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The Analog of Electric and Magnetic Fields in Stationary Gravitational Systems

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Newtonian and Machian aspects of the stationary gravitational field are brought into formal analogy with a stationary electromagnetic field. The electromagnetic vector potential equals (up to a factor) the timelike Killing vector field. The current density is given by the contraction of the Killing vector with the Ricci tensor. A coordinate-dependent split in electric and magnetic field vectors is given, and some results of classical electrodynamics are used to illustrate the analogy. In the linearized theory, the usual Maxwell equations are obtained. The analogy also holds from the point of view of particle motion. The geodesic equation is brought into a special form that exhibits an analog to the Lorentz force. Two examples (which have played an important role in the theoretical discovery of Machian effects) are considered.

1. INTRODUCTION

Roughly speaking, the time component g_{00} of a space-time metric corresponds—when evaluated in appropriate coordinates—to the Newtonian potential (in the linearized theory of gravity it is coupled to the mass density), whereas the mixed components g_{0i} give rise to additional rotational effects (dragging of local reference frames, "Machian effects"). The analysis of the stationary field equations exhibits a "magnetic" character of these phenomena, while the Newton-like properties could be regarded as the "electric" aspect. These two aspects can properly be separated (by means of different physical effects described by different physical quantities) only when a flat background metric is used, i.e., in a weak-field approximation.

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Nevertheless a formal separation with the aim to obtain Maxwell-like equations may be done in the exact case.

In Section 2, the field equations are written down in two different ways. Section 3 describes the physical interpretation of the Thirring–Lense frequency, which may be extracted from the g_{0i} components. In Section 4, a short motivation for the analogy in question is given, and in Section 5 the connection between an electromagnetic field tensor in a stationary space-time and the electric and magnetic field vectors is written down. In Section 6, the analogy is developed by an appropriate *ansatz* for the electromagnetic potential vector. The corresponding current density turns out to be a kind of matter current density. Section 7 specializes to the linearized theory and to the slow motion limit and shows that each stationary solution of the Maxwell equations with positive charge density provides a model for a weakly gravitating system. In Section 8, two examples (a slowly rotating spherical star and a slowly rotating spherical mass shell) are considered. In Section 9, the study of particle motion in a stationary gravitational field exhibits an analog to the Lorentz force in classical electrodynamics. The equations of motion are established for the two examples of Section 8 and are compared with the historical results of Thirring and Lense.

2. THE FIELD EQUATIONS

Let (M, g) be a stationary space-time, i.e., there exists a timelike Killing vector field ξ^α ($\xi^\alpha \xi_\alpha < 0$ everywhere on M); here the signature of the metric is $-{+}{+}{+}$; Greek indices run from 0 to 3 and Latin indices from 1 to 3.

Then there exists a coordinate system[1] in which $\xi^\alpha = \delta^\alpha_0$, $t := x^0$, and $f := -\xi^\alpha \xi_\alpha = -g_{00}$. The line element can accordingly be written as

$$ds^2 = -f(dt - g_i\, dx^i)^2 + \gamma_{ik}\, dx^i\, dx^k$$

The $g_{\mu\nu}$ do not depend on t and γ_{ik} is the "true spatial metric"[2]:

$$\gamma_{ik} = g_{ik} - g_{00}^{-1} g_{0i} g_{0k}$$

Let S be the hypersurface given by $t = 0$; γ_{ik} is considered as a (positive definite) metric on S[3]; its inverse is given by[2]

$$\gamma^{ik} = g^{ik}$$

$g_i := f^{-1} g_{0i}$ is a vector field on S[2]; $g^i = g^{i0}$. In order to establish an appropriate form of the field equations, we replace g_i by another vector field on S:

$$b^m := f^{3/2} \varepsilon^{mik} D_i g_k \tag{1a}$$

where $\varepsilon^{123} = \gamma^{-1/2}$, $\gamma := \det(\gamma_{ik})$, and D_i is the covariant derivative with respect to the γ_{ik} on S. Thus b^m is subject to the condition

$$D_m(f^{-3/2}b^m) = 0 \tag{1b}$$

Expressed in terms of these fields on S, Einstein's field equations $R_{\mu\nu} = 8\pi(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R^{\alpha}{}_{\alpha})$ read (cf. Refs. 2 and 3):

$$f\Delta_\gamma f - \tfrac{1}{2}(D_i f)(D^i f) + b^i b_i = 2fR_{00} = 16\pi f(T_{00} + \tfrac{1}{2}fT^{\alpha}{}_{\alpha}) \tag{2a}$$

$$\tfrac{1}{2}f^{-1/2}\varepsilon^{ikl}D_k b_l = R^i_0 = 8\pi T^i_0 \tag{2b}$$

$$\begin{aligned}\mathscr{R}^{ik} - f^{1/2}D^iD^kf^{1/2} + \tfrac{1}{2}f^{-2}(\gamma^{ik}b_m b^m - b^i b^k) &= R^{ik}\\ &= 8\pi(T^{ik} - \tfrac{1}{2}\gamma^{ik}T^{\alpha}{}_{\alpha})\end{aligned} \tag{2c}$$

where Δ_γ is the Laplacian with respect to γ_{ik}, $\mathscr{R}^{ik}$ the Ricci tensor of γ_{ik} and $R^{\mu\nu}$ the Ricci tensor of $g_{\mu\nu}$.

