractor for (3). Therefore each small neighborhood contains an attractor A_ε for (4).

2.3 Application: Perturbations of the Game

Alternatively we can consider perturbations of the game.

As an example, the global attractor for (3) or (4) for a game close to a two person zero-sum game G is close to the set of optimal strategies of G, as a consequence of [29] and Theorem 2.1.

In [27] games with two identical strategies are perturbed to games with a strictly dominated strategy. Thereby survival of strictly dominated strategies along a nonequilibrium attractor under certain dynamics has been shown.

Finally, note that attractors are in general not lower-semicontinuous against perturbations. This is analogous to components of Nash equilibria of a game. However, the perturbed attractor is (by definition) nonempty. This is not true for components for Nash equilibria. If the attractor consists of equilibria only, then the perturbed attractor—while being nonempty—may contain *no* equilibrium (if it has zero index). Examples of such games are given in [30], [28, Sect. 8.6] and [24, Sect. 7]: The binary choice 3 person game

$-1,-1,-1$	$0,0,\varepsilon$
$0,\varepsilon,0$	$\varepsilon,0,0$

$\varepsilon,0,0$	$0,\varepsilon,0$
$0,0,\varepsilon$	$-1,-1,-1$

can be interpreted, for $\varepsilon = 0$, as a symmetric congestion model with 2 roads and 3 players. Similarly, a variant of Shapley's 2 person game

$(0,0)$	(a,b)	(b,a)
(b,a)	$(0,0)$	(a,b)
(a,b)	(b,a)	$(0,0)$

can be interpreted, for $a = b > 0$, as a symmetric congestion model with 3 roads and 2 players. In both games, the Nash equilibria consist of an unstable symmetric point E and of a connected component C homeomorphic to a circle which is an attractor. A perturbation of the game (i.e., $\varepsilon \neq 0$, or $a \neq b > 0$) has a single equilibrium near E but an attractor in a neighborhood of C.

3 Continuation of Chain-Recurrent Set

3.1 Definitions and Basic Properties

Let $M \subset X$ be an invariant set of (1) and consider $\Phi|_M$, the set-valued flow Φ restricted to M.

For $x, y \in M$, we write $x \hookrightarrow_M y$ if for every $\varepsilon > 0$ and $T > 0$ there exists an integer $n \in \mathbb{N}$, solutions $\mathbf{x}_1, \dots, \mathbf{x}_n$ to (1), and real numbers $t_1, t_2, \dots, t_n$ greater than T such that

(1) $\mathbf{x}_i(s) \in M$ for all $0 \le s \le t_i$ and for all $i = 1, \dots, n$,
(2) $\|\mathbf{x}_i(t_i) - \mathbf{x}_{i+1}(0)\| \le \varepsilon$ for all $i = 1, \dots, n-1$,
(3) $\|\mathbf{x}_1(0) - x\| \le \varepsilon$ and $\|\mathbf{x}_n(t_n) - y\| \le \varepsilon$.

The sequence $(\mathbf{x}_1, \dots, \mathbf{x}_n)$ is called an (ε, T) chain (in M from x to y) for the differential inclusion (1) or the multivalued flow $\Phi|_M$.

Definitions A point $x \in M$ is chain recurrent (in M) if $x \hookrightarrow_M x$.

The set of chain-recurrent points in M is denoted by $\mathrm{CR}(M) = \mathrm{CR}(\Phi|_M)$.

$\mathrm{CR}(M)$ is itself invariant and $\mathrm{CR}(M) = \mathrm{CR}(\Phi|_{\mathrm{CR}(M)})$.

For an attractor $A \subset M$ let $B_M(A)$ denote the basin of attraction of A (in M for $\Phi|_M$) and $A^* = M \setminus B_M(A)$ its complement which is called the dual repellor to A. A^* is closed.

The set of all attractors in M is denoted by $\mathcal{A}(M)$.

