SOME THOUGHTS ON SUSTAINABLE/LEARNABLE EQUILIBRIA

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Abstract. In this note I explore a (group of) new refinement(s) of Nash equilibrium which could be called learnable or sustainable. Every strict equilibrium is sustainable, but many mixed equilibria are not. However, almost every strategic game has at least one sustainable equilibrium. Evolutionary/adaptive dynamic considerations are central in our investigation. Moreover, the special role of potential games and supermodular games in this respect is recognized.
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Myerson (1996) suggested to study 'sustainable' equilibria, which 'can persist when played in a culturally familiar environment'.

More precisely he gropes for a new refinement of Nash equilibrium with something like the following properties:

(1) Every strict equilibrium is sustainable.

(2) For 2×2 games with three equilibria, only the two strict equilibria are sustainable, but not the mixed one.

(3) If a game has a unique equilibrium, it is sustainable.

(4) Almost every (finite) strategic game has at least one sustainable equilibrium.

Hence the concept he is looking for is somewhat between usual equilibrium refinement and equilibrium selection: While equilibrium selection theories aim to single out a unique equilibrium — in contrast to (1) —, most of the classical equilibrium refinement literature deals with properties that hold for every regular equilibrium — thus contrasting (2). A notable exception is the concept of *persistent equilibrium* of Kalai and Samet (1984) which does not fit too well into the hierarchy of other refinements (compare van Damme (1991)) and which enjoys (2). However Myerson (1996) gives some examples where persistent equilibria do not fit his intuition of a sustainable equilibrium (see Ex 4 below).

Another such 'intermediate refinement' is that of an *evolutionarily stable strategy* (ESS)[1] which enjoys so many stability properties that one should require

(5) Every ESS is sustainable.

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[1]Maynard Smith's concept of an ESS is relevant only for symmetric (2 person) games played within one population. For the classical situation of N person strategic games (played between N disjoint player populations) this concept reduces to that of a strict equilibrium. In this note, a 'game' will always refer to one of these two types of finite strategic games.

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However, ESS suffer from an existence problem, and do not satisfy requirements (3) and (4).

Myerson (1997) describes several ideas how to formalize this concept and discusses the problems he encountered. In the present note I further explore his ideas, make them more precise, suggest solutions, and relate them to independent ideas of mine (section III).

I. Axioms

Myerson asks how a formal definition of rational behaviour in culturally familiar context could look like that captures the above intuition. I suggest to consider first the following, maybe crucial property.

(6) *A (regular) equilibrium E is sustainable in a game iff it is sustainable in the game restricted to the support of E.*

This axiom could be called *consistency w.r.t. restrictions of the strategy set*: For an equilibrium to be sustainable it should be completely irrelevant whether we omit unused strategies or, by enlarging the game, add new strategies (for some of the players) that are inferior at that equilibrium. This conforms with and extends the intuition of (1) that every strict equilibrium should be sustainable: In a strict equilibrium, each player uses only one pure strategy. Omitting the others, leads to a trivial game with only one strategy profile which is an equilibrium. Hence (3) and (6) imply (1).

Let us reformulate (6) in terms of an equivalence relation between pairs (G, E) of games and equilibria: $(G, E) \simeq (G', E')$ if the game G restricted to the support[2] of E is the same (i.e. has same payoff functions, modulo renumberings of players and strategies) as the game G' restricted to the support of E'. Then (6) means that sustainability is invariant under this equivalence relation.

We could then define:

(S) *A (regular) equilibrium E of a game G is sustainable iff there exists an equivalent pair (G', E') such that E' is the unique equilibrium of G'.*

Since a strict equilibrium is sustainable in this sense, properties (1) and (3) hold. That the mixed equilibrium in (2) is not sustainable in this sense will follow from the next section. The validity of (4) seems less easy to resolve:

Conjecture 1. *Every regular game[3] has at least one equilibrium which is sustainable according to (S).*

Conjecture 1 would follow from Conjecture 2 below.

II. Dynamics and index

Another of Myerson's approaches is based on **evolutionary/adaptive dynamics**, motivated from the recent evolutionary and learning literature and starts from (5): ESS are asymptotically stable for most adaptive dynamics such as the replicator dynamics, the best response dynamics, the Nash dynamics, etc., see e.g., Hofbauer and Sigmund (1988) and Weibull (1995) for some old results, and Hofbauer

[2]If E is not regular then instead of the support one should consider here the 'extended support', i.e. the set of all best replies to E. Compare Example 3

[3]A game is called regular (see van Damme (1991) or Ritzberger (1994)) if all its equilibria are regular. Almost every strategic game is regular.