In this paper we shall deal with Eq. (1) and (2) that are relevant for f and b^i. As is well known, $\frac{1}{2}f$ plays the role of the Newtonian gravitational potential; cf. Ref 4, p. 452, and Ref. 5). Now let us focus out attention on b^i; b^i can be calculated from the 4-vector[3]

$$\omega^\mu := \tfrac{1}{2}\varepsilon^{\mu\alpha\beta\gamma}\xi_\alpha \nabla_\beta \xi_\gamma \tag{3}$$

(twist of ξ^α, Thirring–Lense frequency). In fact,

$$\omega_i = -\tfrac{1}{2}b_i, \qquad \omega_0 = 0$$
$$\omega^i = -\tfrac{1}{2}b^i, \qquad \omega^0 = -\tfrac{1}{2}g^i b_i$$

Since $\omega^\alpha \xi_\alpha = 0$, ω^α is tangential to S and can therefore be considered as a vector on S.[3] As a consequence, ω^α is spacelike. By contrast, b^i can be extended to the 4-vctor $b_\mu = -2\omega_\mu$.

Equation (3) is the covariant formulation of (1a). With the aid of ω^μ, Eq. (1b) and (22), (2b) can also be expressed covariantly[3]:

$$\nabla_\mu(f^{-2}\omega^\mu) = 0 \tag{1b$'$}$$

$$f\Delta_g f - \nabla^\mu f \nabla_\mu f + 4\omega^\mu\omega_\mu = 2fR_{\alpha\beta}\xi^\alpha\xi^\beta \tag{2a$'$}$$

$$\varepsilon^{\alpha\beta\mu\nu}\nabla_\mu\omega_\nu = 2\xi^{[\alpha}R^{\beta]}{}_\gamma\xi^\gamma \tag{2b$'$}$$

[The different exponentials of f in (1b) and (1b$'$) arise from different ε-tensors of γ_{ik} and $g_{\mu\nu}$: $(-g)^{1/2} = (f\cdot\gamma)^{1/2}$.] Equations (3), (1b$'$), (2a$'$), and (2b$'$) can replace Eqs. (1a), (1b) and (2a), (2b).

3. THE PHYSICAL MEANING OF ω^μ

In the case of axial symmetry, the physical meaning of ω^μ is as follows: A stationary observer at rest on S (4-velocity $u^\alpha = f^{-1/2}\xi^\alpha$) will carry torque-free gyroscopes along his world line. The gyroscopes' angular momenta S^μ are Fermi–Walker transported (Ref. 4, p. 1117):

$$u^\alpha \nabla_\alpha S^\mu = u^\mu S_\beta u^\alpha \nabla_\alpha u^\beta$$
$$u_\mu S^\mu = 0$$

When regarded in the observer's proper time, they revolve uniformly around the vector ω^μ (telescopes that are directed toward fixed points on the rotational axis of M serve as a kinematic reference frame; mathematically speaking, it is Lie-transported). The angular velocity of this rotation is $(\omega^\mu \omega_\mu)^{1/2}$. Here ω^μ is the angular velocity vector of the gyroscopes' angular momenta (including the right-hand rule for the orientation). If S^μ coincides with ω^μ, it will be at rest in the kinematic frame; ω^μ itself is also Fermi–Walker transported.[6]

The vector ω^μ describes the dragging of local reference frames by rotating masses ("Machian effects"[7–11]). Each timelike Killing vector field ξ^α provides a ω^μ by construction of a coordinate system in which $\xi^\alpha = \delta^\alpha_0$. Thus, all stationary observers are included. For a chosen ξ^α, ω^μ is a vector field on M whose properties we are interested in.

In the stationary case without axial symmetry, the S^μ will no longer revolve uniformly around ω^μ, but some aspects of the statement given above remain valid; the kinematic frame is Lie-dragged.[6]

In the vacuum case, (2b) states that b^i is curl-free and therefore locally the gradient of a function $\mathscr{F}$ on S:

$$b_i = \partial_i \mathscr{F}$$

Analogously we may conclude[3]

$$\nabla_\mu \omega_\nu - \nabla_\nu \omega_\mu = -\varepsilon_{\mu\nu\alpha\beta} \xi^\alpha R^\beta_\gamma \xi^\gamma$$

from (3), and thus ω_μ is a gradient. The identification of b_μ with ω_μ gives

$$\omega_\mu = -\tfrac{1}{2}\partial_\mu \mathscr{F}$$

$\mathscr{F}$ is therefore called the Thirring–Lense (or twist) potential. For the connection between f, $\mathscr{F}$, the angular potential, and the multipole moments, see Ref. 12. The field equations (2a′) and (1b′) read

$$f\Delta_g f = \nabla^\mu f \nabla_\mu f - \nabla^\mu \mathscr{F}\, \nabla_\mu \mathscr{F}$$
$$f\Delta_g \mathscr{F} = 2\nabla^\mu f \nabla_\mu \mathscr{F}$$

They can be unified by a complex function:

$$\operatorname{Re} \varepsilon \Delta_g \varepsilon = \nabla^\mu \varepsilon \nabla_\mu \varepsilon$$
$$\varepsilon := f + i\mathscr{F}$$

This is the Ernst equation.[13] Thus, in the vacuum case, the Thirring–Lense frequency is given by the gradient of the imaginary part of the Ernst function ε.[14]

4. ω^μ FOR A ROTATING STAR

In the Kerr metric[15] (Boyer–Lindquist coordinates[16]), $\mathscr{F}$ is given by

$$\mathscr{F}(r, \Theta) = \frac{2Ma \cos \Theta}{r^2 + a^2 \cos^2 \Theta}$$

The flow lines of $\partial_i \mathscr{F}$ for large r resemble those of a stationary magnetic field which is generated by a charged rotating sphere. Condition (1b) means that $f^{-3/2} b_i$ is divergence-free on S. For large r ($f \to 1$), b_i itself becomes divergence-free. Relation (2b) is an equation of type $\operatorname{div} \mathbf{B} = \mathbf{j}$. These observations suggest that b_i has properties resembling those of a magnetic field. The linearized theory of gravity shows that ω_k is a dipole field at large distances from the source (cf. Ref. 4, p. 1119).