Note that for $x \in B_M(A) \setminus A = M \setminus (A \cup A^*)$, *all* solutions $\mathbf{x}(t)$ for $\Phi|_M$ with $\mathbf{x}(0) = x$ and $\mathbf{x}(t) \in M$ for all $t \in \mathbb{R}$ satisfy $d(\mathbf{x}(t), A) \to 0$ for $t \to +\infty$ and $d(\mathbf{x}(t), A^*) \to 0$ for $t \to -\infty$. But note that—in contrast to classical dynamical systems—there may exist solutions $\mathbf{x}(t)$ with $\mathbf{x}(0) \in A^*$ and $\mathbf{x}(t) \in B_M(A)$ for $t > 0$ and even $\mathbf{x}(t) \in A$ for t large.

The following result is a generalization of Conley [12], proved in [9, Theorem 1]. It characterizes the chain-recurrent points in terms of the attractor–repellor pairs (A, A^*).

Proposition 3.1

$$\mathrm{CR}(\Phi|_M) = \bigcap_{A \in \mathcal{A}(M)} \big(A \cup A^*\big).$$

This also shows that $\mathrm{CR}(\Phi|_M)$ is invariant. Let now $M_0 \subseteq X$ denote the global attractor of the flow Φ. We write $\mathrm{CR}(\Phi) = \mathrm{CR}(\Phi|_{M_0})$.

3.2 Upper-Semicontinuity of Chain-Recurrent Set

The characterization given in Proposition 3.1 allows one to extend the upper-semicontinuity result for attractors to $\mathrm{CR}(\Phi)$.

Consider the enlarged/perturbed flow Φ^δ on X.

Theorem 3.1 *Let V be an open neighborhood of $\mathrm{CR}(\Phi)$. Then there exists $\delta_0 > 0$ such that for all $\delta \in [0, \delta_0)$, $\mathrm{CR}(\Phi^\delta) \subset V$.*

Proof For each attractor A of Φ there is the dual repeller $A^* \subset M_0$ which is also the largest invariant set in $X \setminus V(A)$ where $V(A)$ is an open fundamental neighborhood of A. Given a closed set $Y \subseteq X$, the largest Φ-invariant subset of Y is upper semicontinuous against perturbations of Φ. Therefore the dual repeller A^* and the union $A \cup A^*$ are upper semicontinuous against perturbations of Φ.

Suppose $K \subset X$ is a compact set disjoint from $\mathrm{CR}(\Phi)$. We need to show that for small δ, K is disjoint from $\mathrm{CR}(\Phi^\delta)$. By compactness, there are finitely many attractors $A_i, i \in I$, such that K is disjoint from $\bigcap_{i \in I}(A_i \cup A_i^*)$. Hence for small $\delta > 0$, K is disjoint from $\bigcap_{i \in I}(A_i^\delta \cup A_i^{*\delta})$ (where the A_i^δ are associated to Φ^δ from Theorem 2.1) and hence from $\bigcap_{A \in \mathcal{A}(\Phi^\delta)}(A \cup A^*) = \mathrm{CR}(\Phi^\delta)$. □

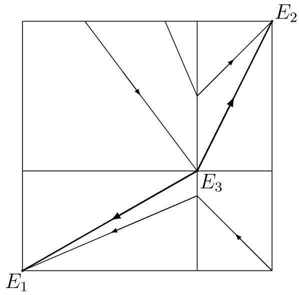

Fig. 2 A 2×2 coordination game

3.3 Application: Potential Games

A specific class where the structure of the chain-recurrent set is well understood are potential games. Consider again the best response dynamics (3) but any other myopic adjustment dynamics [41] could be taken as well. As shown in [6], the potential function of the game increases monotonically along every solution of (3) and is therefore a natural Lyapunov function. If the potential function is smooth enough then Sard's lemma implies that the set of chain-recurrent points, $\mathrm{CR}(\Phi)$, coincides with the set of Nash equilibria. Theorem 3.1 implies that every perturbed dynamics (4) of a game close to a potential game has its chain-recurrent set (and hence *all* ω-limit points of *all* orbits) close to the set of Nash equilibria of the potential game (but not necessarily close to the set of Nash equilibria of the game itself.)