(2000) for more recent ones. He quickly drops this approach because 'no simple adaptive-dynamics models are known for which convergence to equilibrium can be proven from generic initial conditions.' This is right[4], but — I claim — it doesn't matter so much. In Theorem 1 below I describe a result that distinguishes roughly half of the equilibria as being locally asymptotically stable for some adjustment process. Admittedly, the process is not very explicit and might have complicated (non-convergent) dynamics far from the equilibria. But once the distribution of play gets close to one of these equilibria, then it will stay there.

The characterization of these 'good' equilibria uses another approach sketched at the end of Myerson's paper, based on the concept of an **index** for equilibria. Such a concept, envisaged by Myerson (1996) was already formalized by Shapley (1974) in his description of the Lemke–Howson algorithm. For a regular equilibrium, this index is either $+1$ or -1, and the index sum across all equilibria of a given game is 1. In particular this explains the *odd number theorem* of Wilson (1971) and Harsanyi (1973). For the class of 2×2 games in (2), the mixed equilibrium has index -1, while strict equilibria generally have index $+1$. A different definition of the index, based on the replicator dynamics, and drawing on Poincaré and Brouwer, was suggested in Hofbauer and Sigmund (1988) for two-person games and extended by Ritzberger (1994) to N person games and components of equilibria. [This dynamic approach parallels Dierker's (1972) analysis of equilibria of exchange economies.] See also Gul et al. (1993) for yet another definition of the index. Recently, Govindan and Wilson (1997) have shown that these two (or three) definitions agree and (at least for 2 person games) the index of an (isolated) equilibrium is actually the same for EVERY dynamics on the space of mixed strategy profiles whose fixed points are precisely the Nash equilibria. For N person games see deMichelis and Germano (1996). Now it follows directly from the definition of the (Poincaré) index of a regular equilibrium of a differentiable vector field, namely $\mathrm{ind}(E) = \mathrm{sgn}\det(-Df(E))$, and the Hartman-Grobman theorem that index -1 means that E is a saddle point with an odd (and hence at least one) dimensional unstable manifold. Hence *equilibria with index -1 are unstable for* EVERY *dynamics*. Thus they fail to be sustainable. Therefore we may add a further requirement

(7) *An equilibrium with index -1 is not sustainable.*

This extends the intuition of excluding the mixed equilibrium in (2) to arbitrary games.

The question remains whether all equilibria with index $+1$ deserve to be called sustainable. The positive result in this direction, announced above, is the following.

Let us call a *Nash field*[5] any vector field which generates a semiflow on the set of all mixed strategy profiles, whose equilibria are precisely the Nash equilibria.

Theorem 1. *For each regular game there exists a Nash field for which all equilibria with index $+1$ are locally asymptotically stable.*

The index is relevant also for the considerations in section I. If (G, E) and (G', E') are equivalent and both equilibria E, E' are regular then their index is the same. Hence if E is sustainable according to (S) and since the unique equilibrium E' of the game G' has index $+1$, E must have index $+1$. (In particular, the mixed

[4]Such adaptive dynamics are rather unlikely to exist, see Hofbauer and Swinkels (1995).

[5]This definition differs from that given in Ritzberger (1994): He uses the term *Nash field* as a synonym for the replicator dynamics. The above definition may be more reasonable to connect with the name of Nash.

equilibrium in (2), which has index -1, cannot be sustainable in this sense.) I believe the converse is also true:

Conjecture 2. *A regular equilibrium has index* $+1$ *if and only if it is sustainable as defined in (S).*

III. Evolutionary/adaptive dynamics

However, the above result has some deficiencies: The stabilization result of Theorem 1 is only of a purely local nature; and furthermore it would be desirable to choose the Nash field from some class of evolutionary/adaptive dynamics that mushroomed in the last years. In the following I show that this CANNOT be done in general. The obstructions that I present are related to the following, purely strategic objection: There are special classes of games for which the above division into 'good' and 'bad' equilibria provided by the index is not the natural one. Two such classes are *potential games* and *supermodular games*.