Analogously, for weakly gravitating systems, $-\frac{1}{2}\partial_i f$ plays the role of the Newtonian gravitational field strength [$f = \exp(2\psi)$, $\psi =$ Newtonian potential[4]], and $\partial_{ii} f \approx 8\pi$ times the mass density [cf. Eq. (2a)]. Thus $\frac{1}{2}\partial_i f$ obeys the Poisson equation for a stationary electric field with charge density equalling the mass density of the gravitating system.

This analogy will be subject of the following considerations.

5. ELECTROMAGNETIC FIELD TENSOR ON M

Let $F_{\mu\nu} = -F_{\nu\mu}$ be an electromagnetic field tensor in a stationary space-time, i.e.,

$$\begin{gathered} \nabla^\mu F_{\alpha\mu} = 4\pi j_\alpha \\ \nabla_{[\alpha} F_{\beta\gamma]} = 0 \\ F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \qquad \nabla_\alpha j^\alpha = 0 \end{gathered} \tag{4}$$

We use the same coordinate system as before. On can construct generalizations of the electric and magnetic field vectors with respect to the hypersurface S (cf. Ref. 17):

$$E_{\mu} = F_{\mu\alpha}\xi^{\alpha}$$
$$B^{\mu} = -\tfrac{1}{2}\varepsilon^{\mu\alpha\beta\gamma}\xi_{\alpha}F_{\beta\gamma} \tag{5}$$

Since $E_{\mu}\xi^{\mu} = B^{\mu}\xi_{\mu} = 0$, E^{μ} and B^{μ} are tangential to S and therefore in fact three-dimensional $(E_0 = B_0 = 0,\ E_k = F_{k0},\ F^{ik} = f^{-1/2}\varepsilon^{ikm}B_m)$. The spatial part of the current density is calculated as

$$4\pi j^{k} = f^{-1/2}\varepsilon^{kis}D_{i}B_{s} \tag{6}$$

while its projection along ξ^{μ} is given by the more complicated formula

$$4\pi j_{\alpha}\xi^{\alpha} = -\nabla_{\mu}E^{\mu} + F_{\mu\alpha}\nabla^{\mu}\xi^{\alpha} \tag{7}$$

If $f \approx 1$ and $F_{\mu\alpha}\nabla^{\mu}\xi^{\alpha} \approx 0$, then we obtain the flat stationary Maxwell source equations for $F_{\mu\nu}$.

6. THE ANALOGY WITH THE STATIONARY ELECTROMAGNETIC FIELD

If we want to identify $\partial_i f$ and b_i formally with a stationary electromagnetic field, we have to make an *ansatz* for $F_{\mu\nu}$. Thus $-\frac{1}{2}f$ could take over the role of the electrostatic potential (whereas $\frac{1}{2}f$ corresponds to the Newtonian potential; the different signs arise from the formal identification of the mass density with the electric charge density). Equation (1a) suggests a quantity like g_k as the vector potential of the magnetic field. A possible covariant combination of f and g_k is $\xi_{\mu} = g_{\mu 0}$. With an appropriate factor we define

$$A_{\mu} := -\tfrac{1}{2}\xi_{\mu}$$

which leads to

$$F_{\mu\nu} = -\tfrac{1}{2}(\nabla_{\mu}\xi_{\nu} - \nabla_{\nu}\xi_{\mu}) = \nabla_{\nu}\xi_{\mu} \tag{8}$$

Using the identities

$$\nabla_{\mu}\xi_{\nu} + \nabla_{\nu}\xi_{\mu} = 0$$
$$\nabla_{\alpha}\xi^{\alpha} = 0 \text{ (corresponds to the Lorentz gauge)}$$
$$\nabla_{\mu\nu}\xi_{\gamma} = R_{\alpha\mu\nu\gamma}\xi^{\alpha}$$

for Killing vector fields and

$$\nabla_\mu T^{\mu\nu} = 0$$

$$\xi^\mu \nabla_\mu T^\alpha{}_\alpha = \frac{\partial}{\partial t} T^\alpha{}_\alpha = 0$$

we obtain

$$\begin{aligned} &\nabla_{[\alpha} F_{\beta\gamma]} = 0 \\ &\quad 4\pi j_\alpha = \nabla^\mu F_{\alpha\mu} = -R_{\alpha\beta}\xi^\beta = -8\pi(T_{\alpha\beta} - \tfrac{1}{2} g_{\alpha\beta} T^\gamma{}_\gamma)\xi^\beta \\ &\nabla_\alpha j^\alpha = 0 \end{aligned} \tag{9}$$