As a simple example consider a 2×2 coordination game with two strict equilibria E_1, E_2 and a mixed equilibrium E_3, see Fig. 2. There are four attractors: (1) the global attractor (= the maximal invariant set), consisting of the three equilibria and two line segments connecting them (its dual repellor is empty); (2) E_1, its dual repellor being a line segment connecting E_2 and E_3; (3) E_2, its dual repellor being a line segment connecting E_3 and E_2; (4) $\{E_1, E_2\}$, its dual repellor being E_3. The chain-recurrent set is $\{E_1, E_2, E_3\}$. Theorem 3.1 implies that for every perturbed dynamics, and slight perturbations of the game, every orbit will converge to a small neighborhood of one of the three equilibria.

3.4 Application: Global Game Dynamics

Consider the space of all games $\mathcal{G}$ on the state space X (a simplex or a product of simplices) and a game dynamics $\dot{x} \in \phi(g, x)$ given by a multi-valued map $\phi : \mathcal{G} \times X \rightrightarrows TX$ (with TX denoting the tangent space of X) with closed graph and convex values such that, for each game $g \in \mathcal{G}$, X is forward invariant and every Nash equilibrium is a fixed point of the corresponding set-valued (semi-)flow $\Phi(g, \cdot)$ on X:

$$\mathrm{Fix}\Phi(\mathrm{g}, \cdot) \supseteq \mathrm{NE}(g) \quad \text{for all } g \in \mathcal{G}.$$

Examples are the BR dynamics (3), replicator dynamics [28], the Brown–von Neumann–Nash dynamics [23], the Smith dynamics [26], etc. For such a game dynamics Φ we can consider the map that associates to each game g the set $\mathrm{CR}(\Phi(g, \cdot))$. By Theorem 3.1, this map is upper semicontinuous. It is a natural dynamic analog of the equilibrium correspondence which associates to every game the set of Nash equilibria. This equilibrium correspondence has particularly nice properties [33].

Since every Nash equilibrium is a fixed point for Φ, and every fixed point is chain recurrent, the $\mathrm{CR}(\Phi)$ correspondence contains the equilibrium correspondence. In fact, the inclusion is always strict.

There is no game dynamics Φ for which CR(Φ) *equals the set of Nash equilibria for all games.*

Proof Consider the game from [33, p. 1034].

1, 1	0, −1	−1, 1
−1, 0	0, 0	−1, 0
1, −1	0, −1	−2, −2

This game has a single connected component of equilibria, hence with index 1, but homeomorphic to a circle C, hence its Euler characteristic is 0. If the chain-recurrent set CR(Φ) equals this set, then it is connected and hence an attractor (actually the unique attractor, by using the representation in Proposition 3.1). This contradicts theorems[1] in [13, 14]. In fact for some dynamics the set CR(Φ) is a disc bounded by C. □

The CR(Φ) correspondence depends heavily on the game dynamics. A natural choice may be to take the best response dynamics (3). Then, by the result from [31], this CR(BR) contains also the time averages of every interior solution of the replicator dynamics.

Associating to each game its global attractor (the maximal invariant set) is less useful. For example, for the replicator dynamics, this is always the whole simplex X.

Associating the union of all minimal attractors is not a good idea, as minimal attractors need not exist.

4 Discretizations with Small Step Size

4.1 Continuous Time and Discrete Time Trajectories

Consider a discrete time dynamics of the form

$$x_{n+1}^{\varepsilon} - x_n^{\varepsilon} \in \varepsilon F^{\delta(\varepsilon)}\big(x_n^{\varepsilon}\big), \quad x_0^{\varepsilon} = x \tag{9}$$

where ε is a positive small parameter, and $\delta : (0, +\infty) \to [0, +\infty)$ is a positive function with $\delta(\varepsilon) \to 0$ as $\varepsilon \to 0$. Denote by $\mathbf{x}^{\varepsilon}$ the associated continuous trajectory on $[0, +\infty)$ defined by $\mathbf{x}^{\varepsilon}(n\varepsilon) = x_n^{\varepsilon}$ and extended by linear interpolation.