A *partnership game* (Hofbauer and Sigmund, 1988) or *team game* (van Damme, 1996) is a game where every player has the same payoff function. A (weighted) *potential game* (Monderer and Shapley, 1996) or rescaled partnership game (Hofbauer and Sigmund, 1988) is any game that is strategically equivalent to a partnership game. The potential function of such a game is the common payoff function of the equivalent game. Now a natural refinement of Nash equilibrium for such potential games which captures all the desirable properties (1-5) are the (strict) local maxima of the common payoff or potential function[6].

Supermodular games, on the other hand, have a smallest and a largest equilibrium (with respect to the order defined by stochastic dominance). Both are pure and generically even strict equilibria. Hence for these games only the strict equilibria look like good choices[7].

One class of evolutionary/adaptive dynamics, introduced by Swinkels (1993), are myopic adjustment dynamics (MAD), where the vector field points towards a (myopic) better reply for each player position. This class is natural and probably broad enough to serve our purpose of an evolutionary justification of (a suitable subset of) Nash equilibria.

Here we collect some results[8] in this direction.

Theorem 2.
a) *Every strict equilibrium is asymptotically stable for EVERY MAD.*
b) *Let E be the unique equilibrium of a game with linear incentives. Then there exists at least one MAD for which E is globally asymptotically stable.*

[6] Actually this seems the natural concept of equilibrium in potential games. The *global* maximum provides a natural choice for selecting a unique equilibrium in generic potential games, see Monderer and Shapley (1996), van Damme (1996), Hofbauer and Sorger (1999) and Ui (2000). However in the spirit of the present paper, if a population sits at a local maximum it cannot move away by small mutation steps and hence will stay in this 'familiar environment'. So every local maximum is considered sustainable.

[7] This is supported by the dynamic approach: In supermodular games, all mixed equilibria are unstable for most reasonable dynamics, and for some dynamics convergence to strict equilibria can be shown from almost every initial condition.

[8] As a curiosity, I remark that for a symmetric game with an interior equilibrium p, the straightforward dynamics $\dot{x} = p - x$, which points straight towards the equilibrium from each point, is a MAD iff p is an ESS.

c) *If E is an equilibrium which is robust against equilibrium entrants (REE)*[9] *and the game has linear incentives, then there exists a MAD for which E is asymptotically stable.*

Motivated by these results, I propose the definition

(L) *An equilibrium E of a game is* **learnable** *if there exists a myopic adjustment dynamics for which E is asymptotically stable.*

Then the above can be summarized as:

Every REE is learnable. Every learnable equilibrium has index +1.

Example 1. The rock–scissors–paper game with cyclic symmetry, considered as symmetric one population game.

$$\begin{array}{cccc} & R & S & P \\ R & a & b & c \\ S & c & a & b \\ P & b & c & a \end{array} \qquad (c < a < b) \tag{Ex1}$$

This game has a unique equilibrium $E = \frac{1}{3}(R+S+P)$. E has index +1 and satisfies (S). For $b + c > 2a$, E is an ESS, and it is globally asymptotically stable for most 'classical' dynamics, like replicator dynamics, BR dynamics, or Nash dynamics. For $b+c < 2a$, on the other hand, E is unstable for all these dynamics. Solutions spiral out to some 'limit cycle', compare Gilboa and Matsui (1991), Gaunersdorfer and Hofbauer (1995), Hofbauer (1995), Berger and Hofbauer (1996), Oechssler (1997). Hence these 'classical' evolutionary dynamics do not support equilibrium in this case[10]. This example illustrates why we have to allow for a much larger class of adaptive dynamics above, such as MADs, if we want to justify equilibrium in this way. According to Theorem 2b there is at least one MAD for which E is globally asymptotically stable. In this sense, E is learnable.

After these positive results we now turn to obstructions: games whose special structure imply that not every index +1 equilibrium is learnable.

Potential games.

For every potential game, the potential function increases monotonically along every solution of EVERY MAD (see e.g., Hofbauer (1995b), Hofbauer and Swinkels (1995), Sandholm (2000)). This implies convergence to equilibrium from every initial condition. Moreover, the maxima of the potential function are locally asymptotically stable, while all other extreme points of the potential function are unstable for all MADs.