$F_{\mu\nu}$ obeys Eqs. (4); the corresponding conserved current density j^α is composed of the energy current density $T^\alpha_\beta \xi^\beta$ and a vector parallel to ξ^α. This fact is the stronger reason for the analogy outlined here. The "matter current" j^α may formally be regarded as the source for the field $F_{\mu\nu}$, i.e., as the charge current density (which is not coupled to the metric by the Einstein equations!). Comparison of (5) with (3) gives

$$B^\mu = \omega^\mu = -\tfrac{1}{2} b^\mu \tag{10}$$

and the electric field vector is given by

$$E_\mu = -\tfrac{1}{2}\nabla_\mu(\xi^\alpha \xi_\alpha) = \tfrac{1}{2}\partial_\mu f \tag{11}$$

As a gradient E_μ is curl-free, and (1b) or (1b′) can be considered as an analog of the divergence-freeness of the magnetic field. Equation (6) is identical with (2b), and (7) with (2a). When expressed in S, the four equations relevant for E_μ and B_μ read (note that $E^i = \gamma^{ik} E_k$, $B^i = \gamma^{ik} B_k$):

$$\begin{aligned} &D_i(f^{-3/2} B^i) = 0 \qquad (D_i B^i = 3f^{-1} E_i B^i) \\ &\varepsilon^{kis} D_i E_s = 0 \\ &f^{-1/2} \varepsilon^{kis} D_i B_s = 4\pi j^k = -R^k_0 \\ &D_i E^i - f^{-1} E_i E^i + 2f^{-1} B_i B^i = -4\pi j_0 = R_{00} \end{aligned} \tag{12}$$

The quantity that corresponds to the electric charge,

$$Q := \int_S j^\mu \, d\sigma_\mu = -2 \int_S (T^\mu_\alpha - \tfrac{1}{2}\delta^\mu_\alpha T^\beta_\beta)\xi^\alpha \, d\sigma_\mu$$

is identical to the expression for the total gravitational mass given by Komar.[18,19] If S is asymptotically flat, Q reduces to the well-known surface integral

$$Q = \frac{1}{4\pi} \int_S \nabla_\mu F^{\alpha\mu} \, d\sigma_\alpha = -\frac{1}{4\pi} \oint_{\partial \mathbb{R}^3} \nabla^{[\alpha} \xi^{\beta]} \, d\sigma_{\alpha\beta}$$

Then Q is identical to the ADM mass.[20,21]

The source equation for E_i is typical for general relativity, because E_i and B_i appear as sources for E_i.

Because of (12) we expect the flow lines of B_i to be closed in topologically reasonable cases (except for those lines that run to infinity). Furthermore we obtain an analogy with Ampère's law[22] (total current passing through a closed curve := surface integral of the current density = closed line integral of the magnetic field =: "magnetic curcuit voltage"). Let $\mathscr{A}$ be a 2-area in S ($d\sigma_k := \gamma^{1/2} \, d\sigma_k^{\text{flat}}$ is the covariant surface element on $\mathscr{A}$ with respect to γ_{ik}). Then

$$4\pi \int_{\mathscr{A}} f^{1/2} j^k \, d\sigma_k = \int_{\mathscr{A}} \varepsilon^{kis} D_i B_s \, d\sigma_k = \oint_{\partial \mathscr{A}} B_s \, dx^s \tag{13}$$

If we take $\partial\mathscr{A}$ as a closed flow line of B_i (Fig. 1) and chose the orientation such that the right-hand side of (13) becomes positive, then we may conclude in a very heuristic manner, generalizing the well-known theorem of classical electrodynamics: Each flow line of j^k is surrounded by closed flow lines of B_i. The analogy of the "magnetic curcuit voltage" $\oint B_s \, dx^s$ is given by a kind of total "matter current"

$$I := \int_{\mathscr{A}} j^k (-g)^{1/2} \, d\sigma_k^{\text{flat}}$$

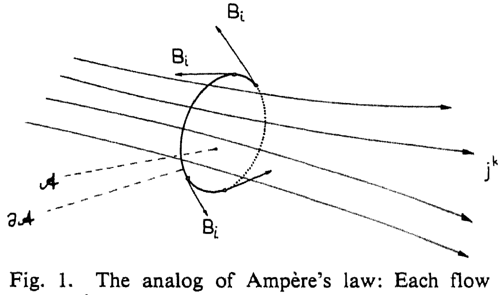

Fig. 1. The analog of Ampère's law: Each flow line of j^k generates closed flow lines of B_i.

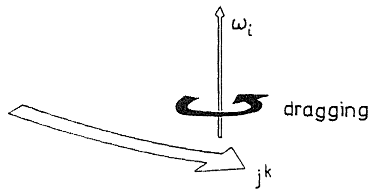

Fig. 2. In cases of simple matter curent densities, the orientation of the frame dragging is easily found:

$$j^k \xrightarrow[\text{right-hand rule}]{} \omega^k \xrightarrow[\text{right-hand rule}]{} \text{dragging}$$

For illustration we may calculate j^k for an ideal fluid with 4-velocity u^μ:

$$j^k = -2(\rho + p)u^k u_\alpha \xi^\alpha = 2(\rho + p)u^k f(u^0 - g_i u^i)$$

In the case of rigid rotation, j^k is zero if S is the rest frame of the fluid ($u^\mu = f^{-1/2}\xi^\mu$). Then B_i is a gradient and has zero curcuit voltage; B_i is identically zero only in the static case ($\omega^\mu = 0$ is the condition for ξ^α being hypersurface-orthogonal[23]).

