KU LEUVEN

ARENBERG DOCTORAL SCHOOL
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Almost inner derivations of Lie algebras

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Dissertation presented in partial fulfillment of the requirements for the degree of Doctor of Science (PhD): Mathematics

September 2020

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Dankwoord

Bijna exact vier jaar nadat ik aan mijn doctoraat mocht beginnen, werk ik met dit dankwoord het laatste deel af. De laatste maanden vormden een vreemde afsluiter van een boeiende en interessante periode waarin ik veel heb bijgeleerd en het goed naar mijn zin heb gehad. Heel wat mensen verdienen een woord van dank, omdat ze elk op hun manier hebben bijgedragen aan deze doctoraatsthesis.

Op de eerste plaats wil ik Karel, mijn promotor, van harte bedanken. Karel, zowel voor mijn masterthesis als mijn doctoraat stelde je een project voor over bijna-inwendige derivaties van Lie-algebra's. Dat onderwerp boeide me meteen en fascineert me nu nog altijd. Ik kreeg veel vrijheid om mijn werk te plannen en om te onderzoeken wat ik zelf interessant vond. Je legde me geen druk op, apprecieerde wat ik deed en hielp me verder wanneer ik daar om vroeg. Bedankt voor het nalezen en beter maken van de verschillende voorlopige versies van de artikels en van deze thesis. Karel, ik had me geen betere promotor kunnen voorstellen. Bedankt voor alles.

Dietrich, you deserve special thanks as well. Thank you for answering my questions, for the interesting discussions, the productive collaboration and the nice talks during my stays in Vienna. It was a great pleasure working with you. Paul, bedankt voor de voortdurende steun, het vertrouwen en de waardering voor mijn werk. Samen met Karel heb je mij je passie voor algebra overgebracht. Ik druk ook graag mijn oprechte appreciatie uit voor Jonas om mijn mentor te zijn. Bedankt voor de interessante inzichten over Lie algebra's en voor de maand in Wenen. Ook bedank ik graag de andere leden van de jury voor het nalezen en beoordelen van dit proefschrift en voor de constructieve vragen en opmerkingen tijdens en na de preliminaire verdediging.

Andries, Charlotte, Hilde, Iris, Jonas, Kaidi, Laura, Lore, Marcos, Paula, Pieter, Sam, Stijn en Timur, bedankt voor de leuke sfeer in de onderzoeksgroep de voorbije jaren. Sam, in het bijzonder wil ik jou bedanken om als peter en kantoorgenoot me steeds te helpen. Daarnaast was je ook mijn voornaamste metgezel tijdens studentikoze activiteiten. Bedankt aan de fysici en de mensen van VWO en usolv-it voor de ontspannende momenten in de koffiekamer. Daarnaast waren het kleurenwiezen en nakaarten een grote troef van de middagpauzes. Ik wil alle collega's bedanken om van de Kulak zo een aangename werkplek te maken. Ook voor mij geldt: "unief van mijn dromen, 'k vergeet u nooit meer."

Naast mijn collega's zijn er nog heel wat mensen die de voorbije vier jaren zo leuk maakten. Daarom wil ik heel graag de oud-leiding van KSA Pius X, de mensen van West en Nationaal en de vriendengroepen uit het middelbaar en uit mijn studententijd aan de Kulak bedanken voor de toffe momenten samen. Michiel, bedankt om de voorbije twee en een half jaar naast huisgenoot ook mijn klankbord te zijn, vooral toen je tijdens het schrijfwerk aan deze thesis lange tijd de enige persoon was die ik 'in het echt' mocht zien.

Tot slot verdient ook mijn familie een enorme bedanking voor de interesse, de steun en de mooie tijd samen. Bedankt voor de goede raad en het feit dat ik altijd op jullie kan rekenen.

Bert

Abstract

A classical problem in spectral geometry was to determine whether or not isospectral manifolds are necessarily isometric. It turns out that the answer to this question is negative and several counterexamples have been given. For the construction of continuous families of isospectral and non-isometric manifolds, class preserving automorphisms of nilpotent Lie groups were crucial.

An automorphism of a group is called class preserving if and only if every element is conjugate to its image. So this condition is related to, but less strict than the one for an inner automorphism. A nilpotent Lie group admitting a discrete and cocompact subgroup and a class preserving automorphism which is not inner can be used to construct a continuous family of isospectral but non-isometric nilmanifolds. Class preserving automorphisms of a Lie group are very closely related to almost inner derivations of the corresponding Lie algebra. These are derivations for which each element is mapped to the Lie bracket of itself with some other element. The set of all almost inner derivations forms a Lie subalgebra of the derivation algebra and contains the inner derivations.