Theorem 3. *For a (weighted) potential game an isolated equilibrium E is learnable if and only if it is a (strict local) maximum of the potential function. In this case, E is asymptotically stable for ALL myopic adjustment dynamics.*

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[9]An equilibrium E was called REE by Swinkels (1992), if is the unique equilibrium of the game restricted to all best replies to E. Hence an REE satisfies (S) and has index +1. REE, like ESS, seems like a good candidate for the notion of sustainable equilibrium. REE captures the essential intuition of ESS that is relevant for rational game theory. Even though it is a much weaker concept it is still too restrictive for our purpose as it doesn't satisfy the existence axiom (4). Anyway, thanks to the result c) I feel licenced to add the requirement

(7) Every REE is sustainable.

[10]With good reason: Along these 'limit cycles' the population earns more on average than at the equilibrium E.

For (asymmetric) N person games the potential function is N–linear. Hence maxima are attained only at pure equilibria, and only those can be learnable. For symmetric 2 person games, the potential function is a quadratic form which can have maxima also at mixed equilibria. In this case, the maxima correspond to the ESS or the REE of the game.

Hence for potential games the (Poincaré–Shapley) index is not suitable to characterize stability as in Theorem 1. However, for potential games a more refined index is available: the *Morse index* of the potential function at the equilibrium: It is 0 for (local) maxima, and k for extreme points for which the Hessian matrix has Morse index k (k positive eigenvalues). The two concepts of index are related through $(-1)^{\text{Morse index}} = \text{index}$. Hence an equilibrium with even numbered Morse index $2, 4, \ldots$ (e.g. an interior saddle point of the potential function in $n \times n$ bimatrix games, if n is odd) has index $+1$, but is not learnable. The simplest example of such a discrepancy between Theorems 1 and 3 is

Example 2. Consider the symmetric 3×3 coordination game with payoff matrix

$$\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} \qquad a, b, c > 0. \tag{Ex2}$$

This game has 7 equilibria (if viewed as a one population game[11]) all being regular: Three strict, three pairwise mixtures (index -1) and a completely mixed one, say E, with index $+1$. The completely mixed equilibrium E is the global minimum of the potential function $ax_1^2 + bx_2^2 + cx_3^2$, and as such is a repellor for each MAD. Hence it is not learnable. However, according to Theorem 1 it can be made locally asymptotically stable for a suitably chosen Nash field. Such a Nash field must have a closed orbit around E to bound its basin of attraction. It is somewhat doubtful whether oscillatory dynamics is appropriate for a potential game.

Hence the concept of 'learnable' equilibrium seems more reasonable for this game than the notions of 'sustainability' considered in sections I and II.

For games without potential function, the above discrepancy seems to disappear, at least for two–person games.

Theorem 4.
a) For generic asymmetric two person games, every equilibrium with index $+1$ is learnable.
b) For every (symmetric or asymmetric) 3×3 game that is NOT a potential game, every regular equilibrium with index $+1$ is learnable.

In a) the genericity assumption is needed to exclude the rare set of potential games for which the conclusion is wrong, according to Theorem 3. I believe (and hope to give a complete proof soon) that the more precise statement in b) holds for two person games with any number of strategies. This means that for 2 players, potential games would be the only exception for 'learnable = sustainable'.

For N person games with $N > 3$ this is not true. This is due to further obstructions, as will be shown below.

The special role of potential games is highlighted also by the following result.

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[11] A similar analysis applies to the corresponding two population game. E is then a saddle point of Morse index 2 of the potential function $\pi(x, y) = ax_1y_1 + bx_2y_2 + cx_3y_3$, and again not learnable.

Theorem 5. *Let E be an interior equilibrium of a symmetric 2 person game which is asymptotically stable for* every MAD. *Then the game has a (concave) potential function.*

Supermodular (binary choice) games.

Consider an N person binary choice game (each of the N players has two pure strategies: A_i and B_i). Suppose both strategy profiles A and B are strict equilibria, and that the best reply correspondence is monotone, meaning that the game is strictly supermodular. Then we have the following generalization of (2).

Theorem 6. *Let E be an interior equilibrium of a supermodular binary choice game. Then E is not learnable, i.e., it is unstable for all MADs.*

Whereas for $N = 2, 3$ such an interior equilibrium has index -1, for $N \geq 4$ players it is easy to give examples of bipolar games with an interior equilibrium with index $+1$. According to Theorem 6 these are not learnable. More precisely we have

Corollary. *For $N \geq 4$ there is an open set of 2^N games that have interior equilibria with index $+1$ which are NOT learnable.*

Hence for $N \geq 4$ players, index $+1$ does not even generically imply learnability — in contrast to Theorem 4a for two players. Still, there might be a positive answer to the following

Conjecture 3. A generic N person game has at least one learnable equilibrium.