The analogy between the Thirring–Lense potential and a stationary magnetic field may be used to speculate about the nature of Machian effects in more complicated matter distributions. The orientation of the frame dragging is given by the direction of ω_μ (near an isolated tube of j^k: right-hand rule applied twice: $j^k \to \omega^k \to$ dragging; Fig. 2). For example, near the equator of a rotating star, the frame dragging is antiparallel to the star's angular velocity. Of course, these arguments cannot replace an exact analysis, but they might help imagination and intuition. In Section 8 we shall consider two examples in the weak-field limit.

7. THE LINEARIZED THEORY

In the linearized theory of gravity (flat background, $g_{\mu\nu} = \eta_{\mu\nu} + \psi_{\mu\nu}$, $|\psi_{\mu\nu}| \ll 1$; $O(\psi^2) = 0$; cf. Ref. 4, p. 435), Eqs. (12) take the form of the stationary Maxwell equations:

$$\begin{aligned} \operatorname{div} \mathbf{B} &= \operatorname{rot} \mathbf{E} = 0 \\ \operatorname{rot} \mathbf{B} &= 4\pi \mathbf{j} \\ \operatorname{div} \mathbf{E} &= 4\pi j^0 \end{aligned} \qquad (14)$$

The additional assumption of small velocities (slow motion limit, cf. Ref. 4) and small pressures gives $T^0_0 = -\rho$ and $T^i_0 = -\rho v^i$ (ρ = mass density, v^i = velocity). The current density then becomes

$$j^\mu = (\rho, 2\rho v^i)$$

and thus

$$\text{rot}\,\mathbf{B} = 8\pi\rho\mathbf{v} =: 4\pi\rho\mathbf{v}_{\text{eff}}$$

$$\text{div}\,\mathbf{E} = 4\pi\rho$$

In this case, the Bianchi identity for $T^{\mu\nu}$ reduces to the matter continuity equation,

$$\text{div}(\rho\mathbf{v}) = 0$$

which is valid because $\partial_\alpha j^\alpha = 0$.

Each solution of the stationary Maxwell equations describing the field of moving particles with positive charges provides a model for a weakly gravitating system when its Newtonian and Machian aspects are considered. The magnetic field corresponds to the Thirring–Lense frequency, and the electric field corresponds to the negative of the gravitational field strength. In the slow-motion limit the velocity of the charges corresponds to the double velocity of matter, and the charge density corresponds to the matter density.

If large velocities are allowed, the connection between charge density/velocity of the charges and mass density/velocity of matter becomes slightly more complicated.

8. EXAMPLES

As a first example, let us consider a uniformly rotating charged sphere with constant charge density ρ, radius R, and angular velocity $\Omega > 0$. In spherical coordinates and orthogonal components, the velocity field is given by $v_r = v_\theta = 0$, $v_\varphi = \Omega r \sin\Theta$ for $r \leqslant R$ (we assume slow motion limit, i.e., $\Omega R \ll 1$). Thus we have to solve the Maxwell equations (14) with the three-dimensional current density

$$j_r = j_\Theta = 0$$

$$j_\varphi = \rho\Omega r \sin\Theta$$

We are especially interested in the magnetic field. The integration of the Maxwell equations with the boundary condition $|\mathbf{B}| \to 0$ as $r \to \infty$ gives

$$
\begin{aligned}
B_r &= 4\pi\rho\Omega(\tfrac{1}{3}R^2 - \tfrac{1}{5}r^2)\cos\Theta \\
B_\Theta &= 4\pi\rho\Omega(\tfrac{2}{5}r^2 - \tfrac{1}{3}R^2)\sin\Theta \qquad (15a) \\
B_\varphi &= 0
\end{aligned}
$$

for the interior region $r < R$, and

$$
\begin{aligned}
B_r &= \frac{8\pi}{15}\rho\Omega\left(\frac{R}{r}\right)^3 R^2\cos\Theta \\
B_\Theta &= \frac{4\pi}{15}\rho\Omega\left(\frac{R}{r}\right)^3 R^2\sin\Theta \qquad (15b) \\
B_\varphi &= 0
\end{aligned}
$$

for the exterior region $r > R$. The B_i are continuous across the surface $r = R$. As stated in Section 7, the slowly rotating charged sphere is a model for a slowly rotating star with constant mass density ρ, radius R, and angular velocity $\Omega_{\text{star}} = \frac{1}{2}\Omega$. In the weak-field and slow-motion limit Eqs. (15) describe the Thirring–Lense frequency $\omega_i = B_i$. The magnitude of the frame dragging is given by

$$\omega_{\text{drag}} = (\omega^\mu \omega_\mu)^{1/2} = |\boldsymbol{\omega}|$$

The flow lines of ω_i are given in Fig. 3. They provide an illustrative picture of Machian effects inside and outside the star.

