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A DIFFERENTIAL GAME APPROACH TO EVOLUTIONARY EQUILIBRIUM SELECTION*

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The equilibrium selection model of Matsui and Matsuyama (1995), which is based on rational players who maximise their discounted future payoff, is analysed with the help of an associated differential game. Equilibrium selection results are derived for games with a $\frac{1}{2}$-dominant equilibrium, for games with a potential function, and some simple supermodular games.

Keywords: Equilibrium Selection; Perfect Foresight; Potential Game; Supermodular Game; Pontryagin Maximum Principle.

1. Introduction

Whenever a strategic game has more than one equilibrium, the players face a problem: which equilibrium should they play? Departing from Nash's solution of the bargaining problem, Harsanyi and Selten (1988) invented a sophisticated method to select a unique equilibrium for each finite strategic game. Since this pioneering work, many other alternative methods of equilibrium selection were developed, but usually only for certain subclasses of games, see for example Güth and Kalkofen (1989), Selten (1995), or Carlsson and van Damme (1993a). The latter, particularly interesting approach was only recently further developed by Morris and Shin (2000) and Ui (2001).

Parallel to these classical approaches, ideas from evolutionary game theory turned out to be useful for this problem. The very powerful and popular variants of stochastic stability are excellently surveyed in Young (1998). Other models include

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a spatial structure, such as Ellison (2000) or Hofbauer (1999). In the present paper, we investigate a dynamic model due to Matsui and Matsuyama (1995). A related stochastic model is due to Blume (1995). Each player is represented by a large population of rational agents which are endowed with perfect foresight. Every now and then these agents revise their strategy in order to maximise their discounted future payoff. The Nash equilibria of the game correspond to stationary states of this dynamic model. Certain dynamic stability properties allow to select a unique equilibrium for many interesting classes of games.

This dynamic model is explained in Sec. 2. In Sec. 3, two basic results about this model are proven. First, we provide a characterisation of perfect foresight paths as the open loop Nash equilibria of a certain differential game, which is determined by the discounted payoff functions of the original strategic game. Using this characterisation, we then obtain a simple proof for the existence of such paths. Section 4 is devoted to equilibrium selection results. First, we consider games with a $\frac{1}{2}$-dominant equilibrium. Then, we consider games with a potential function. We extend our earlier result from Hofbauer and Sorger (1999) to N-person games, showing that the global maximisier of the potential function of the game is selected by this method. This general result is applied to three classes of binary choice games with two strict equilibria. For two-person games, we show that the global maximiser coincides with the risk dominant equilibrium; we thus recover the result of Matsui and Matsuyama (1995). For symmetric games (such as the N-person stag hunt game) we re-derive a selection result of Kim (1996). For games with linear incentives, a class considered earlier by Selten (1995), we obtain a different selection criterion. We conclude with a selection result for symmetric 3×3 supermodular games. Furthermore, we compare our results with those obtained via different equilibrium selection methods.

2. Definitions

2.1. *N-person games*

An N-person strategic game is given by its payoff function $\pi : S_1 \times S_2 \times \cdots \times S_N \to \mathbb{R}^N$. Here, $\pi^i(s_1, s_2, \ldots, s_N)$ denotes the payoff to player i, if player k uses pure strategy $s_k \in S_k$ $(i, k = 1, \ldots, N)$. We assume that each S_i is a finite set, and denote also the N-linear extension to mixed strategy profiles $x = (x^i)_{i=1}^N$, with x^i in

$$\Delta(S_i) = \left\{ (x^i_s)_{s \in S_i} : x^i_s \geq 0 \quad \forall s \in S_i, \quad \sum_{s \in S_i} x^i_s = 1 \right\},$$

by $\pi : \Delta(S_1) \times \cdots \times \Delta(S_N) \to \mathbb{R}^N$. As usual, the vector of mixed strategy profiles $x = (x^i)_{i=1}^N$ will sometimes be written in the form $x = (x^i; x_{-i})$, with $x_{-i} = (x^j)_{j \neq i}$ collecting the mixed strategies of all opponents of player i.

2.2. *Perfect foresight equilibrium paths*

We will now describe the model of rational players with perfect foresight as proposed by Matsui and Matsuyama (1995) in the general framework of N-person games. There are N distinct large populations of players. Time $t \in [0, \infty)$ is a continuous variable. At each point in time, the players are matched randomly (one from each population) to form N-tuples, which then play the N-person game anonymously. Players are not able to choose their strategy at every point in time. Instead, each player has to commit to a particular pure strategy for an exogenously given (random) time interval. Time instants at which a player can switch between strategies follow a Poisson process with mean arrival rate p. These processes are assumed to be independent across players. Without loss of generality, we choose the unit of time in such a way that $p = 1$.

