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ROBUST PERMANENCE FOR ECOLOGICAL DIFFERENTIAL EQUATIONS, MINIMAX, AND DISCRETIZATIONS*

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Abstract. We present a sufficient condition for robust permanence of ecological (or Kolmogorov) differential equations based on average Liapunov functions. Via the minimax theorem we rederive Schreiber's sufficient condition [S. Schreiber, *J. Differential Equations*, 162 (2000), pp. 400–426] in terms of Liapunov exponents and give various generalizations. Then we study robustness of permanence criteria against discretizations with fixed and variable stepsizes. Applications to mathematical ecology and evolutionary games are given.

Key words. population dynamics, average Liapunov functions, robust permanence, discretizations of Kolmogorov type, invariant measures, Liapunov exponent, minimax, Conley index, evolutionary games

AMS subject classifications. Primary, 37B25, 65L05; Secondary, 37B30, 92D25, 90C47, 91A22

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1. Introduction. The concept of permanence (also known as uniform persistence) emerged in the late seventies as the appropriate mathematical description of coexistence in deterministic models of interacting species, replacing the previously used, but far too restrictive, global asymptotic stability of an equilibrium. It simply requires that the boundary of the state space, or the set of all extinction states, be a repeller for the dynamics of the ecological system.

In the late eighties it was realized [16], [28] that the proper framework for permanence (for the boundary as a whole) in topological dynamics was already developed by Zubov, Ura, Kimura, and others (see historical remarks in section 2), while Conley's Morse decompositions allow a finer description. New ideas of Schreiber [57] in a C^r setting are the use of invariant measures and ergodic theory, in particular smooth ergodic theory, and lead to characterizations of a robust form of permanence, meaning that nearby systems are still permanent.

In the present paper we derive sufficient conditions for robust permanence along a more classical approach using topological dynamics, in particular "good" average Liapunov functions (GALF), the Zubov–Ura–Kimura theorem, and Morse decompositions. Our key result is to relate the standard average Liapunov functions $P(x) = \prod x_i^{p_i}$ via the minimax theorem to invariant measures. This allows us to rederive and strengthen Schreiber's [57] sufficient conditions stated in terms of "unsaturated" invariant measures. (Our paper does not concern Schreiber's necessary conditions for robust permanence, based on the deep theory of measurable stable manifolds of Pesin.) Our approach leads to sharper robustness results: First, we allow C^0-perturbations; second, we prove uniform separation of the dual attractor from the repelling boundary. Similar sharper results were recently and independently obtained also by Hirsch, Smith, and Zhao [26] by refining the invariant measure ap-

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proach of [57]. However, our approach, based on GALF, is suitable to derive explicit estimates; see Remark 2.7. It also leads to exponential repulsivity; see section 3, where we also shed light on the relation between GALF and Liapunov functions.

We work out the details for dynamics on the probability simplex and indicate that similar results hold for ecological (or Kolmogorov) systems in $\mathbb{R}^n_+$ (as in [57]) and also for systems with a compact codimension 1 invariant manifold; see section 11.1.

In the second part of the paper we transfer the previous results to discrete-time systems and then turn our attention to discretization problems. We show that natural discretizations of ecological differential equations respect the invariance of boundary faces. For such discretizations of Kolmogorov type we prove robustness of permanence and discuss also Kloeden–Schmalfuss pullback attractor–repeller pairs with free step-size sequences.

In section 10 we give a short review (including open problems) of the literature on index theorems ensuring that isolated invariant sets on the boundary actually repel trajectories into the interior.

In the final section 11 we illustrate the theory with a number of applications to ecological and game theoretic models, such as Lotka–Volterra equations, replicator equations, and imitation dynamics, as well as their discretized versions. Other applications concern invasion of an ecological system by a new species and explicit characterizations of totally permanent systems which are robustly permanent together with all their subsystems.

We use the terminology of standard textbooks like Conley [12] and Nemytzkii and Stepanov [55] without any further notice. In particular, we use the terms attractor and repeller as in [12]. Index theory and ergodic theory of dynamical systems, used in this paper, are contained in these two monographs. We recommend also [2], [51], and [62].

Notation. The nonnegative orthant in $\mathbb{R}^n$ is denoted by $\mathbb{R}^n_+$ and the positive orthant by $\operatorname{int}\mathbb{R}^n_+$. The boundary, closure, and interior of a subset $S \subset X$ are denoted by ∂S, $\operatorname{cl}(S)$, and $\operatorname{int}(S)$. $\mathcal{B}[A,\varepsilon] = \{x : d(x,A) \le \varepsilon\}$ and $\mathcal{B}(A,\varepsilon) = \{x : d(x,A) < \varepsilon\}$ denote the closed and open ε-neighborhood of a set A.

Capital Greek letters Φ, Ψ denote continuous-time dynamical systems. Discretizations are denoted by the respective lowercase Greek letters. In dynamical concepts like $\gamma^+_\Phi(x)$, $\mathcal{A}_\Psi$, $\omega_{\varphi(h,\cdot)}(x)$, etc., the subscripts refer to the corresponding continuous-time or discrete-time dynamical systems.

2. Robust permanence. We consider an autonomous differential equation of Kolmogorov type,

$$\dot{x}_i = x_i f_i(x), \quad x \in X, \tag{1}$$

where X is the probability simplex $\{x \in \mathbb{R}^n : x_i \ge 0, \sum_i x_i = 1\}$ and $f : X \to \mathbb{R}^n$ is a continuous function satisfying $\sum_i x_i f_i(x) = 0$ for each $x \in X$. The standard interpretation in biology is that x_i represents the proportion of the ith species in a given ecosystem, $i = 1, 2, \dots, n$.

Together with (1), we consider its δ-perturbations of the form

$$\dot{x}_i = x_i g_i(x), \;\; x \in X, \quad \text{such that} \quad |g_i(x) - f_i(x)| < \delta \;\text{ for all } x \in X. \tag{2}$$

It is of course assumed that $g : X \to \mathbb{R}^n$ is a continuous function and $\sum_i x_i g_i(x) = 0$ for each $x \in X$. We assume further that both (1) and (2) have the uniqueness property. Denote by $\Phi(\cdot, x)$ and $\Psi(\cdot, x)$ the solutions of (1) and (2) starting in $x \in X$. It is immediate that both $\Phi : \mathbb{R} \times X \to X$ and $\Psi : \mathbb{R} \times X \to X$ are dynamical systems on X.

The boundary of X is denoted by Y. Y is invariant under Φ and Ψ. System (1) is called *permanent* (or uniformly persistent) if Y is a repeller. In ecological equations, permanence means the ultimate survival of all species. If (2) is permanent, then $(\mathcal{A}_\Psi, Y)$ forms an attractor–repeller pair, where $\mathcal{A}_\Psi$ denotes the maximal compact Ψ-invariant set in $X \setminus Y$.

The aim of this section is to give a sufficient condition for *robust permanence*, guaranteeing that every system near (1) is permanent.

DEFINITION 2.1. *Let us call a continuous mapping $P : \mathbb{R}^n_+ \to \mathbb{R}$ a* good average Liapunov function *(GALF)* for (1) *if*

(a) $P(x) = 0$ *for all* $x \in \partial\mathbb{R}^n_+$, $P(x) > 0$ *for all* $x \in \operatorname{int}\mathbb{R}^n_+$*;*

(b) P *is differentiable on* $\operatorname{int}\mathbb{R}^n_+$ *and* $p_i(x) := \frac{x_i}{P(x)}\frac{\partial P}{\partial x_i}$ *can be extended to a continuous function on* X *for every* i*;*

(c) *for every* $y \in Y$ *there is a positive constant* T_y *with the property that*

$$\int_0^{T_y} \sum_i p_i(\Phi(t, y)) f_i(\Phi(t, y))\, dt > 0.$$

Now we are in a position to present the main result of this section.

THEOREM 2.2. *If there is a GALF for* (1), *then* (1) *is robustly permanent: There are a* $\delta > 0$ *and a compact subset* S *of* $X \setminus Y$ *such that every* δ*-perturbation* (2) *of* (1) *is permanent and* $\mathcal{A}_\Psi$ *is contained in* S*.*

Remark 2.3. If the inequality in (c) is reversed, then Y can be shown to be a robust attractor for (1).

The concept of an *average Liapunov function* (ALF) *for* (1) (with (a), (c), and a weaker version of (b), namely, the assumption that

$$\text{(3)} \qquad \text{the function} \quad \frac{\dot{P}}{P} = \sum_{i=1}^n \frac{1}{P(x)} \frac{\partial P}{\partial x_i} x_i f_i(x) \quad \text{is continuous on } X \;)$$

and Theorem 2.2 (without the robustness conclusion) are due to Hofbauer [27], inspired by Schuster, Sigmund, and Wolff [58]. The standard candidate for an ALF satisfying (a) and (b) is $P(x) = \prod_{i=1}^n x_i^{p_i}$ with constants $p_i > 0$.[1] In this case $p_i(x) = p_i$, and Theorem 2.2 reduces to the following.

COROLLARY 2.4. *Suppose there are positive constants* p_i, $i = 1, \dots, n$, *such that for each* $y \in Y$ *of* (1) *there is a time* $T_y > 0$ *such that* $\int_0^{T_y} \sum_i p_i f_i(\Phi(t, y))\, dt > 0$. *Then* (1) *is robustly permanent.*

The concept of an ALF is—like that of a Liapunov function—a topological one: It can be formulated [37] in metric spaces X to show that a closed invariant subset Y is a repeller. The concept of a GALF, on the other hand, makes use of the smooth structure of X. Besides for the simplex, it applies to X being any manifold with corners (i.e., modeled after $\mathbb{R}^n_+$). Theorem 2.2 continues to hold in this more general

[1] In most practical applications this function has been used. Hutson [37] and Hofbauer [28] use more general ALFs (that are not GALFs, however). But in these instances, the standard form $P(x) = \prod_{i=1}^n x_i^{p_i}$ would be sufficient if used as in Theorem 5.5 below, i.e., taking different choices of the vector p on different Morse sets.

setting as long as X is compact. A simple example, arising in section 11.5, is X being a product of simplices. For another simple example let X be a manifold with smooth boundary, as in section 11.1. In this case, the standard GALF is simply the distance to the boundary manifold Y. This standard GALF is even good enough here to *characterize* robust repulsivity of Y.

If the state space is not compact, then some adjustments have to be made. We describe the most important case of (1) defining an ecological differential equation on $\mathbb{R}^n_+$. We restrict ourselves to systems (1) that generate dissipative (semi)flows. Φ need no longer define a flow on $\mathbb{R}^n_+$ (solutions need not be defined for all negative times, as, e.g., in the logistic equation $\dot{x} = x(1-x)$ on $\mathbb{R}_+$). Let X be a compact absorbing subset for the local flow Φ. Then $Y = X \cap \partial\mathbb{R}^n_+$ is also compact and forward invariant under (1). There are at least two ways to define robustness:

1. Consider δ-perturbations only on X and assume that X remains absorbing for the perturbed flow Ψ, i.e., $\Psi(t, X) \subset X$ holds for all $t \geq 0$. This is done in [26, Cor. 4.6], where X is taken as a cube.
2. Allow perturbations of f in (1) in the strong Whitney topology, an approach taken in [57].

Either way, Theorem 2.2 and Corollary 2.4 remain true as stated.

Corollary 2.4 (again without the robustness conclusion) has been widely used to prove permanence of population dynamical systems; see Hofbauer and Sigmund [35]. The new aspect treated in this paper and the difference between ALF and GALF is illustrated by the following example. For a different kind of robustness, see [38].

Example 2.5. Consider $n = 2$, so that (1) is of the form $\dot{x}_1 = x_1(1-x_1)F(x_1)$, $\dot{x}_2 = -x_2(1-x_2)F(1-x_2)$. Both systems

$$\text{(i)} \begin{cases} \dot{x}_1 = x_1(1-x_1)(1/2 - x_1), \\ \dot{x}_2 = -x_2(1-x_2)(x_2 - 1/2) \end{cases} \quad \text{and} \quad \text{(ii)} \begin{cases} \dot{x}_1 = x_1^2(1-x_1)^2(1/2 - x_1), \\ \dot{x}_2 = -x_2^2(1-x_2)^2(x_2 - 1/2) \end{cases}$$

are permanent, since $Y = \{(0,1)\} \cup \{(1,0)\}$ is a repeller. However, (i) is robustly permanent, whereas (ii) is not. The reason is of course that in (i) both $\{(0,1)\}$ and $\{(1,0)\}$ are hyperbolic, but they are not for (ii). This is captured by the auxiliary function $P(x_1, x_2) = x_1 x_2$ (or $x_1^{p_1} x_2^{p_2}$ for any $p_1, p_2 > 0$): Condition (c) reduces to $\min\{F(0), -F(1)\} > 0$, which holds for (i) but not for (ii). Hence P is a GALF for (i) but not (ii). On the other hand, the new auxiliary function $\tilde{P}(x_1, x_2) = e^{-1/x_1 - 1/x_2}$ satisfies

$$x_1(1-x_1)^2\left(\frac{1}{2} - x_1\right)\tilde{p}_1(x_1, x_2) - x_2(1-x_2)^2\left(x_2 - \frac{1}{2}\right)\tilde{p}_2(x_1, x_2) = \frac{(x_1 - x_2)^2}{2}$$

for each $(x_1, x_2) \in X \setminus Y$. Taking continuous extensions, we see that (a), (c), and (3) are satisfied for (ii). But (b) is violated since $\tilde{p}_1(x_1, x_2) = 1/x_1$ and $\tilde{p}_2(x_1, x_2) = 1/x_2$ do not have continuous extensions to X. Thus $\tilde{P}$ is not a GALF (but only an ALF) for (ii).

Next we illustrate the method of GALFs by deriving a stability criterion for a heteroclinic cycle. For further examples, see [27] for the planar case and [31] for higher-dimensional examples.

Example 2.6. Consider the replicator dynamics

$$\dot{x}_i = x_i((Ax)_i - xAx), \quad i = 1, 2, \ldots, n, \tag{4}$$

on the simplex $X = \{x \in \mathbb{R}^n : x_i \geq 0, \sum_{i=1}^n x_i = 1\}$, with $n = 3$ for a rock-scissors-paper game with payoff matrix

$$A = \begin{pmatrix} 0 & -a_2 & b_3 \\ b_1 & 0 & -a_3 \\ -a_1 & b_2 & 0 \end{pmatrix}. \tag{5}$$

Then the boundary Y forms a heteroclinic cycle, with "outgoing" eigenvalues $b_i > 0$ and "incoming" eigenvalues $-a_i < 0$ at the ith corner. Consider the standard function $P(x) = \prod_{i=1}^3 x_i^{p_i}$ (with $p_i > 0$ to be suitably chosen), which satisfies (a) and (b). Since every orbit on the boundary Y converges to one of the corners, it is sufficient to check (c) at these three equilibria. Hence (c) leads to a system of three linear inequalities

$$b_1 p_2 > a_1 p_3, \quad b_2 p_3 > a_2 p_1, \quad b_3 p_1 > a_3 p_2, \tag{6}$$

which can be summarized as

$$A^T p > 0 \quad \text{for suitable} \quad p > 0. \tag{7}$$

Obviously (6) has a solution in $p_i > 0$ if and only if

$$b_1 b_2 b_3 > a_1 a_2 a_3. \tag{8}$$

In this case, by Corollary 2.4, (4) is robustly permanent, i.e., the heteroclinic cycle Y is robustly repelling. If the inequalities in (6), (7), or, equivalently, in (8) are reversed, then, by Remark 2.3, the heteroclinic cycle Y is robustly attracting for (4). Note that the result does not depend on the special dynamics (4) but only on the "external eigenvalues" at the three corner equilibria which correspond to the entries of the matrix A. (Note that the above derivation of the stability criterion (8) using GALFs is much easier compared to other methods, such as finding a true Liapunov function near Y or applying Poincaré sections [35].)

