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Intermingled basins in a two species system*

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Abstract. We present simple examples of (competitive) two species systems with complicated dynamic behaviour. From almost all initial conditions one of the two species dies out. But the survivor is unpredictable: The basins of the two chaotic one-species attractors are everywhere dense and intermingled.

1. A two species system with intermingled basins

Two–species competition systems in continuous time, as modelled by a system of ODEs $\dot{x}_i = x_i\varphi_i(x_1, x_2)$ $(i = 1, 2)$ with

$$\frac{\partial \varphi_i}{\partial x_j} < 0 \qquad \text{for} \quad i, j = 1, 2 \tag{1}$$

have a fairly simple dynamics, as described in every textbook on theoretical ecology.Three different basic types are possible: coexistence at a (locally or even globally) stable equilibrium, one species outcompeting the other species, and finally bistability (with two disjoint, open basins of attraction, separated by a one-dimensional curve). For discrete time systems, modelled by a map $x \in \mathbf{R}^2_+ \mapsto x' \in \mathbf{R}^2_+$

$$x_i' = x_i\varphi_i(x_1, x_2) \qquad i = 1, 2 \tag{2}$$

the situation is more complicated and much less understood. The competition conditions (1) do not effectively restrict the dynamics as they do in continuous time: Already the single species dynamics $x \mapsto x' = T(x) = x\varphi(x)$ can lead to complex behaviour, as known since [19]. Typical examples are the wellknown quadratic map

$$T(x) = Rx(1 - x) \quad x \in [0, 1], \tag{3}$$

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or the ecologically more relevant Ricker–Moran model [20]

$$T(x) = Rxe^{-ax} \quad (x > 0, \quad a > 0) \tag{4}$$

or Hassel's [12]

$$T(x) = \frac{Rx}{(1+ax)^b} \quad (x > 0, \quad R, a, b > 0). \tag{5}$$

In this note we present discrete time two-species competition systems (2) which have the following property: *From almost all initial conditions one of the two species dies out. But the survivor is unpredictable: The basins of the two chaotic one–species attractors are everywhere dense. Even stronger, each basin has positive Lebesgue measure in each open subset of* $\mathbf{R}^2_+$.

This phenomenon has been called *intermingled basins* [2] and was rigorously demonstrated so far only for a few dynamical systems [1], [17],[29]. Here we generalize and adapt Kan's [17] model system to a class of two species models for which we give a rigorous proof of the intermingling of two basins.

Our examples are constructed as suitable perturbation from degenerate competition systems

$$x_i' = x_i \varphi(x_1 + x_2) \qquad i = 1, 2 \tag{6}$$

This degenerate competition model has two special features. First, $\frac{x_2'}{x_1'} = \frac{x_2}{x_1}$, which means that the rays through the origin are invariant under (6), and the proportion of each species in the whole system does not change from one generation to the next. Second, the total population density, $x = x_1 + x_2$, changes according to the law $T(x) = x' = x\varphi(x)$, its value in the next generation depends only on the present value of x. Biologically, the two species are essentially identical. We now consider perturbations of the form

$$\begin{aligned} x_1' &= x_1\varphi(x_1 + x_2)(1 + \varepsilon x_2 G(x_1, x_2)), \\ x_2' &= x_2\varphi(x_1 + x_2)(1 - \varepsilon x_1 G(x_1, x_2)). \end{aligned} \tag{7}$$

If $\varphi'(x) < 0$ for all $x \geq 0$ and ε is small then (7) still defines a competitive system (2) with (1). Expressed in the new coordinates $x = x_1 + x_2$ (total density) and $y = \frac{x_1}{x}$ (proportion of species 1 in the total population), (7) simplifies to

$$x' = T(x) = x\varphi(x) \tag{8}$$

$$\Delta y = y' - y = \varepsilon y(1-y)g(x, y) \tag{9}$$

for some continuous function $g(x, y) = xG(xy, x(1-y))$. Because of (8) the second special feature of degenerate competition (6) still pertains. It is systems of the form (8–9) that will be studied in the next section and throughout the paper. For more concrete examples we will assume that g depends only on x, $g(x, y) = g(x)$, or equivalently, G depends only on $x_1 + x_2$.

For (7) to have intermingled basins we need essentially three ingredients. First, the one-dimensional map $T : x \mapsto x\varphi(x)$ (which governs the dynamics of each species and the total population in (7)) has a chaotic attractor A which is a union of intervals and which carries an absolutely continuous natural invariant measure μ. The logistic map (3), the Ricker–Moran map (4) and the Hassel map (5) satisfy these assumptions for parameters R from a set of positive measure. This gives rise to two attractors A_1, A_2 in the axes of $\mathbf{R}_+^2$. Second, the normal Liapunov exponents at the two axes with respect to the natural measure μ is negative. Biologically this means that a small amount of the competitor typically will die out again. Mathematically this implies that both basins have positive Lebesgue measure. As we will show below, see (24), this holds if $\int g(x) d\mu(x)$ is sufficiently close to 0.

Third we need that the attractors are not stable in the sense of Liapunov, but some nearby orbits converge to the other attractor. We guarantee this by normally unstable periodic points in each attractor. The simplest assumption is $g(p) < 0$ and $g(q_1) + g(q_2) > 0$ for the fixed point p and a period two orbit q_1, q_2 of the map T. It turns out that these two inequalities together with the above integral condition on g suffice. Thus we find that for every chaotic map T there exist many interaction functions g such that for small $\varepsilon \neq 0$ the system (7) has intermingled basins. The precise assumptions and the result are explained in detail in the next section.

Our result sheds new light on the concept of *indeterminate competition* introduced after Park's [25] experiments on the competition of two species of flour beetles, *Tribolium castaneum* and *T. confusum*, see [6] (ch. 8) for a survey. In each of these experiments one of the two competitors was extirpated. But interestingly not always the same species vanished, even when the experiment was repeated from seemingly identical initial conditions. The standard explanation employs stochastic effects due to small population numbers. Deterministic systems have been generally believed to be incapable to explain such observations. Our result indicates that this opinion is unfounded: already for extremely simple two species interactions the survivor may be in principle unpredictable even in a deterministic model.

We expect that this phenomenon occurs also for more realistic ecological models, maybe even in a robust way. Huisman and Weissing's [16] investigations of a simple resource competition model support these expectations.

We conclude with some applications and warnings for the literature on adaptive dynamics and invasion of a resident system where our finding seems to be relevant.