There exists a map $\bar{\delta}$ from $(0, \varepsilon_0)$ to $(0, +\infty)$ with $\bar{\delta}(\varepsilon) \to 0$ as $\varepsilon \to 0$ and such that $\mathbf{x}^{\varepsilon}$ is a solution of

$$\dot{\mathbf{x}} \in F^{\bar{\delta}(\varepsilon)}(\mathbf{x}).$$

[1]Consider a differential inclusion $\dot{x} \in F(x)$ on a compact domain $A \in \mathbb{R}^n$, where F is a correspondence with convex values in $\mathbb{R}^n$ and compact graph. Let C be a 'regular' compact subset of A for F namely: (1) there exists a basis of neighborhoods which deformation retracts on C, i.e. C is a deformation retract, (2) its Euler characteristic $\chi(C)$ exists, (3) there exists a neighborhood $V(C)$ of C such that $0 \notin F(a)$ for all $a \in V(C) \setminus C$, which is moreover asymptotically stable for the dynamics induced by F, then its Euler characteristic coincides with the index $\mathrm{Ind}(F, C)$. This is in particular the case when F is semi-algebraic and C is a (uniformly) attracting component of its set of zeroes. This extends a previous result by Demichelis and Ritzberger [13] dealing with set of zeroes of differential equations.

Proposition 4.1 *If M_0 is the global attractor for Φ, then for any $\eta > 0$, there exists ε_0 such that for every $\varepsilon \in (0, \varepsilon_0)$, the set of accumulation points of the trajectory $\{x_n^\varepsilon\}$ belongs to $N^\eta(M_0)$.*

Proof An elementary proof was given in [29]. □

Proposition 4.2 *Let V be an open neighborhood of* $\mathrm{CR}(\Phi)$. *Then there exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$ the set of accumulation points of $\{x_n^\varepsilon\}$ (as $n \to \infty$) belongs to V.*

Proof By Theorem 4.3 of [6] the set of accumulation points of $\{x_n^\varepsilon\}$ is internally chain transitive for $\Phi^{\bar{\delta}(\varepsilon)}$, and hence is contained in V for small ε by Theorem 3.1. □

For C^2 differential equations, this result has been shown in [18].

4.2 Applications: Discretizations of BR and Other Game Dynamics

A simple discretization of the BR dynamics with constant step size ε is

$$x(t+\varepsilon) \in \varepsilon \mathrm{BR}\big(x(t)\big) + (1-\varepsilon)x(t). \tag{10}$$

This models a population where in each time unit a small proportion of the population switches to a best response.

More general is a discretization with variable step sizes

$$x(t_{n+1}) \in \varepsilon_n \mathrm{BR}\big(x(t_n)\big) + (1-\varepsilon_n)x(t_n), \quad t_n + \varepsilon_n = t_{n+1}. \tag{11}$$

For $\varepsilon_n = \frac{1}{n}$ this is fictitious play. For $\varepsilon_n = \frac{1-\rho}{1-\rho^n}$ (with $0 < \rho < 1$) this is *geometric fictitious play* with discount rate ρ (see [29]) which tends to (10) with $\varepsilon = 1 - \rho$, as $n \to \infty$.

As an example consider the matching pennies games, the standard 2×2 zero-sum game, with a unique, completely mixed Nash equilibrium E. This equilibrium E is the global attractor for the BR dynamics (3). Then Proposition 4.1 shows that the global attractor M_ε of (10) approaches E as $\varepsilon \to 0$. In [4] the structure of M_ε has been studied. M_ε is contained in a neighborhood of E whose radius is of order $\sqrt{\varepsilon}$. Besides the repelling equilibrium E, M_ε contains stable periodic orbits of period $4n$. Orbits of period $4n$ exist for $0 < \varepsilon < \varepsilon_n$, where $\varepsilon_n \to 0$. So for each $\varepsilon > 0$, there are only finitely many periodic orbits of periods $4, 8, 12, \ldots$, but the smaller ε, the more complicated the structure of the attractor M_ε.