Now we turn to the proof of Theorem 2.2. We shall make use of Corollary 6.1.2 of [8], which is a reformulation of Theorem 9 of the 1957 Russian edition[2] of Zubov's monograph [68].

Zubov–Ura–Kimura theorem. *Let (W, d) be a locally compact separable metric space and let Θ be a dynamical system on W. Finally, let $\emptyset \neq M$ be a compact isolated Θ-invariant set in W. Suppose that M is not a repeller. Then $\emptyset \neq \omega(x) \subset M$ for some $x \notin M$.*

Neither Zubov's work [68] nor the paper by Ura and Kimura [64] had been generally known before the 1970 monograph of Bhatia and Szegö [8]. Had they been known before, they might have led to essential simplifications in establishing such important notions of topological dynamics as the Auslander–Seibert duality between stability

[2]The proof of Theorem 9 in [68] is based on Theorem 7 of that work. Unfortunately, this latter statement is false. As it is remarked in the 1964 English edition [69], the error was pointed out by S. Lefschetz to V. I. Zubov. A corrected version of Theorem 7 was published by Bass [5], an associate of Lefschetz. A corrected version of Theorem 7 appears also in [69] and (although the last sentence on p. 35 of [69] is still false) makes the derivation of Theorem 9 correct, too. From Theorem 8 onward, section 11 of the English edition is a word-for-word translation of the Russian edition and contains several interconnected results on the local behavior of continuous-time dynamical systems near compact isolated invariant sets. More or less the same set of results was obtained by Ura and Kimura [64] independently in 1960. What we call the Zubov–Ura–Kimura theorem is a collection of several technical lemmas of [64, pp. 26–31].

and boundedness, Bhatia's concept of weak attraction, and the Wilson–Yorke hyperbolic Liapunov function in the early sixties.[3] A large number of much later results in persistence theory during the eighties (including the Butler–McGehee lemma in the appendix of [15]) followed easily from those in [68] and [64].

Proof of Theorem 2.2. Suppose P is a GALF for (1). We claim that there are three constants $c, \delta, T > 0$ and a compact subset S of $X \setminus Y$ with the following property. Given $x \in \mathrm{cl}(X \setminus S) \setminus Y$ arbitrarily, there exists a time $T_x \in (0, T]$ such that

$$P(\Psi(T_x, x)) > (1 + c)P(x) \quad \text{for every } \delta\text{-perturbation (2) of (1).} \tag{9}$$

In fact, condition (c) plus an easy compactness argument imply that there are positive constants c, d, and T such that for all $x \in I(d) = \{x \in X : P(x) \leq d\}$ (= a small compact neighborhood of Y) there is a time $T_x \in (0, T]$ with

$$\int_0^{T_x} \sum_i p_i(\Phi(t, x)) f_i(\Phi(t, x))\, dt > 3c > 0. \tag{10}$$

Uniform continuity of $p_i f_i$, $i = 1, 2, \ldots, n$, provides an $\varepsilon > 0$ such that

$$|p_i(z) f_i(z) - p_i(w) f_i(w)| < \frac{c}{nT} \quad \text{whenever} \quad z, w \in X \text{ and } |z - w| < \varepsilon.$$

Since the p_i's are bounded and $|g - f| < \delta$, we obtain for δ small enough by the triangle inequality that

$$|p_i(z) g_i(z) - p_i(w) f_i(w)| < \frac{2c}{nT} \quad \text{whenever} \quad z, w \in X \text{ and } |z - w| < \varepsilon.$$

By a standard Arzelà–Ascoli argument, the uniqueness property of (1) implies there is a $\delta > 0$ such that $|\Psi(t, x) - \Phi(t, x)| < \varepsilon$ for $t \in [0, T]$, $x \in X$, and every δ-perturbation (2) of (1). In view of inequality (10), we conclude via condition (b) that

$$\log(P(\Psi(T_x, x)) - \log(P(x)) = \int_0^{T_x} \sum_i p_i(\Psi(t, x)) g_i(\Psi(t, x))\, dt$$

$$\geq - \int_0^{T_x} \left| \sum_i p_i(\Psi(t, x)) g_i(\Psi(t, x)) - \sum_i p_i(\Phi(t, x)) f_i(\Phi(t, x)) \right| dt$$

$$+ \int_0^{T_x} \sum_i p_i(\Phi(t, x)) f_i(\Phi(t, x))\, dt \geq -\frac{2T_x c}{T} + 3c \geq c \qquad \text{for each } x \in I(d) \setminus Y.$$

Set $S = \mathrm{cl}(X \setminus I(d))$ and note that $\mathrm{cl}(X \setminus S) = I(d)$. Since $e^c > 1 + c$, inequality (9) follows.

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[3] All proofs of the Zubov–Ura–Kimura theorem work equally well for discrete-time dynamical systems. However, the first discrete-time version of the Zubov–Ura–Kimura theorem was discovered independently of [68] and [64]. It is Lemma 1 in Browder [10] (termed "a crucial one" by Browder himself), stating that a strongly ejective fixed point is repulsive. Establishing his famous existence theorem on nonejective fixed points, Browder worked out a great deal of basic topological dynamics using his own terminology. His crucial lemma is a direct consequence of the (discrete-time semidynamical version of the) Zubov–Ura–Kimura theorem. (Multivalued and various discretization aspects are investigated in [61] and [22], respectively.)

Next we point out that

$$I(d) \setminus \gamma_\Psi^+(x) \neq \emptyset \qquad \text{for each} \qquad x \notin Y. \tag{11}$$

In fact, suppose there is a $z \in I(d)$ such that $\gamma_\Psi^+(z) \subset I(d)$. Then P attains its maximum value on $\mathrm{cl}(\gamma_\Psi^+(z))$ at some point w. In particular, if $z \notin Y$, this implies $P(w) > 0$ and the existence of a time sequence $\{\tau_n\} \subset \mathbb{R}_+$ such that $P(\Psi(\tau_n, z)) \to P(w)$ as $n \to \infty$. Applying (9), we obtain that $P(\Psi(T_{\Psi(\tau_n,z)}, \Psi(\tau_n, z))) \geq (1 + c)P(\Psi(\tau_n, z)) \to (1 + c)P(w)$, a contradiction to the choice of w.

As a byproduct of (11), $I(d)$ is an isolating neighborhood of Y and for each $x \in I(d) \setminus Y$, inclusion $\emptyset \neq \omega_\Psi(x) \subset Y$ is impossible. By the Zubov–Ura–Kimura theorem, Y is a repeller for (2) and the dual attractor $\mathcal{A}_\Psi$ is contained in S. ◻

Remark 2.7. As for any proof using compactness considerations, the proof of Theorem 2.2 is also nonconstructive. However, it is not hard to see that all "intrinsically nonconstructive ingredients" of the proof are contained in assumption (c). To be more precise, assume that the conditions of Theorem 2.2 are all satisfied. In addition, assume that

(H1) there exist positive constants c, T with the property that, given $y \in Y$ arbitrarily, $\int_0^{T_y} \sum_i p_i(\Phi(t, y)) f_i(\Phi(t, y))\, dt > 4c$ for some $T_y \in (0, T]$.

Finally, assume that (no extra assumptions on the g_i's are needed!)

(H2) the functions p_i, f_i, $i = 1, 2, \ldots, n$, are (globally) Lipschitz.

Reconsidering the proof of Theorem 2.2, it is routine to check that all compactness arguments including the Zubov–Ura–Kimura argument can be replaced by Gronwall inequalities. The final conclusion is that the parameters δ and the distance of S from Y are both larger than $\Lambda c \exp(-\lambda T)$, where $\lambda, \Lambda > 0$ are computable constants and do not depend on c, T (provided by (H1)) and on the perturbation g, but only on the various Lipschitz constants (provided by (H2)). Hence the GALF assumption, together with (H1) and (H2), provides a way of estimating the distance between S and Y. Thus we have a feasible approach to the problem of "practical persistence" discussed by Hutson and Mischaikow [39] in two dimensions.

3. Exponential repulsivity. In this section we explore the concept of an average Liapunov function and its relation to exponential repulsion and existence of (ordinary) Liapunov functions.

THEOREM 3.1. (1) *If P is an ALF for* (1), *then there exist an open neighborhood $\mathcal{N}$ of Y in X and positive constants κ_1, κ_2 such that*

$$P(\Phi(t, x)) \leq \kappa_1 e^{\kappa_2 t} P(x) \quad \text{for each} \quad x \in \mathcal{N} \quad \text{and} \quad t \leq 0. \tag{12}$$

(2) *If P is a GALF for* (1), *then there exist an open neighborhood $\mathcal{N}$ of Y in X and positive constants $\delta, \kappa_1, \kappa_2$ such that for each δ-perturbation*

$$P(\Psi(t, x)) \leq \kappa_1 e^{\kappa_2 t} P(x) \quad \text{for each} \quad x \in \mathcal{N} \quad \text{and} \quad t \leq 0.$$

(3) *If $P(x) = \prod_{i=1}^n x_i^{p_i}$ is a GALF for* (1) *and letting $\delta > 0$ be the same constant as in Theorem* 2.2, *then there exist an open neighborhood $\mathcal{N}$ of Y in X and positive constants $\kappa_1, \kappa_2, \kappa_3$ such that*

$$d_E(\Psi(t, x), Y) \leq \kappa_1 e^{\kappa_2 t} (d_E(x, Y))^{\kappa_3} \quad \text{for each} \quad x \in \mathcal{N} \quad \text{and} \quad t \leq 0.$$

Here $d_E(x, Y)$ denotes the Euclidean distance between a point $x \in X$ and the set Y.

Proof. The proof of assertion 1 will be omitted since it is the same as that of 2 but ignores the robustness.

(2) The application of the Zubov–Ura–Kimura theorem in the last step of proving Theorem 2.2 will be replaced by an explicit computation, as in the earlier proofs in [27], [37]. Parameters introduced and auxiliary inequalities derived in proving Theorem 2.2 will be used throughout.

Since $\sum |p_i g_i| \le \kappa$ for some $\kappa > 0$, we obtain via integrating the identity

$$\frac{d}{dt}\log P(\Psi(t,x)) = \sum_{i=1}^{n} p_i(\Psi(t,x))g_i(\Psi(t,x)) \quad \text{for } t\in\mathbb{R} \text{ and } x\in X\setminus Y, \tag{13}$$

a consequence of assumption (b) that

$$e^{\kappa\tau}P(x) \ge P(\Psi(\tau,x)) \ge e^{-\kappa\tau}P(x) \quad \text{for every} \quad x\in X\ ,\ \tau\ge 0 \tag{14}$$

and every δ-perturbation (2) of (1).

Suppose now that $x \in I(d)\setminus Y$. Since $\mathcal{A}_\Psi \subset S = \mathrm{cl}(X\setminus I(d))$, there exists a nonnegative integer $K(x)$ with the properties that $\Psi([0,K(x)T],x) \subset I(d)$ but $\Psi(t,x) \notin I(d)$ for some $t\in (K(x)T,(K(x)+1)T]$. Set $T_{x,0}=0$ and, recursively, as long as $T_{x,k}\le K(x)T$, set $T_{x,k+1} = T_{\Psi(T_{x,k},x)}$, $k=0,1,\dots,$ (say) $k(x)$. Inequality (9) can be iterated $k(x)$ times and yields that

$$P(\Psi(T_{x,k},x)) \ge (1+c)^k P(x) \quad \text{for each} \quad k=0,1,\dots,k(x).$$

Recall that $0 < T_{x,k+1}-T_{x,k}\le T$. In view of inequality (14), it follows immediately that

$$P(\Psi(t,x)) \ge e^{-\kappa T}(1+c)^k P(x) \quad \text{whenever} \ \ T_{x,k}\le t\le T_{x,k+1}\ ,$$

and $k=0,1,\dots,k(x)$. By using $T_{x,k+1}\le (k+1)T$, we conclude that

$$P(\Psi(t,x)) \ge \frac{e^{-\kappa T}}{1+c}\cdot (1+c)^{t/T}\cdot P(x) \quad \text{whenever} \quad t\in[0,K(x)T]. \tag{15}$$

Choose $T^*>0$ in such a way that $e^{-\kappa T}\cdot(1+c)^{-1+T^*/T} > 1$ and set $\Delta = e^{-\kappa T^*}d$. We claim that

$$P(\Psi(t,I(\Delta))) \le d \qquad \text{for each} \quad t\le 0. \tag{16}$$

Suppose this is not the case. Then there exist a $t^*>0$ and an $x^*\in X$ with $P(x^*)\le\Delta$, $P(\Psi(-t^*,x^*)) = d$ but $P(\Psi(t,x^*)) < d$ for each $t\in(-t^*,0]$. By the construction,

$$e^{-\kappa t^*}d = e^{-\kappa t^*}P(\Psi(-t^*,x^*)) \le P(\Psi(t^*,\Psi(-t^*,x^*))) = P(x^*)\le\Delta,$$

and thus $t^*\ge T^*$. A similar application of (14) and the simple inequality $T<T^*$ show that

$$P(\Psi([0,T],x^*)) \le e^{\kappa T}P(x^*) < e^{\kappa T^*}\Delta = d.$$

We conclude that $K(\Psi(-t^*,x^*))T \ge t^* \ge T^*$, and hence, by using inequality (15) with $t=T^*$ and $x=\Psi(-t^*,x^*)$,

$$P(\Psi(-t^*+T^*,x^*)) \ge e^{-\kappa T}\cdot(1+c)^{-1+T^*/T}\cdot P(\Psi(-t^*,x^*)) > 1\cdot d = d,$$

a contradiction.

Set $\mathcal{N} = \{x \in X \mid P(x) < \Delta\}$. By virtue of (16), we can pass to negative times and obtain from inequality (15) that

$$P(\Psi(t,x)) \leq \frac{1+c}{e^{-\kappa T}} \cdot (1+c)^{t/T} \cdot P(x) \quad \text{whenever} \quad x \in \mathcal{N} \text{ and } t \leq 0. \tag{17}$$

This completes the proof of assertion 2.