Let us denote by $x_s^i(t)$ the fraction of players in population i who are playing the pure strategy $s \in S_i$ at time t. Of course, we must have $x^i(t) \in \Delta(S_i)$ for all $t \in [0, \infty)$. The vector $x^i(t)$ describes the strategy distribution in the ith population at time t. Since players are matched randomly, $x(t)$ can also be thought of as the mixed strategy against which each player plays at time t. It follows that the expected payoff of playing the pure strategy $s \in S_i$ at time τ is given by $\pi^i(s, x_{-i}(\tau))$. It is assumed that all players have perfect foresight so that they correctly anticipate the future evolution of the strategy distribution in the $N - 1$ other populations. Since the time instants at which it is possible to switch between strategies form a Poisson process with mean arrival rate $p = 1$, the period of commitment to a fixed strategy has an exponential distribution with mean 1. Denoting the common discount rate of the players by $\theta > 0$, it follows that the expected discounted payoff of committing to strategy $s \in S_i$ at time t is given by

$$V_s^i(t) = \int_0^\infty \int_t^{t+z} e^{-\theta(\tau - t)} \pi^i(s, x_{-i}(\tau)) d\tau e^{-z} dz$$

which can be simplified to

$$V_s^i(t) = \int_t^\infty e^{-(1+\theta)(\tau - t)} \pi^i(s, x_{-i}(\tau))\, d\tau\,. \tag{2.1}$$

Because of the perfect foresight assumption, a rational player in population i who has the opportunity to switch to a new strategy at time t will switch to a strategy $s \in M^i(t)$ where

$$M^i(t) = \arg\max\{V_s^i(t) \mid s \in S_i\}\,. \tag{2.2}$$

Given the assumption that the switching times follow independent Poisson processes with arrival rate 1, it follows that $x_s^i : [0, \infty) \mapsto \mathbb{R}$ is Lipschitz continuous with Lipschitz constant 1. This implies in particular that $x_s^i(\cdot)$ is differentiable almost everywhere. Because of the way how agents switch between strategies it follows that, for all t where $x_s^i(\cdot)$ is differentiable, the conditions

$$\begin{aligned} \dot{x}_s^i(t) &= -x_s^i(t) && \text{if } s \notin M^i(t)\,, \\ \dot{x}_s^i(t) &\in [-x_s^i(t), 1 - x_s^i(t)] && \text{if } s \in M^i(t) \end{aligned} \tag{2.3}$$

are satisfied. We call a Lipschitz continuous function $x : [0, \infty) \mapsto \Delta(S_1) \times \cdots \times \Delta(S_N)$ such that Eqs. (2.1)–(2.3) hold a *perfect foresight equilibrium path* for the game described by the payoff function π and the discount rate θ.

We note that perfect foresight equilibrium paths are precisely those Lipschitz solutions of the following system of differential inclusions which are defined for all $t \geq 0$ and for which $V(t)$ stays bounded:

$$\begin{aligned} \dot{x}_s^i(t) &\in m_s^i(V(t)) - x_s^i(t) \\ \dot{V}_s^i(t) &= (\theta + 1)V_s^i(t) - \pi^i(s, x_{-i}(t))\,. \end{aligned} \tag{2.4}$$

Here, $m^i(V)$ denotes the set of all mixed strategies for player i giving him his maximum discounted expected payoff, i.e.

$$m^i(V) = \{u \in \Delta(S_i) : u_s = 0 \quad \text{if } V_s^i < V_{\max}^i\}\,, \tag{2.5}$$

where

$$V_{\max}^i = \max_{\sigma \in S_i} V_\sigma^i\,. \tag{2.6}$$

2.3. *Equilibrium paths for the discounted game*

Let

$$\pi_\theta^i(x(\cdot)) = \int_0^\infty e^{-\theta s} \pi^i(x(s))ds \tag{2.7}$$

be the θ-discounted expected payoff for player population i along the path $x(\cdot)$. We fix an initial point $x_0 \in \Delta(S_1) \times \cdots \times \Delta(S_N)$ and consider only *admissible* paths $x(\cdot) = (x^i(\cdot))_{i=1}^N$ such that the motion $x^i(\cdot)$ of population i is from the set

$$\begin{aligned} X_i = \{x^i : [0, \infty) \to \Delta(S_i),\ \text{Lipschitz}, x^i(0) = x_0^i, \\ \dot{x}^i(t) + x^i(t) \in \Delta(S_i) \text{ for a.a. } t \geq 0\}\,. \end{aligned} \tag{2.8}$$

This means that each population distribution may move arbitrarily in its simplex, only its speed of adjustment is limited: $x_s^i(t)$ cannot decrease too quickly or, more precisely, $e^t x_s^i(t)$ never decreases with increasing t. Let $X = X_1 \times \cdots \times X_N$. We call $\bar{x}(\cdot) = (\bar{x}^i(\cdot))_{i=1}^N \in X$ a *θ-equilibrium path* (or *open loop Nash equilibrium*) if

$$\pi_\theta^i(\bar{x}^i(\cdot); \bar{x}_{-i}(\cdot)) \geq \pi_\theta^i(x^i(\cdot); \bar{x}_{-i}(\cdot)) \tag{2.9}$$

holds for all admissible functions $x^i(\cdot) \in X_i$ and all i.