The *Nash map* [39] is the (family of) continuous map(s) $f^\varepsilon : \Delta(I) \to \Delta(I)$ defined by

$$f^\varepsilon(x)_i = \frac{x_i + \varepsilon k_i(x)}{1 + \varepsilon \sum_{j=1}^n k_j(x)}, \quad i \in I \tag{12}$$

where

$$k_i(x) = \big[(Ax)_i - x^T A x\big]_+ \tag{13}$$

and $u_+ = \max(u, 0)$, and $\varepsilon > 0$ is a scaling parameter (Nash had $\varepsilon = 1$). The fixed points of this map are precisely the Nash equilibria of the game. Note that

$$f^{\varepsilon}(x)_i - x_i = \frac{\varepsilon}{1+\varepsilon\sum_{j=1}^{n} k_j(x)} F_i(x) \subseteq \varepsilon F_i^{\varepsilon K}(x)$$

with $F_i(x) = k_i(x) - x_i \sum_{j=1}^{n} k_j(x)$, and $K = \max_{x\in\Delta} \sum_{j=1}^{n} k_j(x)$.

Hence we are in the setting (9): The Nash map (12) is a discretization of the differential equation

$$\dot{x}_i = k_i(x) - x_i \sum_{j=1}^{n} k_j(x). \tag{14}$$

This differential equation is due to Brown and von Neumann [10] (in the case of symmetric zero-sum games $A = -A^T$, for which (13) reduces to $k_i(x) = ((Ax)_i)_+)$ and is known as the Brown–von Neumann–Nash dynamics [23]. It is Lipschitz, but not smooth, so the discretization results from [18] do not apply, but the above results do.

The global attractor of (14) is given by the set of Nash equilibria, for zero-sum games [10], for negative semidefinite games [23], and for stable games [26]. Therefore, by Proposition 4.1, the global attractor M_ε of the Nash map (12) with small step size $\varepsilon > 0$ is contained in a small neighborhood of the set of Nash equilibria of the game. For the matching pennies game, the attractor M_ε has been studied in [3]: It consists of the repelling equilibrium, an attracting invariant closed curve around E whose radius is of order ε, and the connecting orbits in between.

Acknowledgement This research was partially supported by grant SNF 2000-21-138242/1 (Switzerland), grant WWTF-MA09-017 (Austria) and grant ANR-08-BLAN-0294-01 (France).

References

1. Aubin J-P, Cellina A (1984) Differential inclusions. Springer, Berlin
2. Basu K, Weibull J (1991) Strategy subsets closed under rational behavior. Econ Lett 36:141–146
3. Becker RA, Chakrabarti SK, Geller W, Kitchens B, Misiurewicz M (2007) Dynamics of the Nash map in the game of matching pennies. J Differ Equ Appl 13:223–235
4. Bednarik P (2011) Discretized best-response dynamics for cyclic games. Diplomarbeit, University Vienna, Vienna
5. Benaïm M, Hirsch MW (1999) Mixed equilibria and dynamical systems arising from fictitious play in perturbed games. Games Econ Behav 29:36–72
6. Benaïm M, Hofbauer J, Sorin S (2005) Stochastic approximations and differential inclusions. SIAM J Control Optim 44:328–348
7. Benaïm M, Hofbauer J, Sorin S (2006) Stochastic approximations and differential inclusions. Part II: applications. Math Oper Res 31:673–695
8. Blume L (1993) The statistical mechanics of strategic interaction. Games Econ Behav 5:387–424
9. Bronstein IU, Kopanskii AY (1988) Chain recurrence in dynamical systems without uniqueness. Nonlinear Anal Theory Appl 12:147–154
10. Brown GW, von Neumann J (1950) Solutions of games by differential equations. Ann Math Stud 24:73–79
11. Chen H-C, Friedman JW, Thisse J-F (1997) Boundedly rational Nash equilibrium: a probabilistic choice approach. Games Econ Behav 18:32–54
12. Conley C (1978) Isolated invariant sets and the Morse index. CBMS regional conference series in mathematics, vol 38. Am Math Soc, Providence
13. Demichelis S, Ritzberger K (2003) From evolutionary to strategic stability. J Econ Theory 113:51–75
14. Demichelis S, Sorin S (2003) Asymptotic stability for convex-valued differential inclusions. Preprint
15. Fudenberg D, Kreps D (1993) Learning mixed equilibria. Games Econ Behav 5:320–367
16. Fudenberg D, Levine DK (1995) Consistency and cautious fictitious play. J Econ Dyn Control 19:1065–1089
17. Fudenberg D, Levine DK (1998) The theory of learning in games. MIT Press, Cambridge
18. Garay BM, Hofbauer J (1997) Chain recurrence and discretisation. Bull Aust Math Soc 55:63–71