(3) The Euclidean distance between a point $x \in X$ and the set Y equals

$$d_E(x,Y) = \left(\min \left\{ \sum_{j=1}^{n} (x_j - y_j)^2 \mid y \in Y \right\} \right)^{1/2} = \min_{1 \leq j \leq n} x_j.$$

By compactness, there exist continuous, strictly increasing functions $\alpha, \beta : [0, 1/n] \to \mathbb{R}^+$ with the properties that $\alpha(0) = \beta(0) = 0$ and

$$\alpha(d_E(x,Y)) \leq P(x) \leq \beta(d_E(x,Y)) \quad \text{whenever} \quad x \in X. \tag{18}$$

Combining (16) and (18), the rate of repulsion near Y can be estimated *in terms of the Euclidean distance function.*

For example, the standard GALF $P(x) = \prod_{i=1}^{n} x_i^{p_i}$ satisfies

$$(d_E(x,Y))^{\sum_{i=1}^{n} p_i} = \prod_{i=1}^{n} \left(\min_{1 \leq j \leq n} x_j \right)^{p_i} \leq P(x) \leq \min_{1 \leq i \leq n} x_i^{p_i} \leq (d_E(x,Y))^{\min_{1 \leq i \leq n} p_i}$$

for each $x \in X$ and leads to the desired exponential rate of repulsion. In fact, given $x \in \mathcal{N}$ arbitrarily, we obtain that

$$(d_E(\Psi(t,x),Y))^{\sum_{i=1}^{n} p_i} \leq \frac{1+c}{e^{-\kappa T}} \cdot (1+c)^{t/T} \cdot (d_E(x,Y))^{\min_{1 \leq i \leq n} p_i}$$

for each $t \leq 0$. This shows how constants $\kappa_1, \kappa_2, \kappa_3$ in assertion 3 must be chosen. ∎

COROLLARY 3.2. *If P is an ALF for* (1)*, then there exists an exponentially increasing Liapunov function for* (1)*. In other words, there exist a negatively invariant open neighborhood $\mathcal{U}$ of Y in X, a positive constant κ, and a continuous function $V : \mathcal{U} \to \mathbb{R}_+$ such that $V(x) = 0$ if and only if $x \in Y$ and*

$$V(\Phi(t,x)) \leq e^{\kappa t} V(x) \quad \textit{for each} \quad x \in \mathcal{U} \quad \textit{and} \quad t \leq 0. \tag{19}$$

Proof. The standard integration trick [8] is used for eliminating κ_1 from (12). We fix a negatively invariant open neighborhood $\mathcal{U}$ of Y in $\mathcal{N}$ and define

$$V(x) = \int_{-\infty}^{0} e^{-\kappa_2 t/(1+\Delta)} P(\Phi(t,x)) dt \quad \text{for each} \quad x \in \mathcal{U}. \tag{20}$$

Here $\Delta > 0$ is arbitrary and $\kappa = \kappa_2/(1+\Delta)$ in (19). ∎

LEMMA 3.3. *If P is a GALF for* (1) *and $W = \exp(w)$ is any positive C^1 function, then $\tilde{P} = PW$ is also a GALF for* (1).

Proof. $\tilde{P}$ obviously satisfies condition (a) in Definition 2.1. (b) follows from $\tilde{p}_i := \frac{x_i}{\tilde{P}(x)} \frac{\partial \tilde{P}}{\partial x_i} = p_i + x_i \frac{\partial w}{\partial x_i}$, $i = 1, 2, \ldots, n$. And the identity

$$\frac{1}{T} \int_0^T \sum_i (\tilde{p}_i(\Phi(t,y)) - p_i(\Phi(t,y))) f_i(\Phi(t,y))\, dt = \frac{w(\Phi(T,y)) - w(y)}{T}, \quad y \in Y,$$

together with Lemma 4.2 below, shows (c). ∎

THEOREM 3.4. *For $P(x) = \prod_{i=1}^n x_i^{p_i}$ (with $p_i > 0$) and $f \in C^1(X, \mathbb{R}^n)$ the following conditions are equivalent:*

(A) *$P(x)$ is a GALF for* (1).

(B) *There exist a negatively invariant open neighborhood $\mathcal{U}$ of Y in X, a positive constant κ, and a C^1 function $W : \mathcal{U} \to (0, \infty)$ such that $V = PW$ is an exponentially increasing Liapunov function for* (1):

$$\dot{V}(x) \geq \kappa V(x) \quad \textit{for all} \quad x \in \mathcal{U}. \tag{21}$$

Proof. If (B) holds, then V is a GALF, and hence by Lemma 3.3, with $P, W, \tilde{P}$ replaced by $V, 1/W, P$, also P must be a GALF. Now suppose that $P(x) = \prod_{i=1}^n x_i^{p_i}$ is a GALF for (1). We write $\Phi_i(t,x) = x_i Q_i(t,x)$. By Corollary 6.1, $Q_i(t,x) > 0$ for all $t \in \mathbb{R}$ and $x \in X$. Define $q(t,x) := \sum_i p_i \log Q_i(t,x)$. Then (12) implies

$$P(\Phi(t,x)) = e^{q(t,x)} P(x) \leq \kappa_1 e^{\kappa_2 t} P(x) \quad \text{for } t \leq 0. \tag{22}$$

Furthermore $\frac{\partial q(t,x)}{\partial t} = \sum_i p_i f_i(\Phi(t,x)) =: \tilde{f}(\Phi(t,x))$, and for the partial derivatives

$$\frac{\partial}{\partial t} \frac{\partial q}{\partial x_j}(t,x) = \sum_i \frac{\partial \tilde{f}}{\partial x_i}(\Phi(t,x)) \frac{\partial \Phi_i}{\partial x_j}(t,x). \tag{23}$$

Let L be a Lipschitz constant of (1). Then Gronwall's inequality implies $|\frac{\partial \Phi_i}{\partial x_j}(t,x)| \leq e^{L|t|}$, and hence in (23)

$$\left| \frac{\partial}{\partial t} \frac{\partial q}{\partial x_j}(t,x) \right| \leq C e^{L|t|}. \tag{24}$$

After integration this gives

$$\left| \frac{\partial q}{\partial x_j}(t,x) \right| \leq C' e^{L|t|} \tag{25}$$

for some positive constants C, C'. Now use P^α (for any $\alpha > 0$) instead of P in (20) and consider

$$V_\alpha(x) = \int_{-\infty}^0 e^{-\alpha\kappa_2 t/(1+\Delta)} P(\Phi(t,x))^\alpha dt = P(x)^\alpha W_\alpha(x) \tag{26}$$

with

$$W_\alpha(x) = \int_{-\infty}^0 e^{-\alpha\kappa_2 t/(1+\Delta)} e^{\alpha q(t,x)} dt. \tag{27}$$

Then, by (22), for every $\alpha > 0$ and $\Delta > 0$, the function V_α is continuous on X and satisfies (19) with $\kappa = \alpha\kappa_2/(1+\Delta)$. The function W_α is continuous and positive on X. Formal differentiation of (27) gives

$$\frac{\partial W_\alpha}{\partial x_j}(x) = \int_{-\infty}^0 e^{-\alpha\kappa_2 t/(1+\Delta)} e^{\alpha q(t,x)} \alpha \frac{\partial q}{\partial x_j}(t,x) dt. \tag{28}$$

With (22) and (25) we can estimate the integrand up to a constant factor by $\exp(-\frac{\alpha\kappa_2 t}{1+\Delta} + \alpha\kappa_2 t - Lt)$ (for $t < 0$). Hence for

$$\alpha > \frac{1+\Delta}{\Delta}\frac{L}{\kappa_2} \tag{29}$$

the indefinite integral in (28) converges absolutely and uniformly in $x \in X$. This implies that for all these α large enough, W_α is C^1. Then the claim follows for $V = V_\alpha^{1/\alpha} = PW_\alpha^{1/\alpha} =: PW$. ◻

Remark 3.5. We note that W (and hence V) can be made as smooth as the vector field (1) by choosing α sufficiently large: This follows easily by further differentiating (28). The relationship between (12) and (21) is in line with the general observation that, under certain conditions, inequalities can be differentiated with respect to parameters [9], [47].

Remark 3.6. Lemma 3.3 shows that to each GALF P there belongs a whole equivalence class of GALFs differing by a smooth positive factor. Theorem 3.4 shows that for a standard GALF, there is a true Liapunov function among these equivalent GALFs. Still, the advantage of the GALF concept is its considerably easier practical applicability, as compared to a true Liapunov function. Finding a standard GALF for (1) is reduced in the next section to the algebraic problem of finding suitable constants $p_i > 0$. In the setting of manifolds with smooth boundary in section 11.1, there is essentially a unique standard GALF, which is simply the distance to the boundary manifold. The GALF conditions for this simple function *characterize* robust repulsivity of the boundary. Finding an explicit true Liapunov function is considerably more difficult.

Remark 3.7. Theorem 3.4 (B) implies another, very simple proof of robust permanence (under the stronger assumption $f \in C^1$): Write $V = Pe^w$. Then along interior solutions of a δ-perturbation (2)

$$\dot{V}/V = \sum_i p_i \frac{\dot{x}_i}{x_i} + \dot{w} = \sum_i \left(p_i + x_i \frac{\partial w}{\partial x_i}\right) g_i(x)$$

$$= \sum_i \left(p_i + x_i \frac{\partial w}{\partial x_i}\right) f_i(x) + \sum_i \left(p_i + x_i \frac{\partial w}{\partial x_i}\right) (g_i(x) - f_i(x)).$$

The first sum is $\geq \kappa > 0$ by (21) and the second term is less than a constant (since w is C^1) times δ. Hence for δ small enough, V is a local Liapunov function near Y also for (2).

4. GALF and minimax. We need the minimax theorem in the following simplified formulation (see, e.g., [59]).

MINIMAX THEOREM. *Let A, B be Hausdorff topological vector spaces and let $\Gamma : A \times B \to \mathbb{R}$ be a continuous bilinear function. Finally, let C and D be nonempty, convex, compact subsets of A and B, respectively. Then*

$$\min_{a\in C}\max_{b\in D}\Gamma(a,b) = \max_{b\in D}\min_{a\in C}\Gamma(a,b).$$

In what follows $\mathcal{M}_\Phi$ denotes the collection of Φ-invariant Borel probability measures on Y. The collection of all Borel probability measures on Y is denoted by $\mathcal{M}$. Recall that both $\mathcal{M}_\Phi$ and $\mathcal{M}$ are nonempty, convex, weakly-$*$ compact subsets of

$C^*_w(Y,R)$, the dual space of $C(Y,R)$ equipped with the weak-$*$ topology. The subset $\mathcal{M}_\Phi^E$ of ergodic Φ-invariant measures is the set of extreme points of $\mathcal{M}_\Phi$. The characteristic function of a Borel set $B \subset Y$ is denoted by χ_B.

LEMMA 4.1.[4] *Let $h : Y \to \mathbb{R}$ be a continuous function. Then*

$$\min_{\mu\in\mathcal{M}_\Phi} \int_Y h\, d\mu = \min_{y\in Y} \left\{ \limsup_{T\to\infty} \frac{1}{T} \int_0^T h(\Phi(t,y))\, dt \right\}. \tag{30}$$

Proof. Replacing "lim" by "lim sup" in the ergodic theorem, we have that

$$\int_Y h\, d\mu = \limsup_{T\to\infty} \frac{1}{T} \int_0^T h(\Phi(t,y))\, dt \quad \text{for } \mu\text{–almost all } y \in Y. \tag{31}$$

Note that the right-hand side of (31) defines a lower semicontinuous function on Y. Since Y is compact and $\mathcal{M}_\Phi$ is weakly-$*$ compact, we can take minima on both sides. This proves the "$\geq$ part" of (30).

To prove the "$\leq$ part," we argue via reductio ad absurdum and suppose there exist a $y_0 \in Y$, an $\varepsilon_0 > 0$, and a time sequence $\{\tau_n\} \subset \mathbb{R}_+$ such that $\tau_n \to \infty$ and

$$\int_Y h\, d\mu > \varepsilon_0 + \frac{1}{\tau_n} \int_0^{\tau_n} h(\Phi(t,y_0))\, dt \quad \text{for each } \mu \in \mathcal{M}_\Phi \text{ and } n = 1,2,\ldots. \tag{32}$$

By letting $\mu_n(B) = \frac{1}{\tau_n}\int_0^{\tau_n} \chi_B(\Phi(t,y_0))\, dt$ for each Borel set $B \subset Y$, a $\mu_n \in \mathcal{M}$ is defined and the inequality in (32) goes over into $\int_Y h\, d\mu > \varepsilon_0 + \int_Y h\, d\mu_n$. We may assume that, in the weak-$*$ topology, $\mu_n \to \mu_0$ for some $\mu \in \mathcal{M}$. The crucial observation is that $|\mu_n(\Phi(\tau,B)) - \mu_n(B)| \leq \frac{2|\tau|}{\tau_n}$ for each Borel set $B \subset Y$, $\tau \in \mathbb{R}$ and $n = 1,2,\ldots$. By letting $n \to \infty$, we conclude that[5] $\mu_0 \in \mathcal{M}_\Phi$. Hence $\int_Y h\, d\mu_0 \geq \varepsilon_0 + \int_Y h\, d\mu_0$, a contradiction. ∎

LEMMA 4.2. *For any continuous function $h : Y \to \mathbb{R}$, the following properties are equivalent:*

(i) $\min_{y\in Y}\{\limsup_{T\to\infty} \frac{1}{T}\int_0^T h(\Phi(t,y))\, dt\} > 0$.

(ii) *For every $y \in Y$ there is a $T(y) > 0$ with $\int_0^{T(y)} h(\Phi(t,y))\, dt > 0$.*

Proof. (i) $\Rightarrow$ (ii) is trivial. Suppose now that (ii) is satisfied. By an easy compactness argument, we see there is no loss of generality in assuming there are positive constants c_0, T_1, T_2 such that

$$T_1 \leq T(y) \leq T_2 \quad \text{and} \quad \int_0^{T(y)} h(\Phi(t,y))\, dt > c_0 \quad \text{for all } y \in Y.$$

Set $\tau_0 = 0$ and, recursively, $\tau_n(y) = \tau_{n-1}(y) + T(\Phi(\tau_{n-1}(y), y))$. By the construction, $nT_1 \leq \tau_n \leq nT_2$ and

$$\frac{1}{\tau_n(y)} \int_0^{\tau_n(y)} h(\Phi(t,y))\, dt \geq \frac{c_0}{T_2} \qquad \text{for each } n = 1,2,\ldots.$$

Even with lim sup replaced by lim inf, (i) follows immediately. ∎

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[4] This is Exercise 8.5 on p. 57 in [51] (with "limsup" replaced by "liminf"). Proofs were written in [30] and [56], who derived it from a much more general setting. Other generalizations were given in [3] and [11]. We include a proof for completeness.

[5] The argumentation leading to μ_0 is truly fundamental and plays a vital role in the ergodic theory of dynamical systems from its very beginnings in Krylov and Bogoliubov [46] to (proving the first part of) Theorem 4.3 of Schreiber [57].

Remark 4.3. Note that condition (c) in Definition 2.1 has to be checked only for $y \in MC_\Phi(Y)$, the minimal center of attraction of Y (i.e., by definition, the smallest compact Φ-invariant set containing the support of each invariant measure in $\mathcal{M}_\Phi(Y)$). This follows from a twofold application of Lemmas 4.1 and 4.2 with Y and Y replaced by $MC_\Phi(Y)$; cf. [56]. Note that $MC_\Phi(Y)$ is contained in the closure of the set of all $\Phi|_Y$-recurrent points.

THEOREM 4.4. *The following properties are equivalent:*

(α) *For some $p_1, p_2, \dots, p_n > 0$ suitably chosen, $P(x) = \prod_{i=1}^n x_i^{p_i}$ is a GALF for* (1).