The definition of invasibility relying on the normal Ljapunov exponent of the natural measure only (if it exists at all) seems problematic and overly simplified. In our examples these natural normal Ljapunov exponents are negative, yet successful invasion and replacing the resident (say species 1) is possible with small but positive probability: Arbitrarily close to A_1 there start orbits that converge to A_2.

Even worse, there are (sets with positive Lebesgue measure of) orbits starting arbitrarily close to A_1 and converging back to A_1 after (arbitrarily) long excursions. This phenomenon is known as 'the resident strikes back'. Interesting examples are due to [10], [9], [21] in the setting of spatially structured and temporally structured (semelparous) populations, respectively. There the resident system is invaded at one stable attractor and returns to a different stable attractor. Our example shows that bistability of the resident system is not necessary for this phenomenon: Instead of two separate attractors it takes only two ergodic invariant measures whose normal Liapunov exponents have opposite sign to make a species 'invadable, yet invincible'.

2. The main result

The maps T from section 1 are unimodal (eq. (5) at least for $b > 1$): there exists a number $c > 0$ such that T is increasing for $x < c$ and decreasing for $x > c$. We can restrict T to the forward invariant interval $I = [T^2(c), T(c)]$ (with c the unique critical point satisfying $T'(c) = 0$) which every nonzero initial condition enters after finitely many steps. Then together with (9) we obtain a map $F : I \times [0, 1] \to I \times [0, 1]$ of the following 'skew-product' or 'triangular' form

$$F(x, y) = (T(x), f(x, y)), \tag{10}$$

where $f(x, y) = y + \varepsilon y(1 - y)g(x, y)$. The special feature of maps (10) is that vertical lines are mapped to vertical lines, reflecting the assumption in the original two species system that lines of constant total population size $x = x_1 + x_2$ are mapped to such lines of slope -1.

This leads us to the following general setting which we study in the subsequent sections: Let (10) be a map with $T : I \to I$ unimodal (or more generally piecewise monotonic) and let $f : I \times [0, 1] \to [0, 1]$ be continuous and satisfy

$$f(x, 0) = 0, \quad f(x, 1) = 1. \tag{11}$$

This means that the sets $I_0 = I \times \{0\}$ and $I_1 = I \times \{1\}$ are forward invariant under F.

An unimodal map T is called *S-unimodal*, if it is C^3 and satisfies the negative Schwarzian derivative condition

$$S(T) := \frac{T'''}{T'} - \frac{3}{2}\left(\frac{T''}{T'}\right)^2 \leq 0 \quad \text{on} \quad I \setminus \{c\}. \tag{12}$$

The critical point c is *nonflat* if $T^{(n)}(c) \neq 0$ for some n. In our examples $T''(c) \neq 0$.

The map $T : I \to I$ is called *topologically transitive*, if there exists a dense orbit in I.

In order to state the assumptions on f, we introduce the following terminology.

For $0 < \varepsilon < 1$, a differentiable map $u : [0, 1] \to [0, 1]$ is called an *ε–perturbation of the identity*, if $u(0) = 0$, $u(1) = 1$ and $|1 - u'(y)| < \varepsilon$ for all $y \in [0, 1]$.

Now observe that (10) implies that for every $n \in \mathbf{N}$ there exists a function $f_n : I \times [0, 1] \to [0, 1]$, such that

$$F^n(x, y) := \left(T^n x, f_n(x, y)\right) . \tag{13}$$

Suppose that $p \in [0, 1]$ is a point with period ℓ under T, i.e. $T^\ell p = p$. We call p an *I_0–attracting* periodic point, if $f_\ell(p, y) < y$ for all $y \in (0, 1)$, and $\frac{\partial f_\ell}{\partial y}(p, 0) < 1$. This implies that on the vertical lines $x = p, Tp, T^2 p, \ldots$, orbits decrease at a geometric rate to the periodic orbit of $(p, 0) \in I_0$. Similarly, p is called *I_1–attracting* if $f_\ell(p, y) > y$ for all $y \in (0, 1)$, and $\frac{\partial f_\ell}{\partial y}(p, 1) > 1$.

Denoting by $\lambda^2(A)$ the two-dimensional Lebesgue measure of a set $A \subset \mathbf{R}^2$, we can now formulate our main result.

Theorem 1. *Suppose that $T : I \to I$ is S-unimodal with nonflat critical point c, is topologically transitive, and satisfies the* Collet-Eckmann condition

$$\liminf_{n\to\infty} \sqrt[n]{|(T^n)'(Tc)|} > 1. \tag{14}$$

Then there is an absolutely continuous ergodic T–invariant measure μ and a constant $\alpha > 0$ such that for each C^1 function f that satisfies

(a) there is $\varepsilon < \alpha$ such that $y \mapsto f(x, y)$ is an ε-perturbation of the identity for each $x \in I$;
(b) there is an I_0-attracting periodic point $p_0 \in I$ and an I_1-attracting periodic point $p_1 \in I$;
(c) the normal Lyapunov exponents w.r.t. μ are negative:

$$\chi_0 := \int_I \log \frac{\partial f}{\partial y}(x, 0)\, d\mu(x) < 0, \tag{15}$$

$$\chi_1 := \int_I \log \frac{\partial f}{\partial y}(x, 1)\, d\mu(x) < 0; \tag{16}$$

the two-dimensional map (10) has the following property: The state space is split into three pairwise disjoint sets:

$$I \times [0, 1] = B_0 \cup B_1 \cup D, \tag{17}$$

such that
(1) Every orbit starting in B_i converges to the set I_i $(i = 0, 1)$.
(2) For every open set $U \subset I \times [0, 1]$: $\lambda^2(B_0 \cap U) > 0$ and $\lambda^2(B_1 \cap U) > 0$.
(3) The set D has zero Lebesgue measure: $\lambda^2(D) = 0$.

Hence the bottom and top interval I_0 and I_1 (which correspond to the attractors A_1 and A_2 in the x_1 and x_2–axis in the original two species system) are attractors for the map F in a measure theoretic sense: both attract sets of positive measure. This attraction is guaranteed by the negative sign of the normal Ljapunov exponents (15,16) which measure the exponential rate of approach to I_0 and I_1 for suitable 'μ–generic' initial conditions. But these attractors are not stable in the sense of Ljapunov: In particular they contain periodic points which are normally unstable; assumption (b) ensures the existence of such points $(p_0, 1)$ and $(p_1, 0)$ whose unstable manifold lead to the opposite interval I_0 and I_1, respectively. This implies that arbitrarily close to each point in the basin of say I_0 there are orbits which initially tend to I_0 but thereby get too close to this periodic orbit p_1 and are thus repelled away from I_0 again. This may lead such an orbit to converge to I_1, or if it experiences a similar fate near I_1 it may be pushed back to I_0 and maybe, after finitely many such back and forths, settle for one of the attractors. Of course there are many orbits which do not converge to either side, but assertion (3) of Theorem 1 shows that these form a set of Lebesgue measure zero.