Near $r = 0$, $\boldsymbol{\omega}$ points into the positive z direction (upwards). Using $M = (4\pi/3)\rho R^3$ for the mass of the star, the "dragging coefficient"[7,10] becomes

$$\left.\frac{\omega_{\text{drag}}}{\Omega_{\text{star}}}\right|_{r=0} = \frac{2M}{R} = \frac{\text{Schwarzschild radius}}{\text{radius}}$$

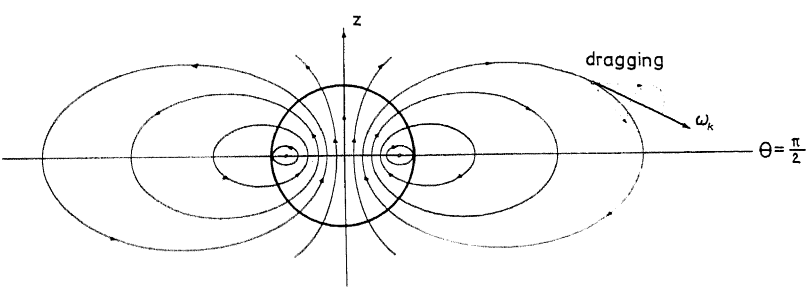

Fig. 3. The flow lines of ω_i for a rigidly rotating star of constant mass density in the slow-motion limit.

In the exterior region, near the equator, $r=R$, $\Theta=\pi/2$, the dragging is antiparallel to the star's angular momentum. Vector $\boldsymbol{\omega}$ points into the negative z direction (downwards). The dragging coefficient is therefore given with a negative sign:

$$\left.\frac{\omega_{\text{drag}}}{\Omega_{\text{star}}}\right|_{\text{equator}} = -\frac{2}{5}\frac{M}{R}$$

Analogously one finds the dragging coefficient near the poles:

$$\left.\frac{\omega_{\text{drag}}}{\Omega_{\text{star}}}\right|_{\text{pole}} = \frac{4}{5}\frac{M}{R}$$

On the ring $\Theta=\pi/2$, $r=\frac{5}{6}R$, the dragging becomes zero.

Comparison with the ω_k in the Kerr metric (Section 4) for large r gives

$$Ma = \tfrac{2}{5}M\Omega_{\text{star}}R^2$$

This is the angular momentum of the star ($\frac{2}{5}MR^2$ is its moment of inertia).[9,10]

The second example is the rotating spherical shell. Setting

$$\rho = \sigma\delta(r-R)$$

($\sigma=$ charge surface density) and

$$j_\varphi = \sigma\Omega\delta(r-R)r\sin\Theta$$

we find the solution of the Maxwell equations

$$B_r = \frac{8\pi}{3}\sigma\Omega R\cos\Theta$$
$$B_\Theta = -\frac{8\pi}{3}\sigma\Omega R\sin\Theta \tag{16a}$$

for the interior region $r<R$, and

$$B_r = \frac{8\pi}{3}\sigma\Omega\left(\frac{R}{r}\right)^3 R\cos\Theta$$
$$B_\Theta = \frac{4\pi}{3}\sigma\Omega\left(\frac{R}{r}\right)^3 R\sin\Theta \tag{16b}$$

for the exterior region $r>R$. Here B_r is continuous, B_Θ is not. The flow lines are given in Fig. 4. Setting $\Omega_{\text{shell}}=\frac{1}{2}\Omega$, we obtain a model for a rotating

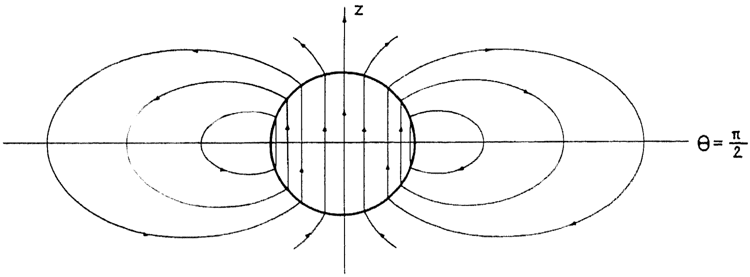

Fig. 4. The flow lines of ω_i for a rigidly rotating spherical shell of constant surface mass density in the slow-motion limit.

spherical mass shell. The frequency ω_{drag} is constant in the interior region. The dragging coefficient is given by ($M = 4\pi\sigma R^2$)

$$\left.\frac{\omega_{\text{drag}}}{\Omega_{\text{shell}}}\right|_{\text{interior}} = \left.\frac{\omega_{\text{drag}}}{\Omega_{\text{shell}}}\right|_{\text{pole}} = \frac{4}{3}\frac{M}{R}$$

in accordance with the result of Thirring[7]; cf. Ref. 4, p. 547. In the exterior region, near the equator, we have

$$\left.\frac{\omega_{\text{drag}}}{\Omega_{\text{shell}}}\right|_{\text{equator}} = -\frac{2}{3}\frac{M}{R}$$

The Newtonian ("electric") field E^i vanishes in the interior region because of the constant surface density of the shell. The result of Thirring[7,8] for the rotating spherical shell is more exact, because he didnot restrict himself to the slow-motion limit. Thus he did not neglect quantities of the order of Ω^2_{shell}. If we want to express our fields within the same accuracy, we have to set

$$u^0_{\text{shell}} = (1 - \mathrm{v}^2_{\text{shell}})^{-1/2} \approx 1 + \tfrac{1}{2}\mathrm{v}^2_{\text{shell}}$$

and thus

$$T^{00} = \sigma\delta(r - R)(1 + \Omega^2_{\text{shell}} R^2 \sin^2\Theta) \qquad (17)$$

for the mass density. Therefore also a nonzero "electric" field E^i is generated in the interior region that gives rise to additional effects (centripetal forces on test particles). The reson for this field is that, due to (17), mass energy appears larger near the equator than near the poles (relativistic mass increase) and thus produces a gratitational field strength pointing toward the equator region.