3. Basic Results

Theorem 3.1 (Existence of equilibrium paths). *For each initial value* $x_0 \in \Delta(S_1) \times \cdots \times \Delta(S_N)$*, there exists an open loop Nash equilibrium.*

Proof. Let the initial value $x_0 \in \Delta(S_1) \times \cdots \times \Delta(S_N)$ be given. The set X_i is convex and compact in the topology of uniform convergence on compact intervals. (By the Ascoli-Arzela theorem, each sequence in X_i has a convergent subsequence, and its limit is again Lipschitz with $e^t x_s^i(t)$ non-decreasing in t.) With this topology, the discounted payoff function $\pi_\theta : X \to \mathbb{R}^N$ defined in (2.7) is easily seen to be continuous. Furthermore, π_θ^i is (affine) linear in $x^i(\cdot)$. Thus, for each $x \in X$ and i,

$$\beta^i(x_{-i}) := \underset{x^i(\cdot)\in X^i}{\arg\max} \quad \pi_\theta^i(x^i(\cdot); x_{-i}(\cdot)) \tag{3.1}$$

is a compact and convex subset of X^i and depends upper semi-continuously on x_{-i}. Hence, Ky Fan's and Glicksberg's extension of Schauder's and Kakutani's fixed point theorem [see e.g. Aliprantis and Border (1999) or Border (1985)] implies the existence of equilibrium paths $x \in \beta(x) = (\beta^i(x_{-i}))_{i=1}^N \subseteq X$ for each initial value x_0. □

Remark. If, more generally than the mixed extension of a finite strategic game, π^i is a concave function of $x^i \in \Delta(S_i)$, then π_θ^i is concave in $x^i(\cdot) \in X^i$ and existence follows in the same way.

Theorem 3.2. *Each open loop Nash equilibrium path is a perfect foresight equilibrium path, and conversely.*

Proof. If $\bar{x}(.)$ is an open loop Nash equilibrium path of the differential game defined by (2.7) and (2.9) then, for each i and given $\bar{x}_{-i}(.)$, $\bar{x}^i(.)$ is an optimal trajectory of the optimisation problem

$$\dot{x}^i(t) = u^i(t) - x^i(t), \quad u^i(t) \in \Delta(S_i) \tag{3.2}$$

$$\int_0^\infty e^{-\theta t}\pi^i(x^i(t), \bar{x}_{-i}(t))dt \to \max\ . \tag{3.3}$$

Since π^i is an N-linear function, the result follows from the following lemma, applied to $a_k(t) = \pi^i(s_k, \bar{x}_{-i}(t))$, $n = |S_i|$, noting that $V_s^i(t)$ and $\psi_s(t)$ differ from each other only by the factor $e^{\theta t}$. □

Lemma 3.1. *Let* $a : [0, \infty) \to \mathbb{R}^n$ *be a bounded Lipschitz function. A path* $u(\cdot)$ *is an optimal control for*

$$\int_0^\infty e^{-\theta t}x(t)a(t)dt \to \max, \quad \dot{x}(t) = u(t) - x(t), u(t) \in \Delta^{n-1}\,, \tag{3.4}$$

if and only if it satisfies $\operatorname{supp} u(t) \subseteq \arg\max_k \psi_k(t)$ *for* $\psi_k(t) = e^t \int_t^\infty e^{-(\theta+1)\tau}a_k(\tau)d\tau$.

Proof. To show necessity, we apply the Pontryagin maximum principle, see Seierstad and Sydsæter (1987). The current value Hamiltonian function for the optimal control problem (3.4) is

$$H(t,\psi,x,u) = \psi_0 e^{-\theta t} x \cdot a(t) + \sum_{k=1}^{n} \psi_k(u_k - x_k) \tag{3.5}$$

where $\psi_0 \in \{0,1\}$ and $\psi = (\psi_1,\psi_2,\ldots,\psi_n) \in \mathbb{R}^n$ is the adjoint variable. H is maximal if and only if $u_i = 0$ for all i for which $\psi_i < \psi_{\max} = \max_k \psi_k$. The adjoint equation is given by

$$\dot{\psi}_k(t) = -\frac{\partial H}{\partial x_k}(t,\psi(t),x(t),u(t)) = \psi_k(t) - \psi_0 e^{-\theta t} a_k(t)\,. \tag{3.6}$$

The limiting transversality condition is

$$\lim_{t\to\infty} \psi_k(t) = 0 \quad \text{for all } k \in \{1,2,\ldots,n\}\,. \tag{3.7}$$

Suppose that $\psi_0 = 0$. This would lead to $\dot{\psi}_k(t) = \psi_k(t)$ and, hence by (3.7), to $\psi_k(t) = 0$ for all k and all t. Since this is a contradiction, we must have $\psi_0 = 1$. The general solution of the linear differential equation (3.6) is given by

$$\psi_k(t) = C_k e^t + e^t \int_t^\infty e^{-(\theta+1)\tau} a_k(\tau)d\tau\,.$$

The second term goes to zero like $e^{-\theta t}$ as $t \to \infty$. Hence (3.7) implies $C_k = 0$.