(β) *For every $\mu \in \mathcal{M}_\Phi$ there exists an $i \in \{1, 2, \dots, n\}$ with $\int_Y f_i \, d\mu > 0$.*

(γ) *There are $p_1, p_2, \dots, p_n > 0$ such that $\sum_{i=1}^n p_i \int_Y f_i \, d\mu > 0$ holds for every ergodic $\mu \in \mathcal{M}_\Phi^E$.*

Proof. We may assume by homogeneity that $p = (p_1, p_2, \dots, p_n) \in X$. Applying the lemmas for $h = \sum_i p_i f_i$, we obtain that

$$(\alpha) \Leftrightarrow \max_{p \in X} \min_{\mu \in \mathcal{M}_\Phi} \sum_{i=1}^n p_i \int_Y f_i \, d\mu > 0. \tag{33}$$

On the other hand, it is elementary to check that

$$(\beta) \Leftrightarrow \min_{\mu \in \mathcal{M}_\Phi} \max_{p \in X} \sum_{i=1}^n p_i \int_Y f_i \, d\mu > 0. \tag{34}$$

With $C = \mathcal{M}_\Phi$, $D = X$, $A = C_w^*(Y, R)$, $B = \mathbb{R}^n$, and $\Gamma(p, \mu) = \sum_i p_i \int_Y f_i \, d\mu$, the minimax theorem implies the equivalence of (α) and (β). Since the minimum in (33) is attained at an ergodic measure, the equivalence of (α) and (γ) follows. ∎

Example 4.5. Returning to the rock-scissors-paper game (4)–(5), note that

$$\mathcal{M}_\Phi = \left\{ \sum_{k=1}^3 q_k \delta_k \;\middle|\; q_k \geq 0, \sum_{k=1}^3 q_k = 1 \right\},$$

where δ_k is the Dirac measure at the kth vertex of the two-dimensional simplex, $k = 1, 2, 3$. By using homogeneity, condition (β) then translates into the requirement that

$$\text{for any } \; q > 0 \; \text{ there exists an } \; i \in \{1, 2, 3\} \text{ with } \; (Aq)_i > 0. \tag{35}$$

The equivalence of (7) and (35), for arbitrary $n \times m$ real matrices, is the well-known Farkas lemma on linear inequalities. Note that in an alternative proof of Theorem 4.4, the minimax theorem can be replaced by using an infinite-dimensional version of the Farkas lemma.

The integrals $\int_Y f_i \, d\mu$ are Liapunov exponents of μ. If μ is ergodic, then there exists a unique nonempty supporting subset $I \subset \{1, 2, \dots, n\}$ such that $\mu(X_I) = 1$ for the (relatively) open face $X_I := \{x \in X : x_i > 0 \text{ for } i \in I \text{ and } x_j = 0 \text{ for } j \notin I\}$. According to Lemma 5.1 in [57], $\int_Y f_i \, d\mu = 0$ for $i \in I$ (compare also Remark 5.4). The integrals $\int_Y f_i \, d\mu$ for $i \notin I$ are called *external Liapunov exponents*. Biologically, they describe the invasion rate of the missing species i at μ. For point measures $\delta_{\bar{x}}$, the external Liapunov exponents reduce to the external eigenvalues $f_i(\bar{x})$ at the boundary equilibrium $\bar{x}$; see [35]. For periodic orbits in Y, the external Liapunov exponents coincide with the (normalized) external Floquet exponents; see [57].

Combining Theorems 2.2 and 4.4, we see that (β) is a sufficient condition for robust permanence. This is a version of the main result in [57]. Strengthening condition (β) leads to the following result on "totally permanent systems" due to Mierczyński and Schreiber [52].

COROLLARY 4.6. *If for every $\mu \in \mathcal{M}_\Phi^E$ all external Liapunov exponents $\int_Y f_i d\mu$ are positive, then* (1) *and each of its subsystems are robustly permanent.*

Proof. This follows immediately from Theorems 2.2 and 4.4, together with the aforementioned Lemma 5.1 of [57] or Remark 5.4 below. Note that $P(x) = \prod_{i=1}^n x_i^{p_i}$ is a GALF for (1) for any choice of the exponents $p_i > 0$. ∎

Using Pesin theory, a converse result can also be shown; see [52].

5. Local GALFs and Morse decompositions. Throughout this section, let K be a nonempty Φ-invariant compact subset of Y, and let U be an open neighborhood of K in $\mathbb{R}_+^n$.

DEFINITION 5.1. *A continuous mapping $P_K : U \to \mathbb{R}$ is a GALF* for (1) on K *if*

(a)$_K$ *$P_K(x) = 0$ for all $x \in U \cap \partial\mathbb{R}_+^n$, $P_K(x) > 0$ for all $x \in U \cap \operatorname{int}\mathbb{R}_+^n$;*

(b)$_K$ *P_K is differentiable on $U \cap \operatorname{int}\mathbb{R}_+^n$ and $p_i(x) := \frac{x_i}{P_K(x)}\frac{\partial P_K}{\partial x_i}$ can be extended to a continuous function on U for every i;*

(c)$_K$ *for every $y \in K$ there is a positive constant T_y with the property that $\int_0^{T_y}\sum_i p_i(\Phi(t,y))f_i(\Phi(t,y))\,dt > 0$.*

THEOREM 5.2. *If P_K is a GALF for* (1) *on K, then there exist an open neighborhood $\mathcal{N}_K$ of K in X and positive constants $\delta, \kappa_1, \kappa_2$ such that for each δ-perturbation*

$$P_K(\Psi(t,x)) \le \kappa_1 e^{\kappa_2 t} P_K(x) \quad \text{whenever} \quad \{\Psi(\tau,x) \mid t \le \tau \le 0\} \subset \mathcal{N}_K.$$

In particular, $\mathcal{N}_K \setminus Y$ does not contain entire trajectories of Ψ and, for each $x \in \mathcal{N}_K \setminus Y$, inclusion $\emptyset \ne \omega_\Psi(x) \subset K$ is impossible.

Proof. Reconsidering the respective proofs in sections 2 and 3, we see that the existence of a local GALF implies that both (17) and (11) remain valid in the local setting. ∎

The collection of Φ-invariant Borel probability measures on K is denoted by $\mathcal{M}_\Phi(K)$. Clearly $\mathcal{M}_\Phi(Y) = \mathcal{M}_\Phi$ and, for a general K, $\mathcal{M}_\Phi(K)$ can be identified with $\{\mu \in \mathcal{M}_\Phi \,:\, \mu(K) = 1\}$. The collection of ergodic measures in $\mathcal{M}_\Phi(K)$ is denoted by $\mathcal{M}_\Phi^E(K)$.

THEOREM 5.3. *The following properties are pairwise equivalent:*

(α)$_K$ *There are $p_1, p_2, \ldots, p_n > 0$ such that $P(x) = \prod_{i=1}^n x_i^{p_i}$ is a GALF for* (1) *on K.*

(β)$_K$ *For every $\mu \in \mathcal{M}_\Phi(K)$ there exists an $i \in \{1, 2, \ldots, n\}$ with $\int_K f_i\, d\mu > 0$.*

(γ)$_K$ *There are $p_1, p_2, \ldots, p_n > 0$ such that $\sum_{i=1}^n p_i \int_K f_i d\mu > 0$ for all $\mu \in \mathcal{M}_\Phi^E(K)$.*

Proof. This is the localized version of Theorem 4.4, replacing Y by K, with the same proof. ∎

Remark 5.4. Assume that $K \subset \{y \in Y \,:\, y_n > 0\}$. Then

$$\int_K f_n\, d\mu = 0 \quad \text{for each } \mu \in \mathcal{M}_\Phi(K).$$

In fact, a twofold application of the Φ-invariance of μ implies via Fubini's theorem that

$$\begin{aligned}\int_K f_n\, d\mu &= \int_0^1 \int_K f_n(\Phi(t,\cdot))\, d\mu dt = \int_K \int_0^1 f_n(\Phi(t,\cdot))\, dt d\mu \\ &= \int_K \{\log(\Phi_n(t,\cdot))\}_{t=0}^{t=1}\, d\mu = \int_K \log(\Phi_n(1,\cdot))\, d\mu - \int_K \log(\Phi_n(0,\cdot))\, d\mu = 0.\end{aligned}$$

The property established above (proved differently in [57, Lem. 5.1], using the ergodic theorem and Poincaré's recurrence theorem) helps check whether $(\beta)_K$ is satisfied or not.

Schreiber [57] (working on $\mathbb{R}^n_+$) defines an invariant probability measure $\mu \in \mathcal{M}_\Phi$ to be *unsaturated* if $\max_{1\le i\le n} \int_Y f_i \, d\mu > 0$, i.e., at least one external Liapunov exponent is positive. (For point measures this reduces to the notion of an unsaturated equilibrium from [35].) He calls a compact invariant set $K \subset Y$ unsaturated if every $\mu \in \mathcal{M}_\Phi(K)$ is unsaturated. By our Theorem 5.3, K is unsaturated if and only if there exists a local GALF near K of the standard form $\prod_i x_i^{p_i}$. One of the main results in [57] and [26] says that if Y has a Morse decomposition with all Morse sets being unsaturated, then (1) is robustly permanent. This result is generalized as follows.

THEOREM 5.5. *Let $M_1, M_2, \dots, M_\ell$ be a Morse decomposition on Y for $\Phi|_Y$. Further, for $k = 1, 2, \dots, \ell$, let U_k be an open neighborhood of M_k in Y and let $P_k : U_k \to \mathbb{R}$ be a GALF for* (1) *on M_k. Then* (1) *is robustly permanent.*

Proof. Arguing as in the first paragraph of the proof of Theorem 2.2, we obtain that there are three constants $c, \delta, T > 0$ and, for $k = 1, 2, \dots, \ell$, there is an open neighborhood N_k of M_k in U_k with $\mathrm{cl}(N_j) \cap \mathrm{cl}(N_k) = \emptyset$ for $j \ne k$ and the property as follows. Given $x \in N_k \setminus Y$, $k = 1, 2, \dots, \ell$, arbitrarily, there exists a time $T_x \in (0, T]$ such that

$$P_k(\Psi(T_x, x)) > (1 + c)P_k(x) \quad \text{for every } \delta\text{-perturbation (2) of (1).}$$

We claim that, for δ sufficiently small,

$$\gamma_\Psi(x) \subset \mathcal{B}[Y, \delta] \Rightarrow \alpha_\Psi(x) \cup \omega_\Psi(x) \subset \bigcup_{k=1}^{\ell} N_k. \tag{36}$$

Since $\bigcup_k M_k$ is the intersection of a finite collection of attractor–repeller pairs, there is no loss of generality in assuming that $\ell = 2$ and that (M_1, M_2) is an attractor–repeller pair for $\Phi|_Y$.

Since M_1 is an attractor for $\Phi|_Y$, there exists a compact neighborhood S_1 of M_1 in N_1 satisfying $\Phi(\mathbb{R}_+, Y \cap S_1) \subset N_1$. We point out next that, for δ sufficiently small,

$$\gamma_\Psi(z) \subset \mathcal{B}[Y, \delta] \text{ plus } z \in S_1 \Rightarrow \gamma^+_\Psi(z) \subset N_1. \tag{37}$$

To the contrary, suppose that, for each $j = 1, 2, \dots$, there exists a $\frac{1}{n}$-perturbation (2) of (1), a $z_j \in S_1$, and a time $t_j > 0$ satisfying $\gamma_{\Psi_j}(z_j) \subset \mathcal{B}[Y, \frac{1}{j}]$ but $w_j = \Psi_j(t_j, z_j) \notin N_1$. We may assume that $z_j \in \partial S_1$, $w_j \in \partial N_1$, $\Psi_j((0, t_j), z_j) \subset N_1 \setminus S_1$ and, by compactness, $z_n \to z_0$ and $w_n \to w_0$ for some $z_0 \in Y \cap \partial S_1$ and $w_0 \in Y \cap \partial N_1$. We distinguish two cases according to whether $\{t_j\} \subset \mathbb{R}_+$ is bounded or not. By passing to a subsequence, we may assume that $t_j \to t_0$ for some $t_0 \in \mathbb{R}_+$ or $t_j \to \infty$. If $t_j \to t_0$, then $\Phi(t_0, z_0) = w_0$, a contradiction. If $t_j \to \infty$, we may assume that $q_j = \Psi_j(t_j/2, z_j) \to q$ for some $q \in Y \cap \mathrm{cl}(N_1 \setminus S_1)$. It is readily checked that $\gamma_\Phi(q) \subset Y \cap \mathrm{cl}(N_1 \setminus S_1)$, a contradiction.

By continuity (and passing to a smaller δ if necessary), we see there exist positive times T_1, $T_2 > T_1$ such that

$$\Psi([T_1, T_2], \mathcal{B}[Y \setminus (N_1 \cup N_2), \delta]) \subset S_1 \quad \text{for every } \delta\text{-perturbation (2) of (1).}$$

In view of property (37), this ends the proof (of case $\ell = 2$) of (36).

The rest is easy. For each $k = 1, 2, \ldots, \ell$, the Zubov–Ura–Kimura argument we used in the last two paragraphs of the proof of Theorem 2.2 applies in $\mathrm{cl}(N_k)$ individually. ∎

Remark 5.6. The proof of Theorem 5.5 shows that any continuous function $P : X \to \mathbb{R}$ satisfying conditions (a) and $P|_{N_k} = P_k|_{N_k}$, $k = 1, 2, \ldots, \ell$, satisfies condition (9), too. Thus, in a technical sense, local GALFs can be joined together to a global "nearly-GALF."

6. Discrete-time analogues. Mutatis mutandis, all the previous results remain valid for discrete-time dynamical systems.

With $F_i : X \to \mathbb{R}$, $i = 1, 2, \ldots, n$, continuous, consider a mapping of the form

$$\mathcal{F} : X \to X, \quad x \to (x_1F_1(x), x_2F_2(x), \ldots, x_nF_n(x)). \tag{38}$$

Throughout this section, it is assumed that $\mathcal{F}$ is a self-homeomorphism of X. Brouwer's open mapping theorem implies that $\mathcal{F}(X \setminus Y) = X \setminus Y$ and $\mathcal{F}(Y) = Y$. In particular, $F_i(x) > 0$ for each $x \in X \setminus Y$, $i = 1, 2, \ldots, n$. Throughout this section, we assume further that $F_i(y) > 0$ for each $y \in Y$, $i = 1, 2, \ldots, n$.

Our next result implies that this latter assumption is quite natural. It will also be crucial in establishing Lemma 7.2, the starting point of the theory of discretizations of Kolmogorov type in the next chapter, and it was used already in the proof of Theorem 3.4.