19. Gaunersdorfer A, Hofbauer J (1995) Fictitious play, Shapley polygons, and the replicator equation. Games Econ Behav 11:279–303
20. Harsanyi JC (1973) Oddness of the number of equilibrium points: a new proof. Int J Game Theory 2:235–250
21. Hirsch MW, Smale S (1974) Differential equations, dynamical systems, and linear algebra. Academic Press, San Diego
22. Hofbauer J (1995) Stability for the best response dynamics. Mimeo
23. Hofbauer J (2000) From Nash and Brown to Maynard Smith: equilibria, dynamics, and ESS. Selection 1:81–88
24. Hofbauer J (2011) Deterministic evolutionary game dynamics. In: Sigmund K (ed) Evolutionary game dynamics. Proceedings of symposia in applied mathematics, vol 69. Am Math Soc, Providence, pp 61–79
25. Hofbauer J, Sandholm WH (2002) On the global convergence of stochastic fictitious play. Econometrica 70:2265–2294
26. Hofbauer J, Sandholm WH (2009) Stable games and their dynamics. J Econ Theory 144:1665–1693
27. Hofbauer J, Sandholm WH (2011) Survival of dominated strategies under evolutionary dynamics. Theor Econ 6:341–377
28. Hofbauer J, Sigmund K (1998) Evolutionary games and population dynamics. Cambridge University Press, Cambridge
29. Hofbauer J, Sorin S (2006) Best response dynamics for continuous zero–sum games. Discrete Contin Dyn Syst B6:215–224
30. Hofbauer J, Swinkels J (1996) A universal Shapley example. Preprint
31. Hofbauer J, Sorin S, Viossat Y (2009) Time average replicator and best reply dynamics. Math Oper Res 34:263–269
32. Hurley M (1982) Attractors: persistence, and density of their basins. Trans Am Math Soc 269:247–271
33. Kohlberg E, Mertens J-F (1986) On the strategic stability of equilibria. Econometrica 54:1003–1038
34. Kojima F, Takahashi S (2007) Anti-coordination games and dynamics stability. Int Game Theory Rev 9:667–688
35. Li D, Zhang X (2002) On dynamical properties of general dynamical systems and differential inclusions. J Math Anal Appl 274:705–724
36. Matsui A (1992) Best response dynamics and socially stable strategies. J Econ Theory 57:343–362
37. McKelvey RD, Palfrey TR (1995) Quantal response equilibria for normal form games. Games Econ Behav 10:6–38
38. Nash, J. (1950) Non-cooperative games. Dissertation, Princeton University, Department Mathematics
39. Nash J (1951) Non-cooperative games. Ann Math 54:287–295
40. Smale S (1967) Differentiable dynamical systems. Bull Am Math Soc 73:747–817
41. Swinkels JM (1993) Adjustment dynamics and rational play in games. Games Econ Behav 5:455–484