Since the Hamiltonian function (3.5) is jointly concave (actually linear) in x and u, the above conditions are also sufficient for optimality of u. □

Remark. The equivalence between perfect foresight equilibrium paths (2.4) and open loop Nash equilibria has not been noted before. It is interesting because in the differential game in Sec. 2.3, each population acts as a single player, whereas in the perfect foresight model of Sec. 2.2, each population consists of infinitely many independent agents endowed with perfect foresight. Theorems 3.1 and 3.2 together show the existence of perfect foresight equilibrium paths for general strategic games. This is a new result as Matsui and Matsuyama (1995) gave only an elementary *ad hoc* construction for 2×2 games, and Hofbauer and Sorger (1999) considered only the special class of potential games.[a]

4. Equilibrium Selection

The state $\bar{x} \in \Delta := \Delta(S_1)\times\cdots\times\Delta(S_N)$ is *globally accessible* if, for every initial point $x_0 \in \Delta$, there exists a perfect foresight equilibrium path $x(\cdot)$ satisfying $x(0) = x_0$ and $\lim_{t\to\infty} x(t) = \bar{x}$.

[a]We are grateful to Professor Akihiko Matsui for pointing us (at the Bilbao meeting) to the paper by his student Oyama (2000) who proved a similar existence result for the class of symmetric two-person games.

A state $\bar{x} \in \Delta$ is called *absorbing* if there exists a neighbourhood U of $\bar{x}$ such that, for all initial states $x_0 \in U$, every perfect foresight equilibrium path $x(\cdot)$ with $x(0) = x_0$ satisfies $\lim_{t\to\infty} x(t) = \bar{x}$.

The selection criterion developed by Matsui and Matsuyama (1995) requires that a Nash equilibrium is globally accessible and the only absorbing state whenever the discount rate θ is small enough. In the following, we present three classes of games for which a unique equilibrium is selected in this way. On the other hand, Oyama (2000) presents an open set of games for which no equilibrium has the required properties.

4.1. $\frac{1}{2}$-*dominance*

A pure strategy profile $\hat{s} = (\hat{s}_i) \in S_1 \times S_2 \times \cdots \times S_N$ is called $\frac{1}{2}$-*dominant*, if it is the unique best reply against any mixed profile $x \in \Delta$ which puts weight at least $\frac{1}{2}$ on $\hat{s}_i$, i.e. $x^i_{\hat{s}_i} \geq \frac{1}{2}$ for all i. In particular, $\hat{s}$ is then a strict Nash equilibrium of the N-person game. A game can have at most one $\frac{1}{2}$-dominant equilibrium, and such an equilibrium usually has very strong equilibrium selection properties, see Maruta (1997) and Ellison (2000).

The game π is said to have *linear incentives* [see Selten (1995)] if, for each i and each pair of strategies $s, s' \in S_i$, the payoff difference $\pi^i(s, x_{-i}) - \pi^i(s', x_{-i})$, which we will also denote by $\pi^i(s - s', x_{-i})$ in the following, can be written as a linear function of x_{-i}. Every two-person strategic game ($N = 2$) has linear incentives. For $N \geq 3$ this is a severe restriction. Such games have also been called polymatrix games and they can be viewed as the additive superposition of two-person games.

Theorem 4.1. *In a game with linear incentives, a $\frac{1}{2}$-dominant strategy $\hat{s}$ is globally accessible for small $\theta > 0$ and absorbing for all $\theta > 0$.*

Proof. (1) For an arbitrary initial condition x_0, we will show that the path

$$x(t) = x_0 e^{-t} + (1 - e^{-t})\hat{s} \tag{4.1}$$

leading straight to $\hat{s}$ is a perfect foresight equilibrium path. First, note that for an arbitrary path $x(t)$, the definition (2.1) and the assumption of linear incentives leads to

$$V^i_{\hat{s}_i}(0) - V^i_{s_i}(0) = \int_0^\infty e^{-(1+\theta)t} \pi^i(\hat{s}_i - s_i, x_{-i}(t))dt = \frac{1}{1+\theta}\pi^i(\hat{s}_i - s_i, \bar{x}_{-i}) \tag{4.2}$$

with

$$\bar{x} = (1+\theta)\int_0^\infty e^{-(1+\theta)t} x(t)dt \in \Delta\,. \tag{4.3}$$