Surjectivity theorem. *Let $F_i : X \to \mathbb{R}_+$, $i = 1, 2, \ldots, n$, be continuous functions, $\sum_i x_iF_i(x) = 1$ for each $x \in X$, and consider the mapping $\mathcal{F} : X \to X$, $x \to (x_1F_1(x), x_2F_2(x), \ldots, x_nF_n(x))$. Then $\mathcal{F}(S) = S$ for each subsimplex S of X.*

Proof. By the particular form of our mapping, this is certainly true for the zero-dimensional subsimplices (vertices) of X. For a k-member subset $\{i_1, i_2, \ldots, i_k\}$ of $\{1, 2, \ldots, n\}$, consider the subsimplex of the form $S = \{x \in X \; : \; x_{i_1} = x_{i_2} = \cdots = x_{i_k} = 0\}$. Applying $\mathcal{F}_i(x) = x_iF_i(x)$ for $i = i_1, i_2, \ldots, i_k$, we obtain that $\mathcal{F}(S) \subset S$. By induction on the subsimplices, we may assume that $S = X$ and $\mathcal{F}(s) = s$ for each facet s of $S = X$. Consider a point $p_0 \in \mathrm{int}(S)$ arbitrarily chosen. For any $\lambda \in [0, 1]$, any facet s of S, and any point $p \in s$, the convexity of s implies that $(1 - \lambda)p + \lambda\mathcal{F}(p) \in s$. It follows that

$$(1 - \lambda)p + \lambda\mathcal{F}(p) \neq p_0 \quad \text{whenever } p \in \partial S \text{ and } \lambda \in [0, 1].$$

By the homotopy property of Brouwer's degree, it follows that

$$\deg(\mathcal{F}, p_0, \mathrm{int}(S)) = \deg(\mathrm{id}_{R^n}, p_0, \mathrm{int}(S)),$$

where id_{R^n} denotes the identity on $\mathbb{R}^n$. Since $\deg(\mathrm{id}_{R^n}, p_0, \mathrm{int}(S)) = 1$, the existence property of the degree implies that $p_0 \in \mathcal{F}(S)$. ∎

Corollary 6.1. *In addition, assume that F_i, $i = 1, 2, \ldots, n$, is of class C^1 (in the sense that F_i admits a C^1 extension $\hat{F}_i : U_i \to \mathbb{R}$ defined on an open neighborhood U_i of X in $\mathbb{R}^n$) and that $\mathcal{F}$ is a C^1 self-diffeomorphism of X. Then $F_i(y) > 0$ for each $y \in Y$, $i = 1, 2, \ldots, n$.*

Proof. Pick $y \in Y$ arbitrarily. For index j satisfying $y_j \neq 0$, inequality $F_j(y) > 0$ is a direct consequence of the surjectivity theorem when applied to the X-facet $S_j = \{x \in X \; : \; x_j = 0\}$. For j satisfying $y_j = 0$, inequality $F_j(y) > 0$ follows from the diffeomorphism assumption. To the contrary, assume that $F_j(y) = 0$ (and $y_j = 0$). A direct computation shows that the jth row of the Jacobian of $\mathcal{F}$ evaluated at y equals $(0, 0, \ldots, 0)$, a contradiction. ∎

The discrete-time version of an ALF for $\mathcal{F}$ [40] is a continuous mapping $R : X \to \mathbb{R}$ with the following properties:

(d) $R(x) = 0$ for all $x \in Y$, $R(x) > 0$ for all $x \in X \setminus Y$.

(e) There exists a continuous function $r : X \to \mathbb{R}$ such that $r(x) = \log(R(\mathcal{F}(x))) - \log(R(x))$ whenever $x \in X \setminus Y$.

(f) For every $y \in Y$ there is a positive integer $N_y > 0$ with the property that $\sum_{k=1}^{N_y} r(\mathcal{F}^{k-1}(y)) > 0$.

In contrast to the continuous-time case, we did not find a reasonable analogue of the notion of GALF for discrete time. Hence for studying robust permanence in discrete-time systems, we restrict ourselves to the standard ALF of the form $R(x) = \prod_{i=1}^n x_i^{r_i}$ with $r_i > 0$, $i = 1, 2, \dots, n$. For this choice of R, condition (e) holds with $r(x) = \sum_i r_i \log F_i(x)$.

Remark 6.2. One can characterize this R by a functional equation. More precisely, if the continuous mappings $r_i, R : X \to \mathbb{R}$ satisfy condition (d) and $\log R(\mathcal{F}(x)) - \log R(x) = \sum_i r_i(x) \log F_i(x)$ for arbitrary $\mathcal{F}$, then $r_i(x) = r_i$ and $R(x) = c_n \prod_{i=1}^n x_i^{r_i}$ for some positive constants r_i, $i = 1, 2, \dots, n$, and c_n. For a proof, see [19].

With $G_i : X \to \mathbb{R}$, $i = 1, 2, \dots, n$, continuous, consider δ-perturbations of $\mathcal{F}$ of the form $\mathcal{G} : X \to X$, $x \to (x_1G_1(x), x_2G_2(x), \dots, x_nG_n(x))$, where $|G_i(x) - F_i(x)| < \delta$, $i = 1, 2, \dots, n$. It is of course assumed that $\sum x_iG_i(x) = 1$ for each $x \in X$. We assume further that $\mathcal{G}$ is a self-homeomorphism of X. If $\mathcal{G}$ is permanent, then $(\mathcal{A}_\mathcal{G}, Y)$ forms an attractor–repeller pair, where $\mathcal{A}_\mathcal{G}$ denotes the maximal compact $\mathcal{G}$-invariant set in $X \setminus Y$. In analogy to the relation between (2) and (1), we say that $\mathcal{G}$ is a δ-perturbation of $\mathcal{F}$ if $|G_i(x) - F_i(x)| < \delta$ for each $x \in X$ and $i = 1, 2, \dots, n$.

THEOREM 6.3. *If there is an ALF for $\mathcal{F}$, then $\mathcal{F}$ is permanent. Moreover, assume that for some constants $r_i > 0$ suitably chosen, $R(x) = \prod_{i=1}^n x_i^{r_i}$ is an ALF for $\mathcal{F}$. Then $\mathcal{F}$ is robustly permanent. There are a $\delta > 0$ and a compact subset S of $X \setminus Y$ with the properties as follows. Every δ-perturbation $\mathcal{G}$ of $\mathcal{F}$ is permanent and $\mathcal{A}_\mathcal{G}$ is contained in S.*

Proof. The proof of Theorem 2.2 can be repeated. The computations are based on the formulae

$$\log(R(\mathcal{F}^{N_x}(x))) - \log(R(x)) = \sum_{k=1}^{N_x} r(\mathcal{F}^{k-1}(x))$$

and

$$\log(R(\mathcal{G}^{N_x}(x))) - \log(R(x)) = \sum_{k=1}^{N_x} \sum_{i=1}^{n} r_i \cdot \log(G_i(\mathcal{G}^{k-1}(x))),$$

respectively. The last step is the application of the discrete-time version of the Zubov–Ura–Kimura theorem. ∎

The first statement of Theorem 6.3 is due to [40]; see also [36]. The robustness result is new.

The set of $\mathcal{F}$-invariant Borel probability measures on Y is denoted by $\mathcal{M}_\mathcal{F}$. When combined with Theorem 6.3, our next result establishes a sufficient condition for robust permanence of $\mathcal{F}$. Note that inequality $\int_Y \log(F_i)\, d\nu > 0$ is stronger than the (seemingly) "more natural" inequality $\int_Y F_i \, d\nu > 1$. A special case of heteroclinic cycles was treated in [24].

THEOREM 6.4. *The following properties are equivalent:*

$(\alpha)^d$ *For some $r_1, r_2, \dots, r_n > 0$ suitably chosen, $R(x) = \prod_{i=1}^n x_i^{r_i}$ is an ALF for $\mathcal{F}$.*

$(\beta)^d$ *For every* $\nu \in \mathcal{M}_{\mathcal{F}}$ *there exists an* $i \in \{1,2,\ldots,n\}$ *with* $\int_Y \log F_i \, d\nu > 0$.

$(\gamma)^d$ *There are* $r_1, r_2, \ldots, r_n > 0$ *such that* $\sum_{i=1}^n r_i \int_Y \log F_i \, d\nu > 0$ *holds for every ergodic* $\nu \in \mathcal{M}_{\mathcal{F}}^E$.

Proof. The proof of Theorem 4.4 can be repeated. Almost no changes are needed. The method of proving Lemma 4.1 yields for each $h \in C(Y, \mathbb{R})$ that

$$\min_{\nu \in \mathcal{M}_{\mathcal{F}}} \int_Y h \, d\nu = \min_{y \in Y} \limsup_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} h(\mathcal{F}^{k-1}(y)).$$

The compactness argument we used in proving Lemma 4.2 implies that inequality $\min_{\nu \in \mathcal{M}_{\mathcal{F}}} \int_Y h \, d\nu > 0$ is equivalent to the following assertion: For every $y \in Y$ there is a positive integer $N_y > 0$ such that $\sum_{k=1}^{N_y} h(\mathcal{F}^{k-1}(y)) > 0$. We may assume by homogeneity that $r = (r_1, r_2, \ldots, r_n) \in X$. With $h = \sum r_i \cdot \log(F_i)$, the minimax theorem applies. □

THEOREM 6.5. *Let* $M_1, M_2, \ldots, M_\ell$ *be a Morse decomposition on* Y *for* $\mathcal{F}|_Y$. *Further, for* $k = 1, 2, \ldots, \ell$, *let* U_k *be an open neighborhood of* M_k *in* Y *and let* $R_k : U_k \to \mathbb{R}$ *be an ALF for* $\mathcal{F}$ *on* M_k. *Then* $\mathcal{F}$ *is permanent. Moreover, assume that each* R_k *is of the special form* $R_k(x) = \prod_{i=1}^n x_i^{r_i^k}$ *for some positive constants* $r_1^k, r_2^k, \ldots, r_n^k$. *Then* $\mathcal{F}$ *is robustly permanent.*

Proof. The method we used in proving Theorem 5.5 applies. □

The formulation of the discrete-time version of Theorems 3.1, 3.4, 5.2, and 5.3 is left to the reader.

7. Discretizations of Kolmogorov type. We discuss definition and basic properties of $\mathcal{P}$th order one-step discretizations of (1).

Let h_0 be a positive constant. Let $\mathcal{P} \geq 1$, $k \geq 0$ be integers with $\mathcal{P} + k \geq 2$. Assume that $f_1, f_2, \ldots, f_n$ are $C^{\mathcal{P}+k+1}$ functions. The $C^{\mathcal{P}+k+1}$ property on closed sets like X (or $[0, h_0] \times X$) is understood as the existence of a $C^{\mathcal{P}+k+1}$ extension defined on an open neighborhood of X in $\mathbb{R}^n$ (or of $[0, h_0] \times X$ in $\mathbb{R} \times \mathbb{R}^n$). Consider a $C^{\mathcal{P}+k+1}$ discretization operator $\varphi : [0, h_0] \times X \to \mathbb{R}^n$. We assume that φ is of order $\mathcal{P}$, i.e., there exists a positive constant K (depending only on $\{f_i\}_{i=1}^n$) such that

$$|\Phi(h, x) - \varphi(h, x)| \leq K h^{\mathcal{P}+1} \quad \text{for all } h \in [0, h_0] \text{ and } x \in X.$$

We require also that φ is locally determined by $\{f_i\}_{i=1}^n$; i.e., we assume the existence of a continuous function $\Delta : [0, h_0] \to [0, \infty)$ such that $\Delta(0) = 0$ and, for all $h \in (0, h_0]$ and $x \in X$, $\varphi(h, x)$ is determined solely by the restriction of $\{f_i\}_{i=1}^n$ to $\mathcal{B}(x, \Delta(h))$. All these assumptions are satisfied if φ comes from a (general r-stage explicit or implicit) Runge–Kutta method. The standard theory of discretization operators (see, e.g., Stuart and Humphries [60]) implies that for all h sufficiently small, say $h \in [0, h_0]$, $\varphi(h, \cdot)$ is a $C^{\mathcal{P}+k+1}$ diffeomorphism of X onto $\varphi(h, X)$.

Now we are in a position to define discretizations of Kolmogorov type. Besides the above requirements on differentiability, consistency, and determinacy (these three were grouped together in [7] for the first time), two further conditions on a general discretization operator are imposed.

DEFINITION 7.1. *We say that our discretization operator is* of Kolmogorov type *on* X *for* (1) *if, for each* $i = 1, 2, \ldots, n$, *there exists a* $C^{\mathcal{P}+k+1}$ *function* $q_i : [0, h_0] \times X \to \mathbb{R}$ *satisfying*

$$\varphi_i(h, x) = x_i q_i(h, x) \quad \textit{whenever } h \in [0, h_0] \textit{ and } x \in X \tag{39}$$

and, in addition, $\varphi(h, X) \subset X$ *for each* $h \in [0, h_0]$.

LEMMA 7.2. *Let φ be a discretization operator of Kolmogorov type on X for* (1). *Then, for all h sufficiently small, say $h \in [0, h_0]$, $\varphi(h,\cdot)$ defines a $C^{\mathcal{P}+k+1}$ discrete-time dynamical system on X.*

Proof. We know already that $\varphi(h,\cdot)$ is a $C^{\mathcal{P}+k+1}$ diffeomorphism of X onto $\varphi(h,X) \subset X$. The surjectivity theorem applies. ∎

Remark 7.3. In accordance with (39), the solution operator of (1) satisfies

$$\Phi_i(h,x) = x_i Q_i(h,x) \quad \text{whenever } h \in [0,h_0] \text{ and } x \in X, \tag{40}$$

where $Q : [0,h_0] \times X \to \mathbb{R}^n$ is a $C^{\mathcal{P}+k}$ function defined by

$$Q_i(h,x) = \int_0^1 \frac{d}{dx_i} \Phi_i(h, x_1, \dots, x_{i-1}, \theta x_i, x_{i+1}, \dots, x_n)\, d\theta, \quad i = 1,2,\dots,n.$$

Actually, Q is of class $C^{\mathcal{P}+k+1}$. Existence and continuity of the last derivative is a consequence of the $C^{\mathcal{P}+k+1}$ parametrized version of the Picard–Lindelöf theorem. In fact, with $x \in X$ as a parameter, let $z(\cdot;x)$ denote the solution of the initial value problem

$$\dot z_i = z_i f_i(x_1 z_1, x_2 z_2, \dots, x_n z_n) \quad \text{and} \quad z_i(0) = 1, \qquad i = 1,2,\dots,n.$$

Since $(x_1 z_1(\cdot;x), x_2 z_2(\cdot;x), \dots, x_n z_n(\cdot;x))$ is a solution to (1), we have by uniqueness that $z(t;x) = Q(t,x)$ for all $t \in R$ and $x \in X$.

Example 7.4. Let $\vartheta : [0,h_0] \times X \to \mathbb{R}^n$ be a discretization operator coming from a (general r-stage explicit or implicit) Runge–Kutta method. It is a straightforward but rather lengthy task to check that, for all h sufficiently small, say $h \in [0,h_0]$, formula

$$\varphi_i(h,x) = \frac{\vartheta_i(h,x)}{\sum_j \vartheta_j(h,x)}\ , \qquad x \ \in X, \quad i = 1,2,\dots,n,$$

makes sense and defines a $\mathcal{P}$th order discretization operator of Kolmogorov type on X for (1). For example, the explicit Euler method leads to

$$\varphi_i^E(h,x) = x_i \frac{1 + h f_i(x)}{1 + h \sum_j x_j f_j(x)}\ , \quad (h,x) \in [0,h_0] \times X, \qquad i = 1,2,\dots,n\ ,$$

a first order discretization operator of Kolmogorov type.

The difference between exact and discretized solutions of (1) on finite-time intervals can be estimated as follows.

LEMMA 7.5. *Let φ be a $\mathcal{P}$th order discretization operator of Kolmogorov type on X for* (1). *Given $T > 0$ arbitrarily, there exists a positive constant $\kappa(T)$ such that for any $M = 0,1,2,\dots$ with $Mh \le T$, the estimate*

$$|\Phi_i(Mh,x) - \{\varphi^M(h,\cdot)\}_i(x)| \le x_i \cdot \kappa(T) \cdot h^{\mathcal{P}}, \quad (h,x) \in [0,h_0] \times X \tag{41}$$

holds true. (Here of course $\{\varphi^M(h,\cdot)\}_i$ denotes the ith coordinate function of the Mth iterate of the discretization mapping $\varphi(h,\cdot)$, $i = 1,2,\dots,n$.)