Interestingly, in the topological sense the situation is opposite: B_0 and B_1 are of first category (i.e., countable unions of nowhere dense sets), and D is residual (i.e., a countable intersection of open dense sets). This will be shown in Proposition 2. The digital world of numerical simulations (and also the natural world), with $y = 0$ and $y = 1$ as absorbing states, seems to cohere more with the measure theoretic result of Theorem 1.

In [2] a related but somewhat simpler phenomenon is considered where one of the two attractors is stable (e.g. a stable fixed point or periodic orbit). Then its basin B_0 will be a dense open set, which is split into countably many smaller and smaller components. The basin B_1 of the unstable attractor I_1 has still positive measure, and density one near the attractor, but this basin is ‘riddled’, the ‘holes’ being the components of the basin B_0 of the other, stable attractor. In fact, B_1 is contained in the (fractal) boundary of B_0. There are many numerical examples, some semi-rigorous, and some rigorous proofs for the existence of such basin riddling, see e.g. [2] (for the two species competition model of [11]), [3], [7], [16], and [22]. These papers also contain nice pictures of such riddled basins. In contrast, our situation with *intermingled basins* is much more complicated, too complicated indeed to be captured in a 2d plot.

The possibility of the phenomenon of intermingled basins was also suggested in [2]. The first rigorous proof is due to Kan [17], for system (10) with $T(x) = 3x(\text{mod}1)$ and $g(x) = \frac{\cos 2\pi x}{32}$. Other cases are shown in [1] and [29].

Our paper is motivated by Kan’s elegant work, and extends it to a large class of more general ‘chaotic’ maps T which are of interest as ecological models. Also the method of proof follows that of Kan. The additional technical difficulties in our more general situation are handled with the methods developed in the last two decades for S-unimodal and expanding maps.

A similar theorem holds also for expanding piecewise monotone maps.

Theorem 2. *Suppose that T is piecewise differentiable, topologically transitive, that $\alpha := \inf \log |T'| > 0$ and that T' is Hölder continuous. Then there is an absolutely continuous ergodic T–invariant measure μ on I. Under the same assumptions (a), (b), (c) on f, the map (10) has the same properties (1), (2), (3) as in Theorem 1.*

Remarks. The assumption of *topological transitivity* is not really restrictive: Under the assumptions of Theorem 1 on T except topological transitivity there is a T–invariant set A which is a finite union of intervals and which contains a dense orbit. Furthermore, for Lebesgue almost all x there is an n such that $T^n x \in A$. It follows that the conclusions (1), (2), (3) of Theorem 1 remain valid, provided that the I_0– and I_1–attracting points p_0, p_1 are in A. A similar result holds in the case of Theorem 2, except that there can be finitely many such topologically transitive sets A, and hence I_0– and I_1–attracting periodic points have to exist in each of these sets.

The assumptions in the two theorems are in some sense stable. In particular, in Theorem 2, the assumptions on T are *robust against smooth perturbations*. Of course this is not the case for Theorem 1. But if one chooses a one-parameter family T_R of unimodal maps, such as (3), (4), or (5), then one can expect that the parameter values R for which T_R satisfies the assumptions of Theorem 1 form a set of positive measure. For the family (3) this was shown in [5], for (4) and (5) see [28].

The *Collet–Eckmann condition* (14) means that the lower critical Lyapunov exponent $\liminf_{n\to\infty} \frac{1}{n} \log |(T^n)'(x)|$ is positive. Several equivalent conditions are given in [24]. If this lower critical Lyapunov exponent is negative then there is a T–periodic point which attracts almost all orbits (whereas on a repelling invariant set of measure zero the dynamics may still be 'chaotic'). It has been conjectured ([8], [24]) that a lower critical Lyapunov exponent equal to zero occurs only for a set of Lebesgue zero of parameter values R in a 'generic' one–parameter family T_R of S-unimodal maps. Hence the Collet–Eckmann condition refers to the 'typical' case of a 'chaotic' attractor for a one-dimensional S-unimodal map. This conjecture was recently proved in [4], in particular for analytic families such as (3), (4) and (5).

For a fixed map T the assumptions on f hold for an open set in $\{f \in C^1(I \times [0,1]) : f(x,0) = 0, f(x,1) = 1\}$.

Construction of Biological Examples. Now we apply Theorem 1 to our original two species system. It is well-known that (3) and (4) are S-unimodal. For (5) an easy calculation shows that this holds for $b \geq 2$ (see also [28]). Note that the crucial assumption (14) can be explicitely checked in some particular cases. For example, if the critical orbit is ultimately periodic, (14) means that this periodic orbit is linearly unstable for T. A simple example is (3) with $R = 4$, for which $T^n(c) = 0$ for $n \geq 2$ and hence the expression in (14) is equal to $T'(0) = 4$. More generally, there are parameter values in the families (3), (4) and (5), such that

$$T^2c < T^3c < \cdots < T^{k-1}c < c < T^kc = p < Tc$$

where p is the unique fixed point in (c, Tc) and $k \geq 3$. These maps are topologically transitive and $\lim \sqrt[n]{|(T^n)'(Tc)|} = |T'(p)| > 1$.