For the path (4.1), this gives

$$\bar{x} = (1+\theta)\left[\frac{x_0}{2+\theta} + \hat{s}\left(\frac{1}{1+\theta} - \frac{1}{2+\theta}\right)\right] \tag{4.4}$$

so that

$$\bar{x}^i_{\hat{s}_i} \geq \frac{1}{2+\theta} \quad \forall\, i\,. \tag{4.5}$$

The $\frac{1}{2}$-dominance assumption and compactness gives a $\delta > 0$ such that

$$\pi^i(\hat{s}_i - s_i, x_{-i}){>}0 \text{ for all } i, \text{ all } s_i{\neq}\hat{s}_i, \text{ and all } x{\in}\Delta \text{ with } x^j_{\hat{s}_j}{\geq}\frac{1}{2} - \delta \;\; \forall\, j\,. \tag{4.6}$$

If we choose $\theta < 4\delta$, then (4.2)–(4.6) show $V^i_{\hat{s}_i} > V^i_{s_i}$ for all $s_i \neq \hat{s}_i$ along the path (4.1). Hence, (4.1) is a perfect foresight equilibrium path.

(2) For the second assertion, we show that for an initial value x_0 close to $\hat{s}$ the straight path (4.1) is the only perfect foresight equilibrium path. Let $x(t)$ be any path with $x(0) = x_0$ and $\dot{x}^i_s(t) \geq -x^i_s(t)$, hence $x^i_{\hat{s}_i}(t) \geq e^{-t}x^i_{\hat{s}_i}(0)$. Then (4.3) implies

$$\bar{x}^i_{\hat{s}_i} \geq (1+\theta)\int_0^\infty e^{-(1+\theta)t}e^{-t}x^i_{\hat{s}_i}(0)dt = \frac{1+\theta}{2+\theta}x^i_{\hat{s}_i}(0) \geq \frac{1}{2}x^i_{\hat{s}_i}(0)\,. \tag{4.7}$$

Hence, for $x^i_{\hat{s}_i}(0) \geq 1 - 2\delta$ [with δ as in (4.6)], the expression in (4.2) is positive so that perfect foresight implies (4.1). □

As will be discussed in Sec. 4.2.2 after (4.20), the assumption of linear incentives is indispensable in Theorem 4.1. This is in contrast to many other equilibrium selection methods. An analogous result for symmetric two-person games was independently shown by Oyama (2000).

4.2. *Potential games*

A *partnership game* (or *game with identical interests*) is a game where every player has the same payoff function: $\pi^i(x) = \pi(x)$. Two games with payoff functions $\pi, \tilde{\pi}$ are *linearly equivalent* if there exist constants $w_i > 0$ such that

$$w_i[\pi^i(s, x_{-i}) - \pi^i(s', x_{-i})] = \tilde{\pi}^i(s, x_{-i}) - \tilde{\pi}^i(s', x_{-i}) \tag{4.8}$$

holds for all i and $s, s' \in S_i$. (This means that the incentive functions of the two games are proportional.) A game that is linearly equivalent to a partnership game has been called a *re-scaled partnership game* in Hofbauer and Sigmund (1988) and a *weighted potential game* by Monderer and Shapley (1996). It is easy to see that both concepts of perfect foresight equilibrium paths and open loop Nash equilibria are invariant under linear equivalence. Therefore, every result for games with identical interests holds automatically for weighted potential games.

For such a game, we can consider the optimal paths of the discounted game:

$$\pi_\theta(x(\cdot)) \to \max \quad \text{for } x(\cdot) \in X\,. \tag{4.9}$$

In analogy to Theorem 2 in Hofbauer and Sorger (1999) (hereafter referred to as HS), we obtain the following lemma.

Lemma 4.1. (1) *Optimal paths exist.*

(2) *Every optimal path is an open loop Nash equilibrium* (2.9).

Proof. (1) Follows either from the proof of Theorem 3.1 — the continuous function π_θ attains its maximum on the compact set X, or from general existence results, as in Seierstad and Sydsæter (1987, Theorem 3.15). (2) follows from comparing the definitions (4.9) and (2.9). □

In analogy to Theorem 1 of HS, we obtain the following theorem.

Theorem 4.2. *Suppose the common payoff function has a unique global maximiser $\bar{x}$, i.e. $\pi(\bar{x}) > \pi(x)$ for all $x \in \Delta(S_1) \times \cdots \times \Delta(S_N)$ with $x \neq \bar{x}$. Then $\bar{x}$ is absorbing for all $\theta > 0$, and $\bar{x}$ is globally accessible for all small enough $\theta > 0$.*

Proof. The proof follows that of Theorem 1 in HS and we describe only the key steps and give details only when a different argument is required.

We first consider the function

$$H(x, V) = \pi(x) + \bar{V} - V \cdot x \tag{4.10}$$

with $\bar{V} = \sum_{i=1}^{N} V_{\max}^i$ and $V \cdot x = \sum_i \sum_{s \in S_i} V_s^i x_s^i$. Note that $H(x, V) \geq \pi(x)$.

The following two lemmata correspond to Lemmas 3 and 4 in HS and have very similar proofs.