Proof. Writing out the coordinate functions explicitly, we find that methods of deriving the standard error estimate $|\Phi(Mh,x) - \varphi^M(h,x)| \le \kappa_0(T) h^{\mathcal{P}}$ (e.g., in [60]) apply and κ is an exponential function of T. For details, see [21]. ∎

LEMMA 7.6. *The previous lemma holds true for variable stepsize sequences. More precisely, given $T > 0$ arbitrarily and $\kappa(T)$ denoting the same constant as in* (41), *the estimate*

$$\left|\Phi_i\left(\sum_{m=1}^{M} h_m, x\right) - \{\varphi(h_M,\cdot)\circ\cdots\circ\varphi(h_1,\cdot)\}_i(x)\right| \le x_i\cdot\kappa(T)\cdot\left(\max_{1\le m\le M} h_m\right)^{\mathcal{P}}$$

holds true whenever $h_m \in (0, h_0]$, $m = 1, 2, \ldots, M$, with $\sum_m h_m < T$, and $x \in X$.

Proof. The proof is almost the same as that for constant stepsizes. ∎

8. Permanence for discretizations. Results in this section fit well in the list of papers in [60] on attraction, Liapunov functions, and discretization. They are particularly closely related to continuity results on exponentially attracting attractors in [4] and on convergence rates of perturbed attracting sets with vanishing perturbation [25]. For the qualitative theory of discretizations in general, see the monograph [60] as well as the fundamental paper [48].

LEMMA 8.1. *Fix $p_i > 0$, $i = 1, 2, \ldots, n$, and consider mapping $P : X \to \mathbb{R}_+$, $P(x) = \prod_{i=1}^n x_i^{p_i}$. Then P is an ALF for* (1) *if and only if P is an ALF for $\mathcal{F} = \Phi(1,\cdot)$.*

Proof. By letting $b = 1$ and $a = k - 1$, $k = 1, 2, \ldots, N$, in the identity $\log(Q_i(b, \Phi(a, y))) = \int_a^{a+b} f_i(\Phi(t, y))\, dt$ (a simple consequence of (40)) and forming the respective linear combinations,

$$\sum_{k=1}^{N}\sum_{i=1}^{n} p_i\cdot\log(Q_i(1, (\mathcal{F}^{k-1}(y)))) = \int_0^N \sum_{i=1}^{n} p_i f_i(\Phi(t, y))\, dt$$

holds for each $y \in Y$ and $N = 1, 2, \ldots$. Consequently, if P is an ALF for $\mathcal{F}$, then P is an ALF for (1) and $T_y = N_y$. Conversely, assume that $\int_0^{T_y} \sum f_i(\Phi(t, y))\, dt > 0$ for some $T_y > 0$. The compactness argument we used in proving Lemma 4.2 implies that $\int_0^{\tau_y} \sum f_i(\Phi(t, y))\, dt > 0$ for some positive integer τ_y. Thus P is an ALF for $\mathcal{F}$ and $N_y = \tau_y$. ∎

Remark 8.2. Together with Lemma 4.1, a similar argument implies that

$$\min_{\nu\in\mathcal{M}_{\Phi(1,\cdot)}} \int_Y \log(Q_i(1,\cdot))\, d\nu = \min_{\mu\in\mathcal{M}_\Phi} \int_Y f_i\, d\mu \qquad \text{for each } \; i = 1, 2, \ldots, n.$$

This is somewhat strange because $\mathcal{M}_\Phi \subset \mathcal{M}_{\Phi(1,\cdot)}$ and the set $\mathcal{M}_{\Phi(1,\cdot)}$ is usually a much larger subset of $\mathcal{M}$ than $\mathcal{M}_\Phi$. It is not hard to establish that the dependence of $\mathcal{M}_{\Phi(t,\cdot)}$ on the parameter $t \in (0, \infty)$ is weakly-$*$ upper semicontinuous. Moreover, if $\mathcal{U}$ is an open neighborhood of $\mathcal{M}_\Phi$ in the weak-$*$ topology of $\mathcal{M}$, then $\mathcal{M}_{\Phi(t,\cdot)} \subset \mathcal{U}$ for $|t|$ sufficiently small. Similarly, if φ is a discretization operator of Kolmogorov type, then there exists a positive constant $h_{\mathcal{U}}$ such that $\mathcal{M}_{\varphi(h,\cdot)} \subset \mathcal{U}$ whenever $0 < h \le h_{\mathcal{U}}$. (A detailed proof of this latter statement is contained in [21].) No upper semicontinuity result holds true for (the closure of the union of) supports of (all) invariant measures. On the general problem of measures and discretization, we recommend [14] and the references therein. Several upper semicontinuity results of numerical dynamics are contained also in [60].

Lemma 8.1 enables us to give a short proof for permanence under discretization.

THEOREM 8.3. *Assume that $P(x) = \prod_{i=1}^n x_i^{p_i}$ is an ALF for* (1). *Let φ be a $\mathcal{P}$th order discretization operator of Kolmogorov type for* (1). *Then, for all h sufficiently* *small, say $h \in (0, h_0]$, Y is a repeller for the discrete-time dynamical system induced by $\varphi(h, \cdot)$. In addition, there is a compact subset S of $X \setminus Y$ with the property that the dual attractor $\mathcal{A}_{\varphi(h,\cdot)}$ is contained in S, $h \in (0, h_0]$.*

Proof. Assume that $0 < h \le h_0 < 1$ and consider the positive integer M_h satisfying $M_h h < 1 \le (M_h + 1)h$. Since[6] Y is $\varphi(h, \cdot)$-invariant, Y is a repeller for $\varphi(h, \cdot)$ if and only if Y is a repeller for $\varphi^{M_h}(h, \cdot)$. Combining Lemma 8.1 and Theorem 6.3, we see it is enough to point out that $\mathcal{G} = \varphi^{M_h}(h, \cdot)$ is a δ-perturbation of $\mathcal{F} = \Phi(1, \cdot)$.

In fact, for each $x \in X$, $h \in (0, h_0]$ and $i = 1, 2, \ldots, n$, inequality $0 < 1 - M_h h \le h$ implies that

$$|\Phi_i(1, x) - \Phi_i(M_h h, x)| = |\Phi_i(1 - M_h h, \Phi(M_h h, x)) - \Phi_i(M_h h, x)| \le x_i \cdot (e^{Kh} - 1)$$

with $K = \max_{1 \le i \le n} \max_{x \in X} |f_i(x)|$. We conclude via (41) that

$$|\Phi_i(1, x) - \{\varphi^{M_h}(h, \cdot)\}_i(x)| \le x_i \left\{(e^{Kh} - 1) + \kappa(1) \cdot h^{\mathcal{P}}\right\}. \tag{42}$$

Note that the coefficient of x_i on the right-hand side of (42) approaches zero as $h \to 0^+$. Thus $\mathcal{G} = \varphi^{M_h}(h, \cdot)$ is a δ-perturbation of $\mathcal{F} = \Phi(1, \cdot)$ for h small enough and Theorem 6.3 applies. ◻

We do not know if Theorem 8.3 is true for a general GALF. The main difficulty is in proving the inequality

$$|P(\Phi(Mh, x)) - P(\varphi^M(h, x))| \le \tilde{\kappa}_1(T) h^{\mathcal{P}} \cdot P(x) \quad \text{if } 0 < T,\ 0 \le Mh \le T, \tag{43}$$

which seems to be a rather delicate matter.

Remark 8.4. Assume that the conditions of Theorem 8.3 are all satisfied. Combining Theorem 4.4 and the discretization result in Remark 8.2, one can establish the existence of positive constants c_*, h_* with the following property: For every stepsize $h \in (0, h_*]$ and $\mu_h \in \mathcal{M}_{\varphi(h,\cdot)}$ there exists an $i \in \{1, 2, \ldots, n\}$ with $\int_Y \log q_i(h, \cdot)\, d\mu_h > c_* h$. An alternative presentation of a great part of sections 8, 9, 10, and 11 can be centered around (the local version of) this inequality.

9. Variable stepsize discretizations. In this section we present a generalization of Theorem 8.3 for variable stepsize discretizations.

The natural framework of handling variable stepsize sequences is that of nonautonomous dynamics. For general considerations, including several attractor definitions in the nonautonomous setting, we refer to [45]. In what follows we restrict ourselves to recalling the concept of cocycle attractors/repellers and to presenting a special case of the key result from Kloeden and Schmalfuss [44].

THEOREM 9.1. *Assume that* (1) *is permanent and let φ be a discretization operator of Kolmogorov type for* (1). *In addition, let $\mathcal{U}_Y$ be an open neighborhood of Y in X, let $\mathcal{U}_{\mathcal{A}_\Phi}$ be an open neighborhood of $\mathcal{A}_\Phi$ in X, and assume that $\mathcal{U}_Y \cap \mathcal{U}_{\mathcal{A}_\Phi} = \emptyset$. Then there are positive constants h_* and τ with the following properties: Given an arbitrary set C with $\mathcal{U}_{\mathcal{A}_\Phi} \subset C \subset X \setminus \mathcal{U}_Y$ and a doubly infinite stepsize sequence $\mathbf{h} = \{h_k\}_{k=-\infty}^{\infty}$ with $\sum_{k=1}^{\infty} h_k = \sum_{k=-\infty}^{0} h_k = \infty$ and $\|\mathbf{h}\| = \sup h_k \le h_*$,*

$$\{\varphi(h_M, \cdot) \circ \cdots \circ \varphi(h_1, \cdot)\}(C)\ ,\ \{\varphi(h_0, \cdot) \circ \cdots \circ \varphi(h_{-M}, \cdot)\}(C) \subset \mathcal{U}_{\mathcal{A}_\Phi} \tag{44}$$

[6]Consider $W = [-1, 1] \subset \mathbb{R}$, $\mathcal{F}_0(w) = -w^3$ for $w \in W$. Then $\{1\}$ is a repeller for $\mathcal{F}_0^2$ but not for $\mathcal{F}_0$. The reason is that $\{1\}$ is not $\mathcal{F}_0$-invariant.

whenever $\sum_{k=1}^{M} h_k \geq \tau$ *and* $\sum_{k=-M}^{0} h_k \geq \tau$*. In addition, the set*

$$A(\mathbf{h}) = \bigcap_{M\geq 0} \mathrm{cl}\left(\bigcup_{m\geq M}\{\varphi(h_0,\cdot)\circ\cdots\circ\varphi(h_{-m},\cdot)\}(C)\right)$$

(45) *is independent of* C *and is contained in* $\mathrm{cl}(\mathcal{U}_{\mathcal{A}_\Phi})$,

$$\mathrm{cl}\left(\bigcup_{m\geq M}\{\varphi(h_0,\cdot)\circ\cdots\circ\varphi(h_{-m},\cdot)\}(C)\right) \to A(\mathbf{h}) \tag{46}$$

in the Hausdorff metric as $M\to\infty$, *and, with* $\theta^m\mathbf{h}$ *denoting the doubly infinite shifted stepsize sequence defined by* $(\theta^m\mathbf{h})_k = h_{k+m}$,

$$\{\varphi(h_m,\cdot)\circ\cdots\circ\varphi(h_1,\cdot)\}(A(\mathbf{h})) = A(\theta^m\mathbf{h}) \quad \textit{for each} \ \ m=1,2,\ldots. \tag{47}$$

Proof. This is a restatement of Theorems 3.1 and 4.5 of [44] within the context of the present paper. Actually, the original results in Kloeden and Schmalfuss [44] are proved under the additional requirement

$$\sup\{h_k/h_\ell \mid k,\ell\in\{0,\pm1,\pm2,\ldots\}\} \leq \mathrm{const.} \tag{48}$$

The starting point of their proof is a classical result in converse Liapunov theory, Theorem 22.5 of Yoshizawa [67] on the existence of Lipschitz continuous Liapunov functions. However, when starting from Conley's C^∞ Liapunov function for the attractor–repeller pair $(\mathcal{A}_\Phi, Y)$, condition (48) turns out to be irrelevant. It is enough to replace Lemma 4.1 of [44] by Lemma 9.2 below and to reconsider the Kloeden–Schmalfuss argumentation. We find that Theorem 9.1 holds true for free stepsize sequences (subject only to the requirements $\sum_{k=1}^{\infty} h_k = \sum_{k=-\infty}^{0} h_k = \infty$ and $\|\mathbf{h}\| = \sup h_k \leq h_*$). ∎

For convenience, recall Lemma 1 of [20], which we "inserted" in the original proof of Theorem 9.1 in [44] above.

LEMMA 9.2. *There exists a* C^∞ *function* $V: X\to[0,1]$ *with the following properties: For every* $x\in X\setminus(\mathcal{A}_\Phi\cup Y)$, *function* $\mathbb{R}\to(0,1)$, $t\to\Phi(t,x)$ *is strictly decreasing, and, in addition,* $V^{-1}(0)=\mathcal{A}_\Phi$, $V^{-1}(1)=Y$. *Finally, for* $c\in(0,1)$ *arbitrarily given, there exists a positive constant* $h^*(c)$ *such that* $V(\varphi(h,x))<c$ *whenever* $h\in(0,h^*(c)]$ *and* $V(x)\leq c$.

Proof. This is a discretization consequence of Theorem 6.12 of Akin [2]. Details can be found in [20]. ∎

Most results in [44], [45] are stated and proved for abstract cocycles with the shift operator θ acting on a compact parameter space. Having applications to stochastic numerics in mind, no attempt is made in these papers to lift/weaken condition (48) in the simplest special case of deterministic discretizations with variable stepsize. The set $A(\mathbf{h})$ is called a *cocycle* or *pull-back attractor*. Properties (44), (45), (46), and (47) are called the *upper semicontinuity, uniqueness, pull-back convergence*, and *equivariance properties*, respectively. Elementary examples show that, together with the stepsize sequence $\mathbf{h}=\{h_k\}_{k=-\infty}^{\infty}$, the accompanying push-forward sequence of sets $\{\varphi(h_M,\cdot)\circ\cdots\circ\varphi(h_1,\cdot)\}(C)$ may also exhibit an oscillating behavior in $\mathcal{U}_{\mathcal{A}_\Phi}$. This explains why cocycle attractors are defined as they are, i.e., by using pull-back convergence. For constant stepsize sequences, Theorem 9.1 reduces to results in [43], the starting point of the theory on numerical attractors.

The dual concept to cocycle attractors is that of a cocycle repeller. Reversing time, Theorem 9.1 establishes the existence of a cocycle repeller $R(\mathbf{h})$. Nevertheless, even for discretizations of Kolmogorov type, the general theory says only that $R(\mathbf{h}) \to Y$ in an upper semicontinuous way. However, in the case that permanence is granted by the standard GALF assumption, $R(\mathbf{h}) = Y$ for every doubly infinite stepsize sequence with $\|\mathbf{h}\|$ sufficiently small.

Actually, a stronger result—a discretized version of Theorem 3.1—holds true.

THEOREM 9.3. *Assume that $P(x) = \prod_{i=1}^n x_i^{p_i}$ is a GALF and let φ be a $\mathcal{P}$th order discretization operator of Kolmogorov type for* (1) *on X. Then there exist an open neighborhood $\mathcal{W}$ of Y in X and positive constants $h_0, \lambda_1, \lambda_2, \lambda_3$ with the properties as follows. Given an infinite stepsize sequence $\{h_k\}_{k=1}^\infty$ with $\sum_{k=1}^\infty h_k = \infty$ and $\sup h_k \le h_0$,*

$$d_E\left(\{\varphi^{-1}(h_1,\cdot) \circ \cdots \circ \varphi^{-1}(h_M,\cdot)\}(x), Y\right) \le \lambda_1 e^{-\lambda_2(h_1+\cdots+h_M)}(d_E(x,Y))^{\lambda_3}$$

whenever $x \in \mathcal{W}$, $M = 1, 2, \ldots$. Here of course $\varphi^{-1}(h_k,\cdot)$ denotes the inverse of $\varphi(h_k,\cdot)$, $k = 1, 2, \ldots, M$, established by Lemma 7.2.