The function f given as in (9)

$$f(x, y) = y + \varepsilon y(1 - y)g(x). \tag{18}$$

is an εa–perturbation of the identity for all continuous functions $g : I \to [-a, a]$. The unimodal map T has a fixed point p and a period two orbit $q = T(q'), q' = T(q)$ which can be chosen in the order $q < p < q'$. Computing (13) for $n = 2$ leads to

$$f_2(q, y) - y = y(1 - y)\big[\varepsilon\left(g(q) + g(q')\right) + O(\varepsilon^2)\big]. \tag{19}$$

Hence p is I_0–attracting if

$$g(p) < 0 \tag{20}$$

and q is I_1–attracting for small $\varepsilon > 0$, if

$$g(q) + g(q') > 0. \tag{21}$$

Since (18) implies $\frac{\partial f}{\partial y}(x, 0) = 1 + \varepsilon g(x)$ and $\frac{\partial f}{\partial y}(x, 1) = 1 - \varepsilon g(x)$, the normal Liapunov exponents (15, 16) are given by

$$\chi_0 := \int_I \log(1 + \varepsilon g(x))\, d\mu(x) < 0, \tag{22}$$

$$\chi_1 := \int_I \log(1 - \varepsilon g(x))\, d\mu(x) < 0. \tag{23}$$

Since $\log(1+\alpha) < \alpha$, both inequalities (22, 23) hold if

$$\int_I g(x) d\mu(x) = 0 \tag{24}$$

holds (at least approximately).

Now it is easy to see that continuous (even monotonic) functions $g : I \to \mathbf{R}$ exist which satisfy the three conditions (20), (21) and (24). The function (18) is then an $\varepsilon \|g\|_\infty$-perturbation of the identity, and Theorem 1 applies.

Hence we have rigorously demonstrated the occurrence of the phenomenon of intermingled basins in ecological equations modelling certain two species interactions. We expect that intermingled basins occur robustly also in more general ecological models. The findings of [16] are promising.

The rest of the paper is devoted to the proof of Theorems 1 and 2. In section 3 we review terminology and some auxiliary results. In section 4 we prove that I_0 and I_1 are attractors, whose basins of attraction have positive measure, are measure-theoretically dense in $I \times [0, 1]$ (its density points are dense) and hence are intermingled. In section 5 we show that the complement of the two basins has measure zero.

3. One-dimensional dynamics

A continuous map T on an interval I, which we will take as $[0, 1]$ in the following, is called *piecewise monotonic*, if there exists a 'partition' $\mathcal{Z}$ of $[0, 1]$ into finitely many pairwise disjoint open intervals with $\bigcup_{Z \in \mathcal{Z}} \overline{Z} = [0, 1]$, such that $T|_Z$ is strictly monotone for all $Z \in \mathcal{Z}$.

The map T is called *piecewise differentiable*, if additionally T' exists on every $Z \in \mathcal{Z}$. If $T'|_Z$ is Hölder continuous with exponent δ for every $Z \in \mathcal{Z}$, then we say that T is piecewise $C^{1+\delta}$. A piecewise differentiable map is called *expanding*, if $\inf |T'| > 1$ holds. Then $\alpha := \log \inf |T'| > 0$ is called the expansion rate of T.

Set $\mathcal{Z}_n = \{\bigcap_{i=0}^{n-1} T^{-i} Z_i \neq \emptyset : Z_i \in \mathcal{Z}\}$, which is a 'partition' of $[0, 1]$ into intervals such that $T^n|_Z$ is monotonic for all $Z \in \mathcal{Z}_n$. For $x \in [0, 1]$ denote the unique element of $\mathcal{Z}_n$ which contains x by $Z_n(x)$, except for the finitely many points for which this does not exist. We denote the diameter of an interval $I \subseteq [0, 1]$ by $|I|$ and set $|\mathcal{Z}_n| := \max\{|Z| : Z \in \mathcal{Z}_n\}$

For a continuous map $F : X \to X$ on a compact metric space X the *ω–limit* set $\omega(\mathbf{x})$ of a point $\mathbf{x} \in X$ is defined as the set of all limit points of the sequence $(F^n \mathbf{x})_{n \in \mathbf{N}_0}$. The basin of attraction of an invariant set $A \subseteq X$ is the set $B(A) := \{\mathbf{x} : \omega(\mathbf{x}) \subseteq A\}$. For standard notions from ergodic theory, like topological transitivity, ergodicity, and the entropy h_μ of an invariant Borel probability measure μ we refer to [30].

Lemma 1. *Suppose that the map* $T : [0, 1] \to [0, 1]$ *is S-unimodal with nonflat critical point* c, $(Tc = 1, T1 = 0)$ *and that* $\liminf_{n\to\infty} \sqrt[n]{|(T^n)'(Tc)|} > 1$.

1. *Then there is an absolutely continuous invariant measure* μ*, which is ergodic and has positive (nonzero) entropy. If* T *is topologically transitive, then its density function* h *satisfies* $\inf h > 0$.

2. $\liminf_{n\to\infty} -\frac{1}{n}\log|\mathcal{Z}_n| =: \alpha > 0$. *The number* α *is called the* expansion rate *of the unimodal map* T.

Proof. The existence, positivity of entropy and ergodicity of μ follows from Theorem V.4.1 in [8]. Theorem V.3.2 in [8] gives the existence of a T–invariant set A which is a finite union of closed intervals, such that $\inf_A h > 0$. If T is topologically transitive we have $A = [0, 1]$. This proves 1. Corollary 21 in [23] gives 2. □

Lemma 2. *Suppose that* T *is expanding, topologically transitive and piecewise* $C^{1+\delta}$. *Then there is an absolutely continuous invariant measure* μ*, which is ergodic and has nonzero entropy and whose density function* h *satisfies* $\inf h > 0$.

Proof. The existence of μ is shown in [18] and [26], using a spectral theorem for the Perron–Frobenius operator P. This gives also a decomposition of μ into finitely many ergodic components supported by invariant sets $A_1, \dots, A_k$ which have nonempty interior, since the density function is of bounded variation. Since T is topologically transitive, there can only be one such set A_1 and hence μ is ergodic. If $\inf_J h > 0$ for some set J then $\inf_{TJ} > 0$ since $Ph = h$ and $\inf \frac{1}{|T'|} > 0$. Since h is of bounded variation, there is an interval K with $\inf_K h > 0$. By Lemma 1 in [14] we get $\bigcup_{j=0}^{m} \overline{T^j K} = [0, 1]$ for some m and therefore $\inf_{[0,1]} h > 0$.

Furthermore, since μ is absolutely continuous w.r.t. the Lebesgue measure, the Hausdorff dimension of μ is 1 and hence $h_\mu = \int \log|T'| d\mu$ by Theorem 1 in [15], which is positive since T is expanding. □

4. The basin of I_0

For a Borel subset A of $\mathbf{R}$ denote by $\lambda^1(A)$ the one–dimensional Lebesgue measure of A, and for a Borel subset A of $\mathbf{R}^2$ denote by $\lambda^2(A)$ the two–dimensional Lebesgue measure of A. If $(x, y) \in [0, 1]^2$, then define $\pi_1(x, y) := x$ and $\pi_2(x, y) := y$.