Lemma 4.2. *Let $(x(\cdot), V(\cdot))$ be a solution of* (2.4). *Then the function $t \mapsto H(x(t), V(t))$ is Lipschitz continuous, satisfies*

$$(d/dt)H(x(t), V(t)) = \theta[\bar{V}(t) - V(t) \cdot x(t)] \geq 0 \tag{4.11}$$

for almost all $t \in [0, \infty)$, and is therefore non-decreasing.

Lemma 4.3. *Let $x(\cdot)$ be a perfect foresight equilibrium path for the common interest game with payoff function π. If x^* is an accumulation point of $x(\cdot)$ as $t \to \infty$, then*

(1) $\pi(x^*) \geq \pi(x(0))$.

(2) *x^* is a critical point of the potential function π on $\Delta(S_1) \times \cdots \times \Delta(S_N)$.*

Here, x^* is a critical point of π if, for all i and all $s, s' \in S_i$ with $x_s^i > 0$ and $x_{s'}^i > 0$, it holds that $\pi^i(s, x_{-i}^*) = \pi^i(s', x_{-i}^*)$. The critical points are precisely the Nash equilibria of all subgames (where each player restricts his pure strategy set to a subset $\tilde{S}_i \subset S_i$). They are precisely the rest points of the replicator dynamics, see Hofbauer and Sigmund (1988). In particular, every pure strategy is a critical point of π.

Now we can conclude the proof. Due to N-linearity of π, the unique global maximiser $\bar{x}$ is a strict equilibrium of the game, and hence of each subgame. Therefore $\bar{x}$ is isolated from all other critical points. Hence, if $x(0)$ is close enough to $\bar{x}$, then

$\pi(x(0)) > \pi(x^*)$ holds for all critical points $x^* \neq \bar{x}$. By Lemma 4.3, the only possible accumulation point of any perfect foresight path $x(t)$ starting at $x(0)$ is $\bar{x}$. Hence $\lim_{t\to\infty} x(t) = \bar{x}$, and $\bar{x}$ is absorbing.

The proof of global accessibility of $\bar{x}$ follows then via a "visiting lemma" analogous to Lemma 1 in HS and the local absorption property of $\bar{x}$. The former shows the existence of an optimal path (which is a perfect foresight equilibrium path by Lemma 4.1) that gets arbitrarily close to $\bar{x}$. □

We now apply this general result to three classes of binary choice games for which potential functions exist.

4.2.1. 2 × 2 *coordination games*

As shown in Hofbauer and Sigmund (1988), every 2 × 2 bimatrix game with three equilibria (two strict, one mixed) is linearly equivalent to a partnership game with payoff bimatrix

$$\begin{array}{ccc} & 1 & 2 \\ 1 & a, a & b, b \\ 2 & c, c & d, d \end{array} \; . \tag{4.12}$$

If

$$a, d > b, c \tag{4.13}$$

then $(1, 1)$ and $(2, 2)$ are the two strict equilibria. The potential function $\pi(x^1, x^2) = ax^1_1x^2_1 + bx^1_1x^2_2 + cx^1_2x^2_1 + dx^1_2x^2_2$ attains local maxima a at $(1, 1)$ and d at $(2, 2)$. If $a > d$ then $(1, 1)$ is the unique (global) maximiser of the potential.

On the other hand, $(1, 1)$ is the risk-dominant equilibrium of (4.12) [and of the original game, since the risk-dominance concept is invariant under linear equivalence, see Harsanyi and Selten (1988)] iff

$$(a - c)(a - b) > (d - b)(d - c)\,, \tag{4.14}$$

which is equivalent to

$$(a - d)(a + d - b - c) > 0\,.$$

Since by (4.13) the second factor is positive, (4.14) is equivalent to $a > d$. Hence *for 2 × 2 games, the risk-dominance selection criterion is equivalent to maximisation of the potential.* Together with Theorem 4.2, this yields the main result of Matsui and Matsuyama (1995).

4.2.2. *Symmetric* 2^N *games*

Consider N-person symmetric binary choice games with the two options called A and B. The corresponding pure strategy profiles "all A" and "all B" are denoted by **A** and **B**. An important example is the N-person stag hunt game, see Carlsson and van Damme (1993b). Denote by a_k (resp. b_k) the payoff for an A player (resp. B player), if k of the N players use A ($k = 0, 1, \ldots, N$, but a_0 and b_N are meaningless).