Proof. The proof is an expanded version of that of Theorem 3.1. For details, see [21]. ◻

A similar result holds true for asymptotically autonomous systems. Consider the ordinary differential equation

$$\dot{x}_i = x_i e_i(t,x), \qquad (t,x) \in \mathbb{R} \times X, \tag{49}$$

where $e_i : \mathbb{R} \times X \to X$ is a continuous function satisfying

$$e_i(t,x) \to f_i(x) \qquad \text{uniformly in } x \in X \text{ as } t \to \infty, \quad i = 1, 2, \ldots, n$$

and $\sum_i x_i e_i(t,x) = 0$ for each $(t,x) \in \mathbb{R} \times X$. Assume that system (49) has the uniqueness property and that function f in the limiting autonomous system (1) is Lipschitz. The solution of (49) through $(t_0, x) \in \mathbb{R} \times X$ is denoted by $\Psi(\cdot, t_0, x)$.

If the limiting autonomous system (1) is robustly permanent due to a standard GALF, then (49) is permanent too. More precisely, the following result holds true.

THEOREM 9.4. *Assume that $P(x) = \prod_{i=1}^n x_i^{p_i}$ is a GALF for* (1) *on X. Let $\mathcal{U}_{\mathcal{A}_\Phi}$ be an open neighborhood of $\mathcal{A}_\Phi$ in X and let C be a compact subset of $X \setminus Y$. Given an initial time t_0 arbitrarily, there exists a time T such that*

$$\Psi(t_0 + t, t_0, x) \in \mathcal{U}_{\mathcal{A}_\Phi} \qquad \textit{whenever } t \ge T \textit{ and } x \in C.$$

Proof. The proof is a simple variation of the proof of Theorem 9.3. Some details are contained in [21]. ◻

10. Connections to index theories. Let K be a nonempty Φ-invariant compact subset of Y. Following Szymczak, Wojcik, and Zgliczynski [63], we say that K is *of repelling type* if $\{x \in X : \emptyset \ne \omega(x) \subset K\} \subset Y$. In view of Theorem 5.2 above, the existence of a GALF for (1) on K implies that K is of repelling type and, for some $\eta > 0$, $\mathcal{B}(K,\eta) \setminus Y$ does not contain entire trajectories. Starting from property $(\beta)_K$, the very same conclusions are derived in the first part of the proof of Theorem 4.4 of Schreiber [57]. By $(\alpha)_K \Leftrightarrow (\beta)_K$ in Theorem 5.3, property $(\beta)_K$ means that $P_K : X \to \mathbb{R}$, $P_K = \prod_{i=1}^n x_i^{p_i}$ defines a GALF for Φ on K. In particular, the existence of a local GALF plus the isolatedness of K with respect to the boundary flow $\Phi|_Y$ imply that K is isolated (i.e., isolated with respect to the entire flow Φ on X).

From now on, assume that $\emptyset \ne K \subset Y$ is a compact isolated invariant set of repelling type. Assume, in addition, that K is a repeller for $\Phi|_Y$. In view of the

Zubov–Ura–Kimura theorem, K is a repeller for Φ (i.e., an attractor for the backward flow Φ^* defined by $\Phi^*(t,x) = \Phi(-t,x)$ for all $(t,x) \in \mathbb{R} \times X$) and thus $\emptyset \neq \alpha(x) \subset K$ for any $x \in X$ with $d_E(x,K)$ sufficiently small. Alternatively, assume that K is an attractor for $\Phi|_Y$. Applying the Zubov–Ura–Kimura theorem again, we find *there exists an* $x \in X \setminus Y$ *with* $\emptyset \neq \alpha(x) \subset K$. Geometrically, the property italicized above means that K repels a trajectory from Y into $X \setminus Y$. However, if K is neither a repeller nor an attractor for $\Phi|_Y$, then the existence of an $x \in X \setminus Y$ with $\emptyset \neq \alpha(x) \subset K$ is a rather delicate matter and requires methods of algebraic topology.

By using standard degree theory, the same problem in $\mathbb{R}^{n-1} \times [0,\infty)$ (and $K \subset \mathbb{R}^{n-1} \times \{0\}$ being a finite collection of equilibria (and $\mathbb{R}^{n-1} \times \{0\}$ invariant)) was first investigated by Hofbauer [29]. Capietto and Garay [13] used the fixed point index (which is an appropriate version of degree theory) and a more general index theory developed by Conley [12]. Their approach, however, worked only for flows induced by vector fields and some special kinds of isolated invariant sets. Both restrictions were removed and much more Conley-type results proved by Wojcik [65]. Generalizations for discrete-time semidynamical systems were given by Szymczak, Wojcik, and Zgliczynski [63]. For details, in particular for the index theories involved, we refer to the original papers [13], [65], [63] and the references cited therein.

The next theorem is a straightforward consequence of the main results of [63] within the context of the present paper.

THEOREM 10.1. *Let* $\emptyset \neq K \subset Y$ *be a compact isolated invariant set of repelling type. Assume that the homotopical Conley index* $I_C(K, \Phi|_Y, Y)$ *of* K *with respect to the boundary flow* $\Phi|_Y$ *in* Y *is nontrivial. Then there exists an* $x \in X \setminus Y$ *with* $\emptyset \neq \alpha(x) \subset K$.

Proof. If $K = Y$, then the Zubov–Ura–Kimura theorem applies.

If $K \neq Y$, then consider a point $y_0 \in Y \setminus K$ and note that the pair $(X \setminus \{y_0\}, Y \setminus \{y_0\})$ is homeomorphic to the pair $(\mathbb{R}^{n-2} \times [0,\infty), \mathbb{R}^{n-2} \times \{0\})$. Modifying the dynamics in a small vicinity of y_0 in X, we may assume that y_0 is an equilibrium point for Φ. Hence all results in [63] (proved for compact isolated invariant subsets of $\mathbb{R}^{n-2} \times \{0\}$, the boundary of the half-space $(\mathbb{R}^{n-2} \times [0,\infty))$ translate into results on compact Φ-invariant subsets of $Y \setminus \{y_0\}$.

By Theorem 2 of [63], the homotopical Conley index $I_C(K, \Phi(1,\cdot), X)$ of K with respect to the time-one map of Φ in X is trivial. If $x \in X \setminus Y$ with $\emptyset \neq \alpha(x) \subset K$ for no $x \in X \setminus Y$, then K is also of attracting type, and thus, by Theorem 1 of [63], $I_C(K, \Phi(1,\cdot), X) = I_C(K, \Phi(1,\cdot)|_Y, Y)$. Since the index map is homotopic to the identity, we conclude that, together with $I_C(K, \Phi(1,\cdot)|_Y, Y)$, also $I_C(K, \Phi|_Y, Y)$ is trivial, a contradiction. ∎

Unfortunately, it is in general very difficult to check whether the homotopical Conley index $I_C(K, \Phi|_Y, Y)$ is nontrivial or not. Note, however, that nontriviality of $I_C(K, \Phi|_Y, Y)$ is a consequence of $I_F(K, \Phi|_Y, Y) \neq 0$, nontriviality of the fixed point index, and that this latter condition can be fairly easily checked [1]. Besides, $I_F(K, \Phi^*|_Y, Y) = (-1)^n I_F(K, \Phi|_Y, Y)$. The time-duality problem for the homotopical Conley index, in particular the question of whether nontriviality of $I_C(K, \Phi^*|_Y, Y)$ is equivalent to the nontriviality of $I_C(K, \Phi|_Y, Y)$, seems to be open. The answer is affirmative on the homology–cohomology level of the Conley index [53].

The following result is a discretization analogue of Theorem 10.1.

THEOREM 10.2. *Let* $\emptyset \neq K \subset Y$ *be a compact isolated invariant set of repelling type. Let* U *be an open neighborhood of* K *in* $\mathbb{R}^n_+$ *and let* $P_K : U \to \mathbb{R}$, $x \to \prod_{i=1}^n x_i^{p_i}$ *be a GALF for* (1) *on* K. *In addition, assume that* K *is the maximal compact* $\Phi|_Y$*-invariant set in* $\mathrm{cl}(U) \cap Y$ *and that the cohomological Conley index* $i_C(K, \Phi|_Y, Y)$ *is* *nontrivial. Finally, let $K_h \subset Y$ denote the maximal compact $\varphi(h,\cdot)|_Y$-invariant set in $\mathrm{cl}(U) \cap Y$. Then, for h sufficiently small, $K_h \neq \emptyset$ and, for some $x_h \in X \setminus Y$ suitably chosen, $\emptyset \neq \alpha_{\varphi(h,\cdot)}(x_h) \subset K_h$.*

Proof. If $K = Y$ and h is small enough, then $K_h = Y$ is a repeller for $\varphi(h,\cdot)$ by Theorem 8.3, and the discrete-time version of the Zubov–Ura–Kimura theorem applies.

If $K \neq Y$ and h is small enough, then $i_C(K_h, \varphi(h,\cdot)|_Y, Y) = i_C(K, \Phi|_Y, Y)$ by the main result in Mrozek and Rybakowski [54] (when applied to $Y \setminus \{y_0\}$, which is locally Lipschitz homeomorphic to $\mathbb{R}^{n-2}$ for any $y_0 \in Y \setminus K$). Hence $K_h \neq \emptyset$. On the other hand, $K_h \subset U \cap Y$ by the upper semicontinuity result in [22], h sufficiently small. Furthermore, combining the proofs of Theorems 5.2 and 8.3, it is not hard to show that K_h (as a subset of X) is isolated with respect to $\varphi(h,\cdot)$ and, for each $x \in U \setminus Y$, inclusion $\emptyset \neq \omega_{\varphi(h,\cdot)}(x) \subset K_h$ is impossible. Thus $\emptyset \neq K_h \subset Y$ is a compact isolated $\varphi(h,\cdot)$-invariant set of repelling type and (as a weakening of the nontriviality of the cohomological Conley index $i_C(K_h, \varphi(h,\cdot)|_Y, Y)$), the homotopical Conley index $I_C(K_h, \varphi(h,\cdot)|_Y, Y)$ is nontrivial. Using Theorems 1 and 2 of [63], the desired result follows immediately. ∎

Discretizations have better topological properties than general discrete-time dynamical systems. For example, they preserve orientation. Near transversal sections, discretizations (for h small enough) embed to continuous-time local dynamical systems [17], [23]. A further nontrivial topological property of discretizations is what we called numerical Wazewski property [22], [18]. In certain applications, as it was pointed out by Conley [12] himself, the classical Wazewski principle is stronger than the index.

Conjecture. *We conjecture that the nontriviality of the cohomological Conley index $i_C(K, \Phi|_Y, Y)$ in Theorem 10.2 can be replaced by the following requirement: Assume that B^+ is not a retract of B, where $B \subset U$ is an isolating block with respect to the boundary flow $\Phi|_Y$ for K in Y and B^+ denotes the entry set of B.*

Combining Theorems 1 in [13] and C4 in [22], we find that the conjecture holds true under the additional conditions that K is contained in a single face of Y and B is an isolating block with corners. One of the major difficulties in proving the conjecture is constructing C^∞ Liapunov functions for attractor–repeller pairs on manifolds with corners (such as Y).

11. Applications. In all the previous sections we worked with the simplex as the phase space and noted only that analogous results are valid in $\mathbb{R}^n_+$ for dissipative flows. In the present section we give applications to systems on other phase spaces such as compact smooth manifolds, half-spaces, products of a simplex with a ray, and products of simplices.

Subsection 11.1 is devoted to differential equations near compact smooth codimension 1 submanifolds of $\mathbb{R}^n$. Note that a much deeper codimension $k \geq 1$ analysis, based on Oseledec's theory, is given in [3], [11] for diffeomorphisms. It is an open problem to extend the concept of GALF (which is at present a codimension 1 object) to codimension k problems.

In subsections 11.2 and 11.3 we study robust permanence of replicator and Lotka–Volterra equations. The problem of successful invasion is discussed in subsection 11.4. Subsection 11.5 is devoted to discretized game dynamics with variable stepsizes.

11.1. Manifolds with smooth boundary. Let Ω be a bounded open set in $\mathbb{R}^n$ and assume that $Z = \partial\Omega$ is a compact smooth codimension 1 submanifold of $\mathbb{R}^n$. Let U be an open neighborhood of Z in $\mathbb{R}^n$ and let $F : U \to \mathbb{R}^n$ be a C^1 function. Assume that Z is invariant with respect to the (local) flow Θ of the ordinary differential equation $\dot{x} = F(x)$, $x \in U$. The set of Θ-invariant Borel probability measures on Z is denoted by $\mathcal{M}_\Theta(Z)$. Finally, for $z \in Z$, let $\nu(z) \in \mathbb{R}^n$ denote the outer normal unit vector of Z at z. It is well known that, for some $\varepsilon > 0$, mapping $(-\varepsilon, \varepsilon) \times Z \to U$, $(\lambda, z) \to x = z + \lambda\nu(z)$ is a coordinate transformation. In this new system of normal and tangential coordinates, $\dot{x} = F(x)$, $x \in U$ can be rewritten as the system $\dot{\lambda} = N(\lambda, z)$, $\dot{z} = T(\lambda, z)$, $(\lambda, z) \in (-\varepsilon, \varepsilon) \times Z$. Since $N(0, z) = 0$ for each $z \in Z$, there exists a continuous function $S : (-\varepsilon, \varepsilon) \times Z \to \mathbb{R}$ satisfying $N(\lambda, z) = \lambda S(\lambda, z)$. With $P(x) = \lambda$ (or, equivalently, $P(x) = \lambda^p$ for any $p > 0$) as GALF and applying (the corresponding analogue, with (X, Y) replaced by $(\mathrm{cl}(\Omega), Z)$, of) Theorem 2.2 and Lemmas 4.1 and 4.2, we obtain the following theorem.

THEOREM 11.1. *If the normal Liapunov exponent*

$$\int_Z S(0, z)\, d\mu > 0 \tag{50}$$

for each (ergodic) $\mu \in \mathcal{M}_\Theta(Z)$, then Z is a repeller for Θ.

It is not hard to compute $S(0, z)$ explicitly:

$$S(0, z) = \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{dF_i}{dx_j}(z) \cdot \nu_j(z) \cdot \nu_i(z) = \langle \nu(z), \mathcal{J}(z)\nu(z) \rangle,$$

where $\mathcal{J}(z)$ denotes the Jacobian of F evaluated at z and $\langle \cdot, \cdot \rangle$ denotes the standard scalar product in $\mathbb{R}^n$.

We note that an analogous result holds for abstract manifolds X with smooth collared boundary $\partial X = Z$. Condition (50) implies via Theorem 3.4 the existence of a Liapunov function in a neighborhood of Z of the form $V(x) = \lambda Q(\lambda, z)$ with Q positive and C^1.