Let $F : [0, 1]^2 \to [0, 1]^2$ be a map of the form (10). Under suitable conditions on T, Lemma 1 and 2 show the existence of an integrable (with respect to λ^1) function $h : [0, 1] \to \mathbf{R}$ with $\inf_{x\in[0,1]} h(x) > 0$, such that $\mu := h\, d\lambda^1$ is an ergodic T–invariant Borel probability measure on $[0, 1]$ with $h_\mu(T) > 0$.

The function $f : [0, 1]^2 \to [0, 1]$ is assumed to be C^1 and to satisfy several assumptions, which are described below. Recall from (13) that the iterates of F take the form

$$F^n(x, y) := \left(T^n x, f_n(x, y)\right) . \tag{25}$$

We obtain by induction, that $\frac{\partial f_n}{\partial y}$ exists for all $(x, y) \in [0, 1]^2$ and all $n \in \mathbf{N}$, and

$$\frac{\partial f_n}{\partial y}(x, y) = \prod_{j=0}^{n-1} \frac{\partial f}{\partial y}\left(F^j(x, y)\right) . \tag{26}$$

For $x \in [0,1]$ we define the *normal Lyapunov exponents* $\chi_0(x)$ and $\chi_1(x)$ by

$$\chi_0(x) = \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \frac{\partial f}{\partial y}\big(T^j x, 0\big) \ , \tag{27}$$

$$\chi_1(x) = \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \frac{\partial f}{\partial y}(T^j x, 1). \tag{28}$$

The ergodic theorem implies that, for μ–almost all $x \in [0,1]$, these limits of time averages exist and coincide with the space averages defined in (15, 16). Therefore for λ^1–almost all $x \in [0,1]$ we have

$$\chi_0(x) = \chi_0 \quad \text{and} \quad \chi_1(x) = \chi_1 \ . \tag{29}$$

Assume now that there is an $\varepsilon \in (0,1)$, such that $y \mapsto f(x,y)$ is an ε–perturbation of the identity for every $x \in [0,1]$. Furthermore suppose that $\chi_0 < 0$, and that there exists an I_0–attracting periodic point $p_0 \in [0,1]$. We will prove that every nonempty open subset of $[0,1]^2$ contains a subset of positive (two-dimensional) Lebesgue measure of initial conditions (x,y) with

$$\omega(x,y) \subseteq I_0 \ .$$

Lemma 3. *There exists a Borel–measurable function* $b_0 : [0,1] \to (0,1)$*, such that for every* $x \in [0,1]$ *with* $\chi_0(x) \le \chi_0 < 0$ *and every* $y \in \big[0, b_0(x)\big]$ *we have*

$$\omega(x,y) \subseteq I_0 \ .$$

Proof. Choose a $\delta \in (0,1)$ such that

$$\big|\log\big(\frac{\partial f}{\partial y}(x,y_1)\big) - \log\big(\frac{\partial f}{\partial y}(x,y_2)\big)\big| < \frac{|\chi_0|}{3} = -\frac{\chi_0}{3}$$

for $x \in [0,1]$ and $y_1, y_2 \in [0,1]$ with $|y_1 - y_2| \le \delta$. For $x \in [0,1]$ define

$$\tau(x) := \sup_{n\in\mathbf{N}} \left(-\frac{2n\chi_0}{3} + \sum_{j=0}^{n-1} \log\Big(\frac{\partial f}{\partial y}\big(F^j(x,0)\big)\Big) \right) \ .$$

As τ is the supremum of a sequence of measurable functions, $\tau : [0,1] \to \mathbf{R} \cup \{+\infty\}$ is measurable. By (27) we get $\tau(x) < +\infty$ for all x which satisfy $\chi_0(x) \le \chi_0$, which by (29) holds for almost all x.

Now define $b_0(x) := \min\big\{\delta, \delta e^{-\tau(x)}\big\}$, which is again a measurable function. Let $x \in [0,1]$ with $\tau(x) < \infty$. Choose a $y \in [0, b_0(x)]$. The mean value theorem and the choice of δ give

$$\big|\pi_2\big(F^n(x,y)\big)\big| \le \big|\pi_2\big(F^{n-1}(x,y)\big)\big| \left(\frac{\partial f}{\partial y}\big(F^{n-1}(x,0)\big)\right) \exp\big(-\frac{\chi_0}{3}\big) \ ,$$

if $\left|\pi_2\big(F^{n-1}(x,y)\big)\right| < \delta$. Hence using the definition of $\tau(x)$ we get by induction

$$\left|\pi_2\big(F^n(x,y)\big)\right| \le |y| \exp\Bigg(-\frac{n\chi_0}{3} + \sum_{j=0}^{n-1} \log\Big(\frac{\partial f}{\partial y}\big(F^j(x,0)\big)\Big)\Bigg)$$
$$\le b_0(x) \exp\Big(\tau(x) + \frac{n\chi_0}{3}\Big) \le \delta \exp\big(\frac{n\chi_0}{3}\big) < \delta.$$

Therefore $\lim_{n\to\infty} \pi_2\big(F^n(x,y)\big) = 0$ and $\omega(x,y) \subseteq I_0$. □

Lemma 3 readily implies $\lambda^2(B(I_0)) > 0$. We now show that $B(I_0)$ is dense in $[0,1]^2$ (in a strong sense).

Proposition 1. *Let* $F : [0,1]^2 \to [0,1]^2$ *be a map defined as in (10). Assume that there is an* $\varepsilon \in (0,1)$*, such that* $y \mapsto f(x,y)$ *is an* ε*–perturbation of the identity for every* $x \in [0,1]$*. Furthermore suppose that* $\chi_0 < 0$*, and that there exists an* I_0*–attracting periodic point* $p_0 \in [0,1]$*. Then for every* $\varrho \in (0,1)$ *there exists a measurable set* $R_0(\varrho) \subseteq [0,1]$ *satisfying the following properties.*

1. *For every* $x \in R_0(\varrho)$ *and for every* $y \in [0,\varrho]$ *we have* $\omega(x,y) \subseteq I_0$.
2. *For every nonempty open set* $U \subseteq [0,1]$ *we have* $\lambda^1(U \cap R_0(\varrho)) > 0$.