Such games have a potential function P. More precisely, there is a linearly equivalent game which is both symmetric and a partnership game, with common payoff function $P : [0,1]^N \to \mathbb{R}$. Let c_k be the common payoff if k of the N players use A, so that $P(\mathbf{A}) = c_N$ and $P(\mathbf{B}) = c_0$. This partnership game is linearly equivalent to the given game (according to (4.8), with weights $w_i = 1$) iff

$$c_k - c_{k+1} = b_k - a_{k+1} \quad k = 0, 1, \ldots, N-1 \,. \tag{4.15}$$

From this we can recursively determine the numbers c_k, up to an additive constant. Adding all equations in (4.15) yields

$$P(\mathbf{B}) - P(\mathbf{A}) = c_0 - c_N = \sum_{k=0}^{N-1} (b_k - a_{k+1}) \,. \tag{4.16}$$

We assume now that both $\mathbf{A}$ and $\mathbf{B}$ are strict equilibria and that there is a unique symmetric mixed equilibrium. This is the case if the sequence

$$a_{i+1} - b_i \quad \text{increases with} \quad i \tag{4.17}$$

and $a_1 - b_0 < 0$ as well as $a_N - b_{N-1} > 0$. The property (4.17) is equivalent to the strict monotonicity (supermodularity) of the game on $[0,1]^N$, i.e. $\partial d_i / \partial p_j > 0$ for all $i \neq j$, where $d_i(p) = \pi^i(B; p_{-i}) - \pi^i(A; p_{-i})$ denotes the incentive function for player i, and $p_j = x_B^j \in [0,1]$. It holds naturally in the stag hunt game: If A means "join the stag hunting group" and B means "hunt a hare by yourself", then a_i increases with i, the number of stag hunters, whereas b_i does not depend on i. The property (4.17) implies also that $\mathbf{A}$ and $\mathbf{B}$ are the only possible local maximisers of the potential P. Hence, by (4.16), $\mathbf{A}$ is the global maximiser if

$$\sum a_i > \sum b_i \,, \tag{4.18}$$

or equivalently, $\int_0^1 d(p)dp < 0$, where $d(\cdot)$ denotes the restriction of any d_i to the diagonal $p = p_1 = p_2 = \cdots = p_N$.

Theorem 4.2 implies that in this case $\mathbf{A}$ is globally accessible and absorbing, while, if the inequality in (4.18) is reversed, then $\mathbf{B}$ has this property. This result was proved earlier in Matsui and Matsuyama (1995) and Kim (1996).[b]

Kim (1996) [see also Carlsson and van Damme (1993b)] shows that also the "global games" method of Carlsson and van Damme (1993a) results in the same selection criterion for this class of games. Making use of the above potential, this follows now also from Ui (2001).

[b]This holds in two senses: In the setting of this paper with N separate populations, or — maybe more realistically for a symmetric game such as stag hunt — in the framework of one population of players. One simply has to restrict the dynamics to the diagonal in $[0,1]^n$. The results in Matsui and Matsuyama (1995) and Kim (1996) deal with the one-population version.

The selection criterion (4.18) is not the only reasonable one for this class of games. Another one is to choose **A** if

$$d(p) < 0 \quad \text{for} \quad 0 \leq p \leq \frac{1}{2}\,, \tag{4.19}$$

i.e. if **A** is $\frac{1}{2}$-dominant on the diagonal. It is easy to see that under the monotonicity assumption (4.17) this is equivalent to $\frac{1}{2}$-dominance on the full hypercube or, compare Kim (1996), to

$$\sum_i (b_i - a_{i+1}) \binom{N-1}{i} < 0\,. \tag{4.20}$$

Whereas this condition is equivalent to (4.18) whenever the incentive function d is linear, it leads to a different criterion for nonlinear d, i.e. for generic N-player games with $N \geq 3$. This shows also that the extra assumption of linear incentives was needed in Theorem 4.1. Criterion (4.19) agrees with the stochastically stable equilibrium [see Kim (1996)], Selten's (1995) method based on generalized Nash products (the present games are a special case of his "equistable biforms"), the ESBORA criterion of Güth and Kalkofen (1989), and the spatial dominance concept of Hofbauer (1999).

The selection criterion based on the general risk-dominance concept of Harsanyi and Selten (1988), on the other hand, leads to even another condition which is different from both (4.18) and (4.19) and considerably more complicated (being nonlinear in the payoffs a_i and b_i), see again Carlsson and van Damme (1993b) and Kim (1996).

4.2.3. 2^N *games with a quadratic potential*

Following Selten (1995), we consider now N-person binary choice games (with pure strategy sets $S_i = \{A_i, B_i\}$) with linear incentives. The incentive functions then take the form (with $p_i = x^i_{B_i}$)

$$d^i(p) = \sum_{j=1}^{N} \alpha_{ij} p_j - s_i \tag{4.21}$$

with $\alpha_{ii} = 0$. We assume further that for all $i \neq j$

$$\alpha_{ij} = \alpha_{ji} > 0\,. \tag{4.22}$$

This class of games is important because it covers two person unanimity games with incomplete information, if the N players in (4.21) are interpreted as the types of the two players in the incomplete information game, see Selten (1995) for details.