Within higher accuracy, the simple relation $\mathbf{v}_{\text{matter}} = \frac{1}{2}\mathbf{v}_{\text{charges}}$ is destroyed, and we have to return to (14).

Finally, a last formula will be given (which may be applied exactly to the exterior regions in our two examples). Applying the analysis of magnetostatics to linearized theory, one finds that the Thirring–Lense frequency far from the source (which is assumed to be rotating with constant angular velotity $\mathbf{\Omega}$) is given by

$$\mathbf{\omega} = \frac{I}{r^3} \frac{3(\mathbf{\Omega x})\mathbf{x} - \mathbf{\Omega} r^2}{r^2}$$

where I is the moment of inertia of the source (cf. Ref. 4, p. 1119); $I\mathbf{\Omega}$ corresponds to the magnetic dipole moment. This equation for $\mathbf{\omega}$ first appeared in Ref. 24.

9. PARTICLE MOTION

The analogy of Newtonian and Machian aspects with an electromagnetic field remains a purely formal one unless it is shown that E^i acts on moving test particles like an electric field and that B^i acts on them like a magnetic field. In special relativity, the motion of a point particle with mass m and charge e in an electromagnetic field is given by the Lorentz force

$$m \frac{du^\alpha}{ds} = F^{\alpha\beta} u_\beta \tag{18}$$

with spatial components ($w^i := dx^i/dt$, ordinary velocity)

$$m \frac{du^i}{ds} = (1 - \mathbf{w}^2)^{-1/2} e(E^i + \varepsilon^{ikl} w_k B_l) \tag{19a}$$

and the time component

$$\frac{d}{dt}(m(1 - \mathbf{w}^2)^{-1/2}) = eE_i w^i \tag{19b}$$

We are interested now in knowing if the geodesic motion of test particles in a stationary gravitational field is somewhat similar to (19a) if E^i and B^i are taken from (10) and (11). We use the same coordinate system as in the previous sections.

Let $u^\alpha = dx^\alpha(s)/ds$ be the 4-velocity of a neutral test particle, $u^\alpha u_\alpha = -1$, and $\nabla_u := u^\alpha \nabla_\alpha$. The motion is described by the geodesic equation

$$\nabla_u u^\alpha \equiv \frac{d^2x^\alpha}{ds^2} + \Gamma^\alpha_{\beta\delta}\frac{dx^\beta}{ds}\frac{dx^\delta}{ds} = 0 \tag{20}$$

The fields E_i and B_i appear in the Christoffel symbols $\Gamma^\alpha_{\beta\delta}$. We hope to give the spatial part of this equation a form analogous to (19a). As the velocity enters linear in (18) but quadratic in (20), we expect the factor $(1-\mathbf{w}^2)^{-1/2}$ to be replaced by a quantity like $(1-\mathbf{w}^2)^{-1}$.

To construct a 3-velocity, we note that the projection

$$v^\alpha := u^\alpha + f^{-1}\xi^\alpha \xi_\gamma u^\gamma$$

is a vector tangential to S with components

$$v^i = u^i, \qquad v_i = u_i + g_i u_0 = \gamma_{ik} v^k$$

$$v_0 = 0, \qquad v^0 = g_i v^i$$

Thus $v^i = dx^i/ds$, $v_k = \gamma_{ki} v^i$ may be regarded as a 3-vector on S (like E^i and B^i). As u^α is a unit vector, u^0 may be expressed as

$$u^0 - g_i v^i = [f^{-1}(1 + v_i v^i)]^{1/2} =: \mu = -f^{-1} u_0 \tag{21a}$$

Diract calculation of $(u^0)^{-1} = ds/dt$ gives another form,

$$u^0 = [f(1 - g_i w^i)^2 - w_i w^i]^{-1/2} = \mu(1 - g_i w^i)^{-1} \tag{21b}$$

expressed through the ordinary velocity $w^i = dx^i/dt$. In the linearized theory, both μ and u^0 reduce to the well-known factor $(1 - w^2)^{-1/2} = (1 + v^2)^{1/2}$. The quantities v^i and w^i are related by $v^i = u^0 w^i$.

The Christoffel symbols may be calculated straightforwardly, using (1a) and (11):

$$\Gamma^i_{00} = E^i$$

$$\Gamma^i_{0k} = -g_k E^i + \tfrac{1}{2} f^{-1/2} b^m \varepsilon_{mk}{}^i$$

$$\Gamma^i_{kl} = {}^{(3)}\Gamma^i_{kl} - \tfrac{1}{2} f^{-1/2} b^m (g_k \varepsilon_{ml}{}^i + g_l \varepsilon_{mk}{}^i) + g_k g_l E^i$$

$$\Gamma^0_{00} = g^i E_i$$

$$\Gamma^0_{0i} = f^{-1} E_i - g_i g^n E_n + \tfrac{1}{2} f^{-1/2} g^n b^m \varepsilon_{\min}$$

$$\Gamma^0_{ik} = g_s{}^{(3)}\Gamma^s_{ik} - f^{-1}(E_k g_i + E_i g_k) + \tfrac{1}{2} f^{-1/2} g^n b^m (g_i \varepsilon_{mnk} + g_k \varepsilon_{mni}) + g^n E_n g_i g_k - \tfrac{1}{2}(\partial_i g_k + \partial_k g_i)$$

where ${}^{(3)}\Gamma^s_{ik}$ are the Christoffel symbols with respect to γ_{ik}.