Using Pesin's theory, one can show, as in [3] or [57], that the converse of Theorem 11.1 is "almost" true: If F is C^2 and the reverse inequality holds in (50) for at least one invariant measure $\mu \in \mathcal{M}_\Theta(Z)$ (i.e., at least one normal Liapunov exponent is negative), then the invariant manifold Z attracts at least one orbit from $\Omega \setminus Z$. If F is only C^1, then a weaker converse result can be obtained from the ergodic closing lemma, as in [30], [57]: There are arbitrarily small C^1-perturbations of the flow, with Z as invariant manifold, that have a periodic orbit in Z, which has negative normal Floquet exponent and is therefore normally attracting. Other converse results can be derived from index theory, as used in section 10.

We finally remark that Theorem 11.1 can be applied also to study dissipativity in a suitable compactification of the state space and to investigate critical cases of stability by analyzing homogeneous differential equations that arise as the principal part of normal forms, such as Molchanov's theorem [42]; for details see [21].

11.2. Consequences for replicator equations. The replicator equation (4) on the simplex X enjoys an important averaging principle (see [35, Thm. 7.6.4], [41]), which can be stated in terms of time averages or space averages.

LEMMA 11.2. (1) *Suppose that for $x \in X$, $\omega(x)$ is contained in some (relatively open) face of the simplex. Then every limit point of the time average of this solution,*

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \Phi(t, x)dt, \tag{51}$$

is an equilibrium point $\bar{x}$ on this face.

(2) *Let* $\mu \in \mathcal{M}_{\mathcal{F}}^{E}$ *be an ergodic invariant measure for* (4). *Then its mean* $\bar{x} = \int x d\mu(x)$ *is an equilibrium of* (4), *and the external Liapunov exponents of* μ *coincide with that at* $\bar{x}$.

A consequence of this averaging property is that in applying Theorem 4.4 for replicator equations one can restrict oneself to (convex combinations of) point measures instead of all invariant measures. Hence a finitely computable sufficient condition for robust permanence can be established. In particular, Corollary 2.4 simplifies to the following result from [35] and [41], but now strengthened with robustness.

THEOREM 11.3. *If there are* $p_i > 0$ $(i = 1, \ldots, n)$ *such that for every fixed point* $\bar{x}$ *of* (4) *in* Y,

$$p \cdot A\bar{x} > \bar{x} \cdot A\bar{x},$$

then $P(x) = \prod_i x_i^{p_i}$ *is a GALF, and hence* (4) *is robustly permanent.*

Consider now the discrete-time replicator dynamics

$$(\mathcal{F}(x))_i = x_i \frac{1 + h(Ax)_i}{1 + h\ x \cdot Ax}. \tag{52}$$

Here $h > 0$ is such that $1 + ha_{ij} > 0$ for all i, j. Then the map (52) is a diffeomorphism on X [49]. Another discrete-time replicator dynamics is

$$(\mathcal{F}(x))_i = x_i \frac{e^{h(Ax)_i}}{\sum_j x_j\ e^{h(Ax)_j}}, \tag{53}$$

which is a diffeomorphism for h small and, in contrast to (52), enjoys a similar averaging property to that of (4) [35, p. 79, Ex. 7.6.6]. Note that both (52) and (53) are first order discretization operators of Kolmogorov type for (4), but none of them is of the form of those investigated in Example 7.4 (except for zero-sum games, i.e., $A = -A^T$).

The first part of our next result is a simple consequence of Theorem 8.3, whereas the stronger result in the second part follows from Theorem 6.3 and the averaging property. Finding a finitely computable condition for robust permanence for (52) for arbitrary $h > 0$ is an open problem.

THEOREM 11.4. *Let the assumption of Theorem* 11.3 *hold, i.e., there exists a standard ALF* $P(x) = \prod_i x_i^{p_i}$ *for the continuous-time replicator dynamics* (4). *Then for small* $h > 0$, *the discrete-time replicator equation* (52) *is robustly permanent. Similarly, for all* $h > 0$, P *is also an ALF for* (53), *and hence* (53) *is robustly permanent.*

11.3. Consequences for Lotka–Volterra equations. Lotka–Volterra systems

$$\dot{x}_i = x_i\big(r_i + (Ax)_i\big), \quad x \in \mathbb{R}_+^n, \tag{54}$$

enjoy a similar averaging property to that of Lemma 11.2 for replicator equations. This goes essentially back to Volterra; see [35], [41], [57, Lem. 7.1]. Hence for Lotka–Volterra equations (54) it is sufficient to consider point measures at boundary equilibria when applying our permanence results. In particular, our Theorem 4.4 shows that the sufficient condition in Schreiber's [57, Thm. 7.2] for robust permanence of Lotka–Volterra equations is *equivalent* to the sufficient condition for permanence based on a standard GALF due to Jansen [41] (see also [35, Ex. 13.6.3]):

There are $p_i > 0$ $(i = 1, \dots, n)$ *such that for every fixed point* $\bar{x}$ *of* (54) *on* $\partial\mathbb{R}^n_+$,

$$p \cdot (r + A\bar{x}) > 0. \tag{55}$$

Now we turn to discrete-time Lotka–Volterra systems such as a higher-dimensional version of the logistic map,

$$(\mathcal{F}(x))_i = x_i(1 + h(r_i + (Ax)_i)), \tag{56}$$

and an exponential version (see [33] and [50]),

$$(\mathcal{F}(x))_i = x_i e^{h(r_i + (Ax)_i)}. \tag{57}$$

The first part of our next result is a simple consequence of Theorem 8.3, whereas the second part follows from Theorem 6.3 and from the averaging principle in [33].

THEOREM 11.5. *Suppose* (56) *and* (57) *are (robustly) dissipative and* (55) *holds true; i.e., there exists a standard ALF* $P(x) = \prod_i x_i^{p_i}$ *for* (54). *Then for small* $h > 0$, (56) *is (robustly) permanent. Similarly, for all* $h > 0$, P *is an ALF for* (57), *which is (robustly) permanent.*

Dissipativity of (57) is discussed in [33] and [50].

As a further application we rederive and strengthen a recent result of Mierczyński and Schreiber [52] on totally permanent Lotka–Volterra systems. They established the equivalence (L2) ⇔ (L4) with their weaker meaning of robust permanence. Note that (L3) and (L4) are computable conditions.

THEOREM 11.6. *The following conditions are equivalent:*

(L1) *Equation* (54) *as well as all its subsystems are permanent.*

(L2) *Equation* (54) *as well as all its subsystems are robustly permanent.*

(L3) $-A$ *is a P-matrix (i.e., all principal minors of* $-A$ *are positive) and each (relatively open* k*-dimensional,* $k = 1, 2, \dots, n$*) face of* $\mathbb{R}^n_+$ *contains an equilibrium.*

(L4) *Equation* (54) *is dissipative, each face contains a unique equilibrium, and all its external eigenvalues are positive.*

Proof. Every permanent Lotka–Volterra system has a unique interior equilibrium and $\det(-A) > 0$; see [35, Thms. 13.5.1 and 13.5.2]. Applying this to all subsystems we conclude that (L1) implies (L3). The P-matrix property implies the dissipativity of (54) and also uniqueness of saturated equilibrium; see [35, Thms. 15.2.1 and 15.4.5]. Hence no boundary equilibrium can have an external eigenvalue ≤ 0. This shows (L3) ⇒ (L4). Finally (L4) implies (L2) by Corollary 4.6, and (L2) ⇒ (L1) is trivial. ◻

11.4. Invasion of a permanent system. Consider a permanent n-species community and a further species which is able to invade that resident community. Will the invader be able to survive, i.e., will the population move towards a new stable community consisting of the invader and a certain subset of the resident population? In the biological literature, e.g., in [66], this question is often phrased as, *Does invasion lead to persistence?*

A positive answer to this question is possible only under stringent assumptions. For example, if in the resident system there are several attractors, and the new species invades at one attractor, it could be driven out again by leading the population to the other attractor. This can be avoided only if all normal Liapunov exponents on the global interior attractor are positive.

Even then, one could imagine that the population evolves to a state where the invader as well as some of the resident species are eliminated. A simple example in two dimensions is the system

$$\dot{x} = x\left(x(1-x) - y\right),$$
$$\dot{y} = y(x-y).$$

The density of the invading species y increases near the resident equilibrium $x = 1, y = 0$. But every interior solution converges to the origin: $\frac{y}{x}$ increases monotonically, and for $y > x$, y decreases. However, this dynamics is degenerate, since the origin is not hyperbolic. If the resident system is robustly permanent (thanks to a GALF), this extinction phenomenon cannot occur. This is the essence of the next theorem, which generalizes an analogous result on Lotka–Volterra equations in [32].

THEOREM 11.7. *We consider a system of* n *resident species on* $X = \mathbb{R}^n_+$ *and an invader whose density we denote by* $y \geq 0$,

$$\dot{x}_i = x_i f_i(x,y), \tag{58}$$
$$\dot{y} = yg(x,y) \tag{59}$$

on the augmented state space $X' = \mathbb{R}^n_+ \times \mathbb{R}_+$. *We identify* X *with the subsystem* $X \times \{0\}$ *of* X' *and assume that* (58)–(59) *give rise to a dissipative dynamical system* $\Theta = (\Phi, \Psi) : \mathbb{R} \times X' \to X' = X \times \mathbb{R}_+$. *In addition, assume there exists a GALF for the resident system* (58) *with* $y = 0$ *which is therefore robustly permanent. Finally, assume that the global attractor* $\mathcal{A} \subset \operatorname{int} X$ *of* (58) *is nonsaturated in the sense that*

$$\int_{\mathcal{A}} g(x,0) d\mu_x > 0 \qquad \textit{for each} \quad \mu_x \in \mathcal{M}_\Theta(\mathcal{A}). \tag{60}$$

Then

$$\limsup_{t\to\infty} \Psi(t,x,y) > 0 \qquad \textit{for all } (x,y) \in \operatorname{int} X'.$$

Proof. To the contrary, suppose that $\Psi(t,z,w) \to 0$ for some $(z,w) \in \operatorname{int} X'$. Hence $\emptyset \neq \omega_\Theta(z,w) \subset X$. Actually, since $(\mathcal{A}, \partial X)$ is an attractor–repeller decomposition of the resident system, $\omega_\Theta(z,w) \subset \mathcal{A}$ or $\omega_\Theta(z,w) \subset \partial X = Y$. Combining Theorems 5.3 and 5.2, condition (60) implies that the first inclusion is impossible. Hence $\omega_\Theta(z,w) \subset Y$.

Consider now the GALF $P : X \to \mathbb{R}$ and observe for each $t \in \mathbb{R}$ that

$$\frac{d}{dt}\log(P(\Phi(t,z,w))) = \sum_{i=1}^{n} p_i(\Phi(t,z,w)) f_i(\Phi(t,z,w), \Psi(t,z,w)).$$

Integrating between 0 and T yields

$$\frac{1}{T}\log\left(\frac{P(\Phi(T,z,w))}{P(z)}\right) = \frac{1}{T}\int_0^T \sum_{i=1}^{n} p_i(\Phi(t,z,w)) f_i(\Theta(t,z,w))\, dt. \tag{61}$$

Since $\Phi(t,z,w)$ approaches Y, $P(\Phi(T,z,w)) \to 0$ as $T \to \infty$. Thus each limit point of the left-hand side of (61) is ≤ 0. On the other hand, property $\Psi(T,z,w) \to 0$, the GALF assumption, Lemma 4.2, and the robustness arguments in the proof of

Theorem 2.2 imply that every limit point as $T \to \infty$ on the right-hand side is positive. This is a contradiction. □

An obvious drawback of Theorem 11.7 is that only positivity of the limsup can be guaranteed. However, one cannot do better in general: It is easy to construct examples with $\liminf_{t\to\infty} \Psi(t,x,y) = 0$. Consider (58)–(59) on $X' = \mathbb{R}^4_+$ with three resident species (1,2,3). Suppose, that the invader eliminates resident species 3 and forms together with 1 and 2 a system with an attracting heteroclinic cycle, such as in Example 2.6. Then $\limsup_{t\to\infty} \Psi(t,x,y) > 0$ but $\liminf_{t\to\infty} \Psi(t,x,y) = 0$. Moreover, the invader gets arbitrarily close to 0 for arbitrarily long times. Hence, practically, the invader is not safe from extinction. This example shows that information on the global dynamics of the full system (58)–(59) is needed to guarantee persistence after invasion.

11.5. Discretizations and diminishing stepsizes. Hofbauer and Schlag [34] studied imitation dynamics for two-person (bimatrix) games. These led to recurrence relations of the following form ($k = 0, 1, \dots$):

$$p_i^{k+1} = p_i^k \left(1 + hf_i\left(p^k, q^k\right)\right), \quad q_j^{k+1} = q_j^k \left(1 + hg_j\left(p^k, q^k\right)\right). \tag{62}$$

Here the state space X is a product of two simplices, $X = X_n \times X_m \subseteq \mathbb{R}^n_+ \times \mathbb{R}^m_+$, and $f_i, g_j : X_n \times X_m \to \mathbb{R}$ ($1 \le i \le n$, $1 \le j \le m$) are appropriately chosen functions such that $(p^k, q^k) \mapsto (p^{k+1}, q^{k+1})$ defines a map from $X_n \times X_n$ into itself. In the limit $h \to 0$ these discrete-time models tend to the differential equation

$$\dot{p}_i = p_i f_i(p,q), \quad \dot{q}_j = q_j g_j(p,q). \tag{63}$$

The functions f_i, g_j are given by

$$f_i(p,q) = (\pi^1(i,q) - \pi^1(p,q))\phi^1(\pi^1(p,q)), \quad g_j(p,q) = (\pi^2(p,j) - \pi^2(p,q))\phi^2(\pi^2(p,q)), \tag{64}$$

where π^1, π^2 are the payoff functions for the two players, and ϕ^i are strictly decreasing functions with positive values.

For $n = m = 2$, i.e., each of the two players has two pure strategies, X is simply the square $[0,1]^2$. Of particular interest are games with a cyclic structure: For these games, the boundary of the square, Y, forms a heteroclinic cycle for the dynamics (62) and (63). They have a unique Nash equilibrium E which lies in the interior of X and which has been shown [34, Thm. 1] to be globally asymptotically stable for (63); i.e., E is the dual attractor to the repeller Y. Furthermore, there exists a standard GALF for (63), so that Y is a robust repeller.

For small enough $h \in (0, h_0)$ (h_0 being the minimal slope of the reciprocal of ϕ^i), the map (62) still has a standard GALF, so that Y remains a robust repeller; see [34, Prop. 1]. The dual attractor, $\mathcal{A}_h$, contains E (which is unstable for the maps [34, Prop. 3]) as proper subset.

Combining now Theorem 8.3 and the upper semicontinuity result [20] for the attractor–repeller pair (E, Y) of (63) shows that for small enough $h > 0$, the numerical attractor $\mathcal{A}_h$ arising from E attracts all of the interior of the square X, i.e., is dual to the robust repeller Y. This confirms the conjecture in [34, p. 535, footnote 4].

On the other hand, if instead of a constant stepsize h a decreasing sequence of stepsizes satisfying $h_k \to 0$ and $\sum h_k = \infty$ is chosen in (62), then every orbit starting in the interior of the square converges to the equilibrium E. This conjecture from [34, p. 539 and footnote 9] follows now from the permanence result in Theorem 9.3 above, Lemma 4 from [20] (or Benaim and Hirsch [6]), and the attractor–repellor decomposition (E, Y) for the limiting differential equation (63).

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