The positivity of the coefficients in (4.22) implies the monotonicity (supermodularity) of the game. The symmetry of the "interaction matrix" α_{ij} implies that the game is a potential game. The potential function $V : [0,1]^N \to \mathbb{R}$ satisfies

$\frac{\partial V(p)}{\partial p_i} = d^i(p)$ and is given (up to a constant) by the quadratic function

$$V(p) = \frac{1}{2}\sum_{i,j} \alpha_{ij} p_i p_j - \sum_i s_i p_i \,.$$

Note that $V(\mathbf{A}) = 0$ and $V(\mathbf{B}) = \frac{1}{2}\sum_{i,j}\alpha_{ij} - \sum_i s_i$. If **A** and **B** are (strict) equilibria, then they are (strict) local maximisers of V. Hence, **B** is selected over **A** by the potential method if

$$V(\mathbf{B}) > V(\mathbf{A}), \quad \text{i.e. iff} \quad \frac{1}{2}\sum_{i,j}\alpha_{ij} > \sum_i s_i \,, \tag{4.23}$$

and **A** is selected over **B** if the inequalities in (4.23) are reversed. If there are no other pure Nash equilibria besides **A** and **B**, then (4.23) together with Theorem 4.2 implies that **B** is globally accessible and absorbing. Ui's (2001) result implies that in this case **B** is "robust to incomplete information", i.e. the equilibrium selected by the global games method. Hofbauer (1999) showed that in this case **B** is the spatially dominant equilibrium. Hence, (4.23) is the selection criterion emerging from four completely different equilibrium selection methods! On the other hand, Selten (1995) suggested a different selection criterion, in terms of generalised Nash products.

4.3. *Supermodular games*

While the paper dealt up to now with (asymmetric) N-person games, we conclude it with a class of 3×3 symmetric games played within one population, strategy set $S_1 = S_2 = \{1, 2, 3\}$ and payoff function given by the matrix $A = (a_{ij})_{i,j=1,2,3}$. This is the setting of Hofbauer and Sorger (1999) and Oyama (2000).

We assume that this game has three strict equilibria, i.e. $a_{ii} > a_{ji}$ holds for all $j \neq i$. Such a game is supermodular if additionally the two inequalities

$$a_{22} + a_{13} < a_{12} + a_{23}, \qquad a_{22} + a_{31} < a_{32} + a_{21} \tag{4.24}$$

hold.

Let us write $i \gg j$ if strategy i pairwise risk-dominates j, i.e. $a_{ii} - a_{ji} > a_{jj} - a_{ij}$. Denote furthermore

$$q_1 = \frac{a_{11} + a_{12} - a_{21} - a_{22}}{a_{21} + a_{23} - a_{11} - a_{13}}, \qquad q_3 = \frac{a_{33} + a_{32} - a_{23} - a_{22}}{a_{21} + a_{23} - a_{31} - a_{33}}\,. \tag{4.25}$$

Theorem 4.3. *For a symmetric* 3×3 *game with three strict equilibria and* (4.24) *the following holds.*

(a) *If* $2 \gg 1$ *and* $2 \gg 3$, *then* 2 *is globally reachable for small* $\theta > 0$ *and absorbing for all* $\theta > 0$.
(b) *If either* $1 \gg 2 \gg 3$ *or* $1 \gg 2$, $3 \gg 2$, *and* $q_1 > q_3$, *then* 1 *is globally reachable for small* $\theta > 0$.

(c) *If either* $3 \gg 2 \gg 1$ *or* $1 \gg 2$, $3 \gg 2$, *and* $q_3 > q_1$, *then* 3 *is globally reachable for small* $\theta > 0$.

The proof of this theorem is too long to be reproduced here. We only note that case (a) follows from Theorem 4.1, since the strict equilibrium 2 is actually $\frac{1}{2}$-dominant. We believe that in cases (b) and (c) the respective equilibrium is also absorbing.

The selection criterion in Theorem 4.3 may look strange, in particular because in cases (b) and (c) it is not pairwise risk-dominance which decides between the two candidates 1 and 3 but the more complicated expressions q_1 and q_3. Amazingly, the very same selection criterion arises for the spatial dominance concept of Hofbauer (1999) and for the global games approach [see Morris (1999)].

5. Conclusion

We studied the equilibrium selection method introduced by Matsui and Matsuyama (1995). Relating the perfect foresight paths to the open loop Nash equilibria of a simple differential game, i.e. the discounted game associated to the original finite N-person strategic game, we obtain a simple proof of their existence in general.

Turning to equilibrium selection, we show that this method selects the $\frac{1}{2}$-dominant equilibrium in games with linear incentives. Then we prove an analog of our previous result from Hofbauer and Sorger (1999) for potential games, and present several important special cases. We conclude with a result for supermodular games.

The selection criteria obtained so far agree with those of the "global games" method of Carlsson and van Damme (1993a). The recent breakthrough via "higher order beliefs" to extend this intriguing method from 2×2 games to games with more players and strategies is impressively surveyed in Morris and Shin (2000). Furthermore, for games with linear incentives, the criteria agree so far with those arising from the spatial dominance concept of Hofbauer (1999). This agreement of completely different approaches to equilibrium selection is unexpected and surprising, and should encourage further exploration.

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