For supermodular games with more than two strategies per player the above obstruction seems to break apart for the broad class of MADs, even though for the 'standard' dynamics like replicator equation and BR dynamics, again only pure equilibria can be stable, see also Ex 6 below. Hence, while the above definition of 'learnable' equilibrium perfectly matches our intuition about 'good' and 'bad' equilibria for potential games, it is too broad to do so for general supermodular games.

IV. More examples

Here I consider some more examples, in order to illustrate and compare the different versions of 'sustainable' and 'learnable' equilibria.

Example 3. An example from Myerson (1996):

$$\begin{array}{cccc} & A_2 & B_2 & C_2 \\ A_1 & 4,0 & 0,4 & 3,0 \\ B_1 & 0,4 & 4,0 & 3,0 \\ C_1 & 0,3 & 0,3 & 3,3 \end{array} \tag{Ex3}$$

This 3×3 game has one regular equilibrium $E = (\frac{1}{2}A_1 + \frac{1}{2}B_1, \frac{1}{2}A_2 + \frac{1}{2}B_2)$. It satisfies (S) (the 2×2 subgame is a 'matching pennies' game with a unique equilibrium) and hence has index $+1$. The only other equilibrium $C = (C_1, C_2)$ is pure, not perfect, not essential, etc. and has index 0. Hence C cannot be asymptotically stable for any dynamics. In the myopic best response dynamics (and probably most other 'classical' game dynamics), E attracts all orbits, except the constant solution starting at C.

Myerson uses this example as illustration that the naive distinction 'prefer pure equilibria (if exist) over mixed' is unreasonable. Also in the present context, E is the unique sustainable equilibrium of the game. The possibility of unreasonable pure equilibria like C is the reason why in section I, in (6) and (S), I cautiously restrict to regular equilibria.

Example 4. Another example from Myerson (1996) is the following 2×4 game:

$$\begin{array}{ccccc} & A_2 & B_2 & C_2 & D_2 \\ A_1 & 0,6 & 1,4 & 0,3 & 1,0 \\ B_1 & 1,0 & 0,3 & 1,4 & 0,6 \end{array} \tag{Ex4}$$

There are three equilibria: $E_1 = (\frac{3}{5}A_1 + \frac{2}{5}B_1, \frac{1}{2}A_2 + \frac{1}{2}B_2)$, $E_2 = (\frac{1}{2}A_1 + \frac{1}{2}B_1, \frac{1}{2}B_2 + \frac{1}{2}C_2)$, $E_3 = (\frac{2}{5}A_1 + \frac{3}{5}B_1, \frac{1}{2}C_2 + \frac{1}{2}D_2)$.

From the symmetry of the game and index sum $= +1$, Myerson identifies the index of the equilibria: E_1 and E_3 have index $+1$, whereas E_2 has index -1. Then he asks, 'is there any reasonable sense in which E_2 is less sustainable than the other two equilibria'. We can easily answer his question: The equilibria E_1 and E_3 live in 'matching pennies' type subgames hence both are sustainable according to (S). On the other hand, E_2 lives in a coordination type subgame. (This again determines their index.) Both E_1 and E_3 are asymptotically stable for the BR dynamics (see Hofbauer, 1995a). The index -1 equilibrium E_2 is unstable for every dynamics[12]. I believe that from almost all initial conditions, BR solutions will converge to either E_1 or E_3.

Example 5. Consider again a symmetric 3×3 game with payoff matrix

$$M = \begin{pmatrix} 0 & 6 & -4 \\ -3 & 0 & 5 \\ -1 & 3 & 0 \end{pmatrix}. \tag{Ex5}$$

There are three equilibria: A strict one at $E_1 = (1, 0, 0)$, another one at $E_2 = (\frac{4}{5}, 0, \frac{1}{5})$ and an interior one at $E_3 = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. E_2 has index -1, E_1 and E_3 have index $+1$. The best response regions are shown in figure ??. From E_3 there are best response paths leading away to the strict equilibrium E_1, and one path leading to E_2. Hence E_3 is not robust against equilibrium entrants. Is E_3 sustainable? It is in the sense of (S): One can enlarge the game by adding one strategy such that E_3 is the unique equilibrium in this bigger game. E_3 is locally stable for some Nash field according to Theorem 1. According to Theorem 4 it is learnable. Indeed, it is asymptotically stable for the replicator equation, see Hofbauer and Sigmund (1988).