These expressions may be used to calculate the spatial components of (20) ($D_v := v^i D_i$):

$$D_v v^i \equiv \frac{d^2x^i}{ds^2} + {}^{(3)}\Gamma^i_{kl} \frac{dx^k}{ds} \frac{dx^1}{ds} = -\mu^2 E^i + \mu f^{-1/2} \varepsilon^{ikm} v_k b_m \tag{22}$$

The formal analogy to (19a) is obvious; μ^2 plays the role of $(1-\mathbf{w}^2)^{-1}$ as stated above. If we set $v_k = u^0 w_k$ on the right-hand side of (22), the factor μu^0 plays the same role. The left-hand side of (19a) is replaced by the three-dimensional covariant acceleration with respect to γ_{ik}. Here $-E^i$ corresponds to the electric field and $b_m = -2B_m$ to the magnetic field when mass and charge are set equal, $m = e$. The minus signs arise from the fact that the electric repulsion of two positive charges is replaced by the gravitational attraction of two masses. The factor 2, which replaces B_m by b_m, arises from the tensor character of the gravitational potentials $g_{\mu\nu}$:

$$\Gamma^i_{\alpha\beta} u^\alpha u^\beta = \Gamma^i_{00}(u^0)^2 + 2\Gamma^i_{0j} u_0 v^j + \Gamma^i_{kl} v^k v^l$$

The time component of (20) may be brought into the form

$$\frac{d}{ds}(\mu f) = 0 \tag{23}$$

i.e., μf is a constant of the motion. It is easily identified with the energy of the particle: Since ξ^α is a timelike Killing field, its scalar product with the geodesic 4-momentum is conserved:

$$\varepsilon := -m\xi^\mu u_\mu = -mu_0 = mf(u^0 - g_i v^i) = m\mu f \tag{24}$$

The energy conservation may be used to define the proper time s on the hypersurface S without regard to the whole manifold (up to trivial scaling). (If s is replaced by another, arbitrary, parameter, the quantity (24) will not be constant.) Thus (22) and (23) give a full description of geodesic motion, expressed with fields on S.

In the linearized theory, (22) reads

$$D_v v^i = (1-\mathbf{w}^2)^{-1}(-E^i + \varepsilon^{ikl} w_k b_l) \tag{25}$$

which reduces, in the slow-motion limit ($O(\mathbf{w}^2) = 0$), to

$$\frac{dv^i}{ds} = \frac{dw^i}{dt} = -E^i + \varepsilon^{ikl} w_k b_l \tag{26}$$

the exact formula for the nonrelativistic Lorentz force. In this limit, the stationary gravitational field acts on a test particle with mass m exactly like a stationary electromagnetic field (−**E**, **b**) on a test particle with mass m and charge m. The Newtonian approximation of this equation is

$$\frac{dw^i}{dt} = -E^i \tag{27}$$

the "magnetic" term is a post-Newtonian correction (Machian part of the gravitational force).

Let us evaluate (26) for two examples. First consider the rotating spherical shell of Section 8. In the interior region, the equations of motion for a test particle are [using (16a) and (10)], expressed in Cartesian coordinates,

$$\ddot{x} = -\frac{8}{3}\frac{M}{R}\Omega_{\text{shell}}\dot{y}$$

$$\ddot{y} = \frac{8}{3}\frac{M}{R}\Omega_{\text{shell}}\dot{x}$$

$$\ddot{z} = 0$$

in accordance with Thirring's[7] result. As stated in Section 8, Thirring obtained additional terms proportional to Ω^2_{shell}, arising from a nonzero E^i.

As a second example, consider the exterior region of the rotating star of Section 8: B^i is given by (15b), and the "electric" field is

$$E_r = Mr^{-2}$$

Equation (26) reads, in Cartesian coordinates,

$$\ddot{x} = \frac{4}{5}M\Omega_{\text{star}}R^2r^{-3}\left[\dot{y}\left(1 - \frac{3z^2}{r^2}\right) + 3\dot{z}\frac{yz}{r^2}\right] - Mr^{-3}x$$

$$\ddot{y} = -\frac{4}{5}M\Omega_{\text{star}}R^2r^{-3}\left[\dot{x}\left(1 - \frac{3z^2}{r^2}\right) + 3\dot{z}\frac{xz}{r^2}\right] - Mr^{-3}y$$

$$\ddot{z} = \frac{12}{5}M\Omega_{\text{star}}R^2r^{-3}\left[\dot{y}\frac{xz}{r^2} - \dot{x}\frac{yz}{r^2}\right] - Mr^{-3}z$$

This is exactly the result obtained by Thirring and Lense.[9]

In conclusion we state that the Thirring–Lense frequency ω_μ may be regarded as a formal analog of a magnetic field (when the masses are identified with the charges) from two point of views: (i) from its generation by mass-energy, cf. Eqs. (12), and (ii) from its action on test particles, cf. Eq. (22). Twice a factor 2 arises ($2\mathbf{v}_{\text{matter}} = \mathbf{v}_{\text{charges}}$ in Section 7 and $b_i = -2B_i$) as a consequence of the remaining fundamental difference between electromagnetism and gravity.

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