Proof. There exists an open interval U_0 with $p_0 \in \overline{U_0}$ and $T^j|_{U_0}$ is strictly monotonic for $j \in \{1,2,\dots,2l_0\}$, where l_0 is the period of p_0. Obviously $T^{2l_0}|_{U_0}$ is strictly increasing.

Suppose that $x_0 \in U_0$, $x_0 < p_0$ and $T^{2l_0}(x_0) \ge x_0$. Set $U_1 := (x_0, p_0)$. Then $T^{2l_0}U_1 \subseteq U_1$. As T is topologically transitive there exists an x with $\omega(x) = [0,1]$. Hence $T^k x \in U_1$ for some k. Therefore the sequence $(T^{k+2nl_0}x)_{n\ge 0}$ is monotonic, and hence there is a x' with $x' := \lim_{n\to\infty} T^{k+2nl_0}x$. This implies $\omega(x) = \{x', Tx', \dots, T^{2l_0-1}x'\}$, contradicting $\omega(x) = [0,1]$.

Hence $T^{2l_0}x < x$ for all $x \in U_0$ with $x < p_0$. An analogous proof shows $T^{2l_0}x > x$ for all $x \in U_0$ with $x > p_0$. This implies that $T^{2l_0} : V := U_0 \cap T^{-2l_0}U_0 \to U_0$ is bijective. There exists a function $S : U_0 \to V$ with $S = \big(T^{2l_0}|_V\big)^{-1}$. Obviously F^{2l_0} can be extended to a bijective continuous function $\overline{V} \times [0,1] \to \overline{U_0} \times [0,1]$. Define $G := \big(F^{2l_0}|_{V\times[0,1]}\big)^{-1}$. Then G is a continuous function.

Choose an arbitrary $\gamma \in (0,1)$. Let b_0 be as in Lemma 3. Since b_0 is positive and measurable there exists an $\eta_0 > 0$ such that $\lambda^1\big(\{x \in U_0 : b_0(x) \ge \eta_0\}\big) > 0$. Using the fact that p_0 is an I_0–attracting periodic point we get a closed interval C with $p_0 \in \operatorname{int} C$, such that $\pi_2\big(G(x,y)\big) - y > 0$ for all $x \in C \cap \overline{U_0}$ and all $y \in (0,\gamma]$. By a standard compactness argument we obtain a $j \in \mathbf{N}$ and an $\eta_1 \in (0,\gamma)$ with $S^j\overline{U_0} \subseteq C \cap \overline{U_0}$ and $\pi_2\big(G^j(x,y)\big) \ge \eta_1$ for every $(x,y) \in \overline{U_0} \times [\eta_0, 1]$. As $C \cap \overline{U_0} \times [\eta_1,\gamma]$ is compact we obtain that $\inf\{\pi_2\big(G(x,y)\big) - y : (x,y) \in C \cap \overline{U_0} \times [\eta_1,\gamma]\} > 0$. This implies that for every $\gamma > 0$ there exists an $N(\gamma) \in \mathbf{N}$ with

$$[0,\gamma] \subseteq \pi_2\Big(G^n\big(\{x\} \times [0,\eta_0]\big)\Big) \tag{30}$$

for every $n \ge N(\gamma)$ and every $x \in U_0$.

Now we choose a $\varrho \in (0, 1)$. Set $\beta := \inf_{(x,y)\in[0,1]^2} \frac{\partial f}{\partial y}(x, y)$. We get $\beta > 0$, since there is an $\varepsilon < 1$, such that $y \mapsto f(x, y)$ is an ε–perturbation of the identity for all $x \in [0, 1]$. For $k \in \mathbf{N}$ set $N_k := N(\gamma_k)$ for $\gamma_k = 1 - \beta^k(1 - \varrho)$. Define

$$R_0(\varrho) := \{x \in [0, 1] : \chi_0(x) \le \chi_0\} \cap \left(\bigcup_{k=1}^{\infty} T^{-k} S^{N_k}\big(\{x \in U_0 : b_0(x) \ge \eta_0\}\big)\right) . \tag{31}$$

Now let $U \subseteq [0, 1]$ be a nonempty open set. The topological transitivity of T together with Lemma 1 in [14] implies the existence of a $k \in \mathbf{N}$ with $p_0 \in \overline{T^k U}$. Set $U' := \{t \in U : \chi_0(t) \le \chi_0\} \cap T^{-k} S^{N_k}\big(\{t \in U_0 : b_0(t) \ge \eta_0\}\big)$. Then $U' \subseteq R_0(\varrho)$ and since the measure μ is absolutely continuous and satisfies $\inf h > 0$ we get $\lambda^1(U') > 0$. This shows (2).

Finally let $x \in R_0(\varrho)$ and $y \in [0, \varrho]$. Then there exists a k such that $x \in \{t \in [0, 1] : \chi_0(t) \le \chi_0\} \cap T^{-k} S^{N_k}\big(\{t \in U_0 : b_0(t) \ge \eta_0\}\big)$. The mean value theorem, (13), (26) and the definition of β give $1 - \pi_2\big(F^k(x, y)\big) \ge \beta^k(1 - y)$. Hence $\pi_2\big(F^k(x, y)\big) \le 1 - \beta^k(1 - \varrho)$, as $y \le \varrho$. Now (30) and Lemma 3 imply (1). □

Remark. Under the same assumptions as in Proposition 1, but replacing the assumptions on χ_0 and p_0 by $\chi_1 < 0$ and the existence of an I_1–attracting periodic point p_1, we obtain for every $\varrho \in (0, 1)$ a measurable set $R_1(\varrho)$, such that (2) holds (with $R_0(\varrho)$ replaced by $R_1(\varrho)$), and $\omega(x, y) \subseteq I_1$ for every $x \in R_1(\varrho)$ and every $y \in [1 - \varrho, 1]$.

Proposition 2. *The basins $B(I_i)$, $(i = 0, 1)$ are sets of first category.*

Proof. Obviously, $B(I_1) \subseteq \bigcup_n A_n$ with $A_n = \bigcap_{k\ge n} F^{-k}([0, 1] \times [\frac{2}{3}, 1])$. Hence the basin of I_1 is contained in the countable union of the closed sets A_n. Since $B(I_0)$ is dense in $[0, 1]^2$ by Proposition 1, the sets A_n are nowhere dense. Hence $B(I_1)$ is of first category. □

5. The union of the basins has full measure

Let $B := B(I_0) \cup B(I_1)$ be the union of the basins of attraction of I_0 and I_1. In this section we will prove that B has full λ^2-measure. To get this result we need some information about the slope of images under F^n of horizontal line segments, which is established in Lemma 7 and Lemma 8 below.

The following result can be found as "Koebe principle" on page 258 in [8].

Lemma 4. *Suppose T is S-unimodal. Fix an open interval I and a closed interval $J \subset I$. Then there is some constant c such that for any open interval K and any n such that $T^n : K \to I$ is bijective one has*

$$\frac{1}{c} \le \left|\frac{(T^n)'(x)}{(T^n)'(y)}\right| \le c$$

for all $x, y \in K$ whose images $T^n(x)$ and $T^n(y)$ are in J.

A similar result holds if T is expanding (see Lemma 3 of [13]).

Lemma 5. *Let T be expanding and piecewise $C^{1+\delta}$. Then there is a $d > 0$ such that*

$$\frac{1}{d} \leq \left| \frac{(T^n)'(x)}{(T^n)'(y)} \right| \leq d$$

if x and y are in the same element Z of $\mathcal{Z}_n$.

Lemma 6. *Suppose that T is piecewise monotonic . If μ is an ergodic invariant measure with $h_\mu > 0$, then there is an open interval E and a closed interval $C \subset E$ and a set L with $\mu(L) = 1$ such that for all $x \in L$ infinitely many $n \in \mathbf{N}$ exist with $T^n(Z_n(x)) = E$ and $T^n(x) \in C$.*

Proof. In the proof of Proposition 2 in [15] it is shown, that an open interval $\hat{E}$, contained in an element of $\mathcal{Z}$, and a closed interval $\hat{C} \subset \hat{E}$ and a set L with $\mu(L) = 1$ exist such that for all $x \in L$ there are infinitely many n with $T^n(Z_{n+1}(x)) = \hat{E}$ and $T^n(x) \in \hat{C}$. Now the assertion follows with $E = T(\hat{E})$ and $C = T(\hat{C})$. □

For $x \in L$ denote by $\mathcal{N}(x)$ the subset of $\mathbf{N}$ for which the assertion in Lemma 6 holds. Set $p_n = \left(T^n|_{Z_n(x)}\right)^{-1}$. Choose an arbitrary $y \in (0, 1)$ and set $\gamma_n = p_n(C) \times \{y\}$. We define the map $k_n : C \to [0, 1]$ by $F^n(\gamma_n) = \{(t, k_n(t)) : t \in C\}$. Then $k_n(t) = f(F^{n-1}(p_n(t), y))$ follows.

Lemma 7. *Suppose that $T : [0, 1] \to [0, 1]$ is expanding, topologically transitive and piecewise $C^{1+\delta}$. Furthermore suppose that $f : [0, 1]^2 \to [0, 1]^2$ is C^1 and that the maps $y \mapsto f(x, y)$ are ε-perturbations of the identity for all $x \in [0, 1]$ with $\varepsilon < \alpha$, where α is the expansion rate of T. Then we get*

$$\sup_{x \in L} \sup_{n \in \mathcal{N}(x)} \sup_{t \in C} \left| \frac{d}{dt} k_n(t) \right| = S < \infty.$$

Proof. We compute using Lemma 6 in [27] and with $s = p_n(t)$

$$\begin{aligned}
\frac{d}{dt} k_n(t) &= \frac{d}{ds} f(F^{n-1}(s, y)) \frac{d}{dt} p_n(t) \\
&= \frac{1}{(T^n)'(s)} \sum_{i=0}^{n-1} \frac{\partial f}{\partial x}(F^i(s, y)) \prod_{j=0}^{i-1} T'(T^j(s)) \prod_{j=i+1}^{n-1} \frac{\partial f}{\partial y}(F^j(s, y)) \\
&= \sum_{i=0}^{n-1} \frac{\frac{\partial f}{\partial x}(F^i(s, y))}{T'(T^i(s))} \prod_{j=i+1}^{n-1} \frac{\frac{\partial f}{\partial y}(F^j(s, y))}{T'(T^j(s))}.
\end{aligned}$$

Now we get using concavity of the logarithm function and setting $C_1 := \sup |\frac{\partial f}{\partial x}| < \infty$

$$\begin{aligned}
\left| \frac{d}{dt} k_n(t) \right| &\leq C_1 e^{-\alpha} \sum_{i=0}^{n-1} \exp \left(\sum_{j=i+1}^{n-1} \log(|\frac{\partial f}{\partial y}(F^j(s, y))|) - \log |T'(T^j(s))| \right) \\
&\leq C_1 e^{-\alpha} \sum_{i=0}^{n-1} \exp\left((n - i - 1)(\log(1 + \varepsilon) - \alpha)\right) \\
&\leq C_1 e^{-\alpha} \sum_{i=0}^{\infty} \exp\left(i(\varepsilon - \alpha)\right) < \infty
\end{aligned}$$

since $\varepsilon - \alpha < 0$ holds. Since the estimate is independent of $x \in L$, $n \in \mathcal{N}(x)$ and $t \in C$ the lemma is proved. □

Now we give an analogous result if T is an unimodal map.

Lemma 8. *Suppose that* $T : [0,1] \to [0,1]$ *is S-unimodal with nonflat critical point, topologically transitive and such that* $\liminf_{n\to\infty} \sqrt[n]{|(T^n)'(1)|} > 1$. *Furthermore suppose that* $f : [0,1]^2 \to [0,1]^2$ *is* C^1 *and that the maps* $y \mapsto f(x,y)$ *are* ε*-perturbations of the identity for all* $x \in [0,1]$ *with* $\varepsilon < \alpha$ *where* α *is the expansion rate of* T *defined in Lemma 1. Then*

$$\sup_{x\in L} \sup_{n\in\mathcal{N}(x)} \sup_{t\in C} \left|\frac{d}{dt}k_n(t)\right| = S < \infty.$$

Proof. Choose $x \in L$ and $n \in \mathcal{N}(x)$ and $t \in C$. The same estimation as in the proof of Lemma 7 gives with $C_1 = \sup|\frac{\partial f}{\partial x}| < \infty$

$$\begin{aligned}\left|\frac{d}{dt}k_n(t)\right| &\le \sum_{i=0}^{n-1} \left|\frac{\partial f}{\partial x}(F^i(s,y))\right| \prod_{j=i+1}^{n-1} \left|\frac{\partial f}{\partial y}(F^j(s,y))\right| \prod_{j=i}^{n-1} \frac{1}{|T'(T^j(s))|}\\ &\le C_1 \sum_{i=0}^{n-1} e^{\varepsilon(n-i-1)} \frac{1}{|(T^{n-i})'(T^i(s))|}\end{aligned}$$

with $s = p_n(t)$, since $1 + \varepsilon \le e^{\varepsilon}$.