Up till now, almost inner derivations of Lie algebras have not been studied in detail yet. They have almost only been considered from a differential geometric perspective, where the focus was on constructing some examples. The goal of this thesis is to study this notion in a purely algebraic way. Although the motivation from spectral geometry only makes sense for nilpotent Lie algebras, from an algebraic point of view, there is no reason to restrict to this class only. Hence, we study almost inner derivations of Lie algebras more generally. We also consider non-nilpotent Lie algebras and Lie algebras over an arbitrary field, so not only over the real or complex numbers.

This dissertation consists of three main parts and one appendix. The first part is an introduction, which provides all preliminaries to understand what follows. In Chapter 2, we define Lie algebras and develop the necessary notions which will be important in the study of almost inner derivations. Chapter 3 contains more information about the geometric motivation from spectral theory. We also present properties of the related notion of class preserving automorphisms of groups. Finally, in Chapter 4, we describe some interesting techniques for doing computations on the almost inner derivations of (a class of) Lie algebras.

In the second part, we will focus on the fact that the dimension of the set of almost inner derivations depends on the field over which the Lie algebra is defined. Chapter 5 contains an elaborated example, where a Lie algebra is given by means of the structure constants. The distinction for various fields has to do with a different factorisation of polynomials. In Chapter 6, we show a procedure to construct new almost inner derivations by using finite field extensions. In particular, this gives a way to set up a Lie algebra for which the dimension of the set of almost inner derivations is distinct when we consider different fields. Chapter 7 focuses on Lie algebras related to finite groups. We explain the connection with class preserving automorphisms of finite groups and compare the results we have for the two notions.

In the last part, we will use the observations from the two other parts to compute the set of almost inner derivations for different classes of Lie algebras. In Chapter 8, we give an overview of almost inner derivations for low-dimensional Lie algebras. The appendix contains tables where the non-vanishing Lie brackets for a lot of low-dimensional Lie algebras are collected. Each time, we also provide tables with results, such as the dimension of some subalgebras of the derivation algebra. The next three chapters are devoted to other classes of nilpotent Lie algebras. Two-step nilpotent Lie algebras are studied in Chapter 9. Further, we also consider filiform Lie algebras and free nilpotent Lie algebras (in Chapter 10 respectively Chapter 11). The last chapter contains results for some other classes of (not only nilpotent) Lie algebras.

Beknopte samenvatting

Een klassiek probleem in de spectraalmeetkunde was om na te gaan of variëteiten met hetzelfde spectrum ook isometrisch zijn. Verschillende tegenvoorbeelden tonen aan dat dit niet noodzakelijk het geval is. Voor de constructie van continue families isospectrale en niet-isometrische variëteiten blijken klassebewarende automorfismen van cruciaal belang te zijn.

Een automorfisme van een groep is klassebewarend als en slechts als elk element geconjugeerd is met zijn beeld. Deze voorwaarde is dus heel gelijkaardig aan, maar minder streng dan die voor een inwendig automorfisme. Een nilpotente Lie-groep met een discrete en cocompacte deelgroep waarvoor bovendien ook een klassebewarend automorfisme bestaat dat niet inwendig is, kan gebruikt worden om een continue familie isospectrale en niet-isometrische nilvariëteiten op te bouwen. Klassebewarende automorfismen van een Lie-groep zijn in sterke mate verbonden met bijna-inwendige derivaties van de bijhorende Lie-algebra. Dit zijn derivaties waarbij elk element afgebeeld wordt op de Lie-haak van zichzelf met een ander element. De verzameling van alle bijna-inwendige derivaties vormt een Lie-deelalgebra en bevat alle inwendige derivaties.

Tot nu toe zijn deze bijna-inwendige derivaties van Lie-algebra's nog niet in detail onderzocht. Het werd enkel bestudeerd vanuit meetkundig standpunt, waarbij de focus lag op het construeren van concrete voorbeelden. Het doel van deze doctoraatsthesis is om dit begrip op een puur algebraïsche manier te bestuderen. Ook al geldt de motivatie vanuit de spectraaltheorie enkel voor nilpotente Lie-algebra's, vanuit algebraïsch opzicht is er geen reden om ons tot die klasse te beperken. Daarom is het de bedoeling om bijna-inwendige derivaties van Lie-algebra's meer algemeen te behandelen en ook Lie-algebra's te bekijken die niet nilpotent zijn. Verder bestuderen we niet enkel reële en complexe Lie-algebra's, maar ook Lie-algebra's die gedefinieerd zijn over een willekeurig veld.

Deze thesis bestaat uit drie grote delen en een appendix. Het eerste deel is een inleiding en bevat alle zaken die nodig zijn om de resultaten uit latere hoofdstukken te kunnen begrijpen. In Hoofdstuk 2 worden Lie-algebra's ingevoerd, net als andere begrippen die belangrijk zijn in de studie van bijna-inwendige derivaties. Hoofdstuk 3 bevat meer informatie over de meetkundige motivatie vanuit de spectraalmeetkunde. Daarnaast geven we eigenschappen van klassebewarende automorfismen voor groepen. In Hoofdstuk 4 beschrijven we enkele interessante technieken om de bijna-inwendige derivaties te berekenen voor (een bepaalde klasse van) Lie-algebra's.

In het tweede deel ligt de nadruk op het feit dat de dimensie van de verzameling bijna-inwendige derivaties afhangt van het veld waarover de Lie-algebra gedefinieerd is. Hoofdstuk 5 bevat een uitgewerkt voorbeeld, waarbij een Lie-algebra voorgesteld is aan de hand van de Lie-haken. Het onderscheid bij verschillende velden heeft te maken met een andere veeltermontbinding. In Hoofdstuk 6 tonen we een werkwijze om, aan de hand van eindige velduitbreidingen, nieuwe bijna-inwendige derivaties te construeren. Dit geeft in het bijzonder een manier om een Lie-algebra op te stellen waarbij de verzameling bijna-inwendige derivaties varieert voor verschillende velden. In Hoofdstuk 7 gaat het over Lie-algebra's geassocieerd aan eindige groepen. We leggen het verband met klassebewarende automorfismen van eindige groepen uit en vergelijken de resultaten die we voor beide begrippen hebben.

Voor het laatste deel gebruiken we de observaties uit de twee andere delen om de bijna-inwendige derivaties te bepalen voor verschillende klassen van Lie-algebra's. In Hoofdstuk 8 berekenen we de bijna-inwendige derivaties voor laagdimensionale Lie-algebra's. De appendix bevat een overzicht van de Lie-haken die niet nul zijn voor heel wat Lie-algebra's van lage dimensie. Telkens is er ook een tabel waarin de resultaten weergegeven zijn, zoals de dimensie van een aantal deelalgebra's van de derivatie-algebra. De volgende drie hoofdstukken zijn gewijd aan nilpotente Lie-algebra's. In Hoofdstuk 9 worden twee-staps nilpotente Lie-algebra's bestudeerd. Verder behandelen we ook filiforme en vrije nilpotente Lie-algebra's (in Hoofdstuk 10 respectievelijk Hoofdstuk 11). In het laatste hoofdstuk staan resultaten voor een aantal andere klassen van (niet enkel nilpotente) Lie-algebra's.

Contents

Chapter 1

Overview of the thesis

One of the interesting problems in spectral geometry is to determine to what extent the spectrum of a manifold determines the shape. Hermann Weyl asked whether or not isospectral manifolds have to be isometric. Milnor ([65]) presented two flat tori in dimension 16 which are isospectral but not isometric, thereby answering the question. The following years, several other counterexamples have been given, all of them consisting of finite families. In 1984, Gordon and Wilson constructed continuous families of non-isometric manifolds with the same spectrum. Therefore, they considered a connected and simply connected nilpotent Lie group G with a discrete cocompact subgroup N in such a way that the compact nilmanifold $N\backslash G$ has a metric g. To obtain a continuous family of isospectral and non-isometric nilmanifolds, they used the notion of a 'class preserving automorphism of a group'.

An automorphism φ of a group G is said to be class preserving if and only if $\varphi(x)$ is conjugate to x for any $x \in G$. The set of all class preserving automorphisms of G is denoted with $\mathrm{Aut}_c(G)$. It forms a normal subgroup of $\mathrm{Aut}(G)$ which contains $\mathrm{Inn}(G)$, the group of all inner automorphisms. If $\{\varphi_t\}_t$ is a continuous family of automorphisms in $\mathrm{Aut}_c(G)$ with $\varphi_0 = \mathrm{Id}$, then $(N\backslash G, \varphi_t^* g)$ is called an 'isospectral deformation' of $(N\backslash G, g)$, which means that the nilmanifolds are isospectral. When $\varphi \in \mathrm{Inn}(G)$, then they are isometric as well. However, this is not necessarily the case for $\varphi \notin \mathrm{Inn}(G)$. Hence, to obtain isospectral and non-isometric nilmanifolds, one can search for nilpotent Lie groups admitting non-inner class preserving automorphisms.

The usual approach to do this is by studying 'almost inner derivations', the analogous notion of class preserving automorphisms for Lie algebras. Gordon and Wilson ([41]) introduced this concept and described some examples. Since then, a few other papers have appeared concerning this subject, but all from a differential geometric point of view. The goal was to describe Lie algebras admitting non-inner almost inner derivations and to 'translate' it back to Lie groups, so the notion has not been studied in detail.

In this dissertation, we start an algebraic study of almost inner derivations of Lie algebras. A derivation D of a Lie algebra $\mathfrak{g}$ is said to be almost inner if $D(x) \in [x, \mathfrak{g}]$ for all $x \in \mathfrak{g}$. The set of all almost inner derivations $\mathrm{AID}(\mathfrak{g})$ forms a Lie subalgebra of $\mathrm{Der}(\mathfrak{g})$ and contains the Lie algebra of inner derivations $\mathrm{Inn}(\mathfrak{g})$. The motivation from spectral theory only makes sense for nilpotent Lie algebras, so this will be our starting point. However, we don't restrict to this class. Further, we also consider Lie algebras over general fields instead of only over $\mathbb{R}$ and $\mathbb{C}$. This thesis consists of three parts and one appendix.

Part I The first part is an introduction, which provides the preliminaries to understand the results. Chapter 2 consists of a short algebraic introduction to Lie algebra theory. We describe the concepts which will be used in the rest of this thesis. In Chapter 3, we make the link with Lie groups. This allows us to go in more detail about the constructions of isospectral and non-isometric nilmanifolds. Class preserving automorphisms have been studied for (finite p–)groups as well. We list some of the obtained results in this area and compare them in later chapters with properties of almost inner derivations of Lie algebras.

In Chapter 4, we first prove some basic properties of almost inner derivations. Further, we introduce the notion of 'fixed basis vectors'. With the aid of this concept, we will prove in some cases that the only almost inner derivations are the inner ones. The advantage of this technique is that it can be used on the basis of the structure constants of the given Lie algebra, without having to compute the derivation algebra. We also describe the correspondence between Lie algebras and so-called 'skew-symmetric matrix pencils'. For each (finite-dimensional) Lie algebra $\mathfrak{g}$ over a field $\mathbb{F}$, there exists an associated pencil $\mu_1 A_1 + \cdots + \mu_m A_m$, where μ_i is an indeterminate and A_i is a skew-symmetric matrix for all $1 \leq i \leq m$. We show that the determinant of the matrix pencil plays an important role in computing the dimension of $\mathrm{AID}(\mathfrak{g})$. This method is in particular interesting for 2–step nilpotent Lie algebras, but can also be used more generally. The techniques make it possible to compute for each Lie algebra $\mathfrak{g}$ the set of almost inner derivations $\mathrm{AID}(\mathfrak{g})$ and to compare it with $\mathrm{Inn}(\mathfrak{g})$ and $\mathrm{Der}(\mathfrak{g})$.

Part II In contrast to what one might expect, the algebraic structure of $\mathrm{Der}(\mathfrak{g})$ does not really give much information with respect to the almost inner derivations. As an example of this, consider a finite field extension $k \subseteq K$ and a

Lie algebra $\mathfrak{g}$ over k. We show that $\mathrm{Der}(\mathfrak{g} \otimes K) \cong \mathrm{Der}(\mathfrak{g}) \otimes K$, so the algebraic structures of $\mathrm{Der}(\mathfrak{g})$ and $\mathrm{Der}(\mathfrak{g} \otimes K)$ are very closely related. However, it often happens that $\mathrm{Der}(\mathfrak{g} \otimes K)$ does not contain almost inner derivations which are not inner, while $\mathrm{Der}(\mathfrak{g})$ has a lot of non-trivial almost inner derivations. This illustrates that the dimension of $\mathrm{AID}(\mathfrak{g})$ depends on the field k over which $\mathfrak{g}$ is defined. The main focus of the second part is to study this phenomenon in more detail.

Chapter 5 is devoted to the detailed computations of the almost inner derivations for one example. The Lie algebra $\mathfrak{g}$ is described using structure constants and can be considered over an arbitrary field. In this example, we show that the number of different roots of the determinant of the matrix pencil determines the dimension of $\mathrm{AID}(\mathfrak{g})$. This explains why $\dim(\mathrm{AID}(\mathfrak{g}))$ and $\dim(\mathrm{AID}(\mathfrak{g} \otimes K))$ can be different.

In Chapter 6, we show that if $\mathrm{AID}(\mathfrak{g} \otimes K) \neq \mathrm{Inn}(\mathfrak{g} \otimes K)$ holds, then also $\mathrm{AID}(\mathfrak{g}) \neq \mathrm{Inn}(\mathfrak{g})