Example 6. Consider symmetric 3×3 coordination games

$$M = \begin{pmatrix} 0 & -a_{12} & -a_{13} \\ -a_{21} & 0 & -a_{23} \\ -a_{31} & -a_{32} & 0 \end{pmatrix} \tag{Ex6}$$

with $a_{ij} > 0$ so that all three pure strategies are strict equilibria. Suppose there is also an interior equilibrium E. Then E has index $+1$. Hence it is locally stable for

[12] All three equilibria of this game are persistent in the sense of Kalai and Samet (1984). Hence this refinement concept does not rule out the unreasonable equilibrium E_2.

some Nash field, according to Theorem 1. If Conjecture 2 is true, then E satisfies also the condition (S).

The simplex Δ is divided into three best response regions (quadrilaterals sharing the common vertex E). Hence E looks very unstable. If

$$a_{12} + a_{23} + a_{31} = a_{21} + a_{32} + a_{13} \qquad (*)$$

then the game has a potential function (see Hofbauer and Sigmund, 1988) and E, being its minimum, cannot be learnable according to Theorem 3. However, if (*) is violated, then according to Theorem 4b, E is learnable! There exist MADs for which E is locally asymptotically stable, usually with a rather small basin of attraction, again bounded by a closed orbit. As in Example 1 above, oscillations may look counter–intuitive for such games, no matter whether (*) holds or not. Indeed, one can show that for certain subclasses of MADs such as monotone selection dynamics, or payoff monotone dynamics (see e.g., Weibull, 1995), E is repelling, and almost all orbits converge to one of the strict equilibria. Does E really deserve to be called sustainable or learnable?

An open subset of the payoff matrices (Ex6) give rise to supermodular games. For such games the natural notion of 'sustainability' or 'learnability' selects only pure equilibria, as argued above. Hence the above definition of 'learnable' equilibria eliminates the unreasonable equilibria of potential games but not for other games. This example suggests that finer notions of 'sustainability' or 'learnability' than the two described above are out there.

V. Tracing

Myerson suggests one more interesting possibility to characterize (or define) 'sustainability' – using the **tracing procedure** of Harsanyi and Selten (1988): He suggests to distinguish those equilibria that can be reached from an *open* set of priors. I suggest to call these equilibria **tracible**. They obviously satisfy (1–3). Since tracing leads to an equilibrium from any prior, at least one equilibrium (component) is obviously reached from a set of priors which has positive measure. (Most likely this will also have nonempty interior). In this slightly weaker sense, also existence (4) holds. Tracible equilibria probably do not satisfy the consistency axiom (6).

Conjecture 4. Every ESS and every REE is tracible. Equilibria with index -1 are not tracible.

Problem. Characterize the tracible equilibria of potential and supermodular games.

Here I can only point out that not every index $+1$ equilibrium is tracible. Recall Example 6. Only the three strict equilibria are reached from an open set of priors (namely the corresponding best reply regions). The interior equilibrium E which also has index $+1$, can be reached only from itself, and hence is not tracible.

Summary and Conclusion

I presented some ideas and (partial) results how a formal definition of sustainable/learnable equilibrium could look like. Theorem 1 defines the broadest class, in which (among the regular equilibria) sustainable equilibria are characterized as having index $+1$ or equivalently being asymptotically stable with respect to some dynamics. Conjecture 2 tries to characterize these equilibria in purely strategic terms.

However, this seems not the only reasonable possibility. Evolutionary considerations suggest to confine to equilibria which are asymptotically stable for at least one myopic adjustment dynamics. This leads to a narrower and maybe more useful concept. I showed that this concept of 'learnable' equilibrium does still correspond to index +1 for generic two person games, but not for $N > 3$ players. For $N \geq 3$ players, generic existence (4) remains an open problem. Even so, in some examples the suggested definition of 'learnable' appears too broad. Hence the quest for the right concept continues.

Finally, the 'tracible' equilibria of section V seem to lead to a reasonably narrow concept, for which generic existence holds. This concept deserves further study.

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