For $0 \le i \le n$ set $V_{n-i} = T^i(p_n(C))$. Then $T^{n-i}(V_{n-i}) = C$ and $V_{n-i} \subseteq Z_{n-i}(T^i(x))$ holds. For $0 \le i \le n-1$ the mean value theorem gives

$$\frac{1}{|(T^{n-i})'(\vartheta)|}|T^{n-i}(V_{n-i})| = |V_{n-i}|$$

for a $\vartheta \in V_{n-i}$ and we get using Lemma 4

$$\frac{1}{|(T^{n-i})'(T^i(s))|} \le \frac{1}{|C|c}|V_{n-i}| \le \frac{1}{|C|c}|\mathcal{Z}_{n-i}|$$

for $s = p(t)$, where c is the constant of Lemma 4. Choose $\alpha' \in (\varepsilon, \alpha)$. Using (2) of Lemma 1 we obtain the existence of a constant C_2 such that $|\mathcal{Z}_l| \le C_2 e^{-\alpha' l}$ holds for all $l \ge 0$. This gives

$$\begin{aligned}\left|\frac{d}{dt}k_n(t)\right| &\le C_1 \sum_{i=0}^{n-1} e^{\varepsilon(n-i-1)} \frac{C_2}{|C|c} e^{-\alpha'(n-i)}\\ &\le C_3 \sum_{i=0}^{n-1} \exp((\varepsilon - \alpha')i)\\ &\le C_3 \sum_{i=0}^{\infty} \exp((\varepsilon - \alpha')i) < \infty\end{aligned}$$

which completes the proof since the estimate is independent of $x \in L$, $n \in \mathcal{N}(x)$ and $t \in C$. □

Proposition 3. *Suppose that* $T : [0,1] \to [0,1]$ *is topologically transitive and either expanding and piecewise* $C^{1+\delta}$ *or S-unimodal with nonflat critical point and* $\liminf_{n\to\infty} \sqrt[n]{|(T^n)'(1)|} > 1$. *Furthermore suppose that* $f : [0,1]^2 \to [0,1]^2$ *is* C^1 *and satisfies the conditions (a), (b), (c) from Theorem 1 where* α *is the expansion rate of* T. *Then*

$$\lambda^2(B) = 1.$$

Proof. Denote the complement of B by D. Because of Fubini's theorem and Lebesgue's density theorem it suffices to show for $y \in (0,1)$ that $D_y := \{x : (x,y) \in D\}$ containes λ^1-almost no points with Lebesgue density.

Remember that x is a point of Lebesgue density for a set $A \in [0,1]$ if $\lim_{r\to 0} \frac{\lambda^1(U(x,r)\cap A)}{\lambda^1(U(x,r))} = 1$, where $U(x,r)$ denotes an interval containing x with diameter at most r. Lebesgue's density theorem states that almost every point in a set of positive Lebesgue measure is a point of Lebesgue density for this set.

Fix $y \in (0,1)$. We will show that $D_y \cap L$ contains no points of Lebesgue density, where L is the set found in Lemma 6. Under both assumptions for T there is an ergodic invariant and absolutely continuous measure μ, whose density function is bounded away from zero and which has positive entropy (see Lemma 2 and Lemma 1). Hence $\lambda^1(L) = \mu(L) = 1$ holds.

Fix $x \in D_y \cap L$. Divide the interval C found in Lemma 6 into finitely many subintervals with length at most $\frac{1}{3S}$ where S is the constant from Lemma 7 for the expanding case, or Lemma 8 for the unimodal case. Let I be one of these intervals such that $T^n(x) \in I$ for an infinite subset $\mathcal{M}(x)$ of $\mathcal{N}(x)$. For $n \in \mathcal{M}(x)$ set $U_n(x) = (T^n|_{Z_n(x)})^{-1} I$ and $\gamma_n = U_n(x) \times \{y\}$. Lemma 7, if T is expanding, and Lemma 8, if T is S-unimodal, imply

$$\pi_2(F^n\gamma_n) \subseteq [\pi_2(F(x,y)) - 1/3, \pi_2(F(x,y)) + 1/3].$$

Choose $\varrho > 1/2 + 1/3$ and suppose $\pi_2(F(x,y)) \le 1/2$. Remember the definition of $R_0(\varrho)$ for $\varrho \in (0,1)$ and set $B_0(\varrho) = R_0(\varrho) \times [0,\varrho)$. Using the fact that B is invariant under F we compute

$$\begin{aligned} T^n(U_n(x) \cap D_y^c) &= T^n((\gamma_n \cap D^c)_y) = \pi_1(F^n(\gamma_n \cap B)) \\ &= \pi_1(F^n\gamma_n \cap B) \supseteq \pi_1(F^n\gamma_n \cap B_0(\varrho)) = I \cap R_0(\varrho). \end{aligned}$$

With use of Lemma 5 in the expanding case and Lemma 4 in the unimodal case we get for arbitrary $n \in \mathcal{M}(x)$

$$\begin{aligned} \frac{\lambda^1(U_n(x) \cap D_y^c)}{\lambda^1(U_n(x))} &\ge \frac{\lambda^1(I \cap R_0(\varrho))}{\sup_{\zeta\in U_n(x)} |(T^n)'(\zeta)|} \frac{\inf_{\zeta\in U_n(x)} |(T^n)'(\zeta)|}{\lambda^1(I)} \\ &\ge \frac{1}{c}\frac{\lambda^1(I \cap R_0(\varrho))}{\lambda^1(I)} > 0 \end{aligned}$$

where the last inequality follows from Proposition 1. This implies

$$\limsup_{n\to\infty} \frac{\lambda^1(U_n(x) \cap D_y^c)}{\lambda^1(U_n(x))} > 0$$

and hence

$$\liminf_{n\to\infty} \frac{\lambda^1(U_n(x) \cap D_y)}{\lambda^1(U_n(x))} < 1.$$

If $\pi_2(F(x, y)) > 1/2$ holds then the same argument using $B_1(\varrho) = R_1(\varrho) \times (1 - \varrho, 1]$ instead of $B_0(\varrho)$ completes the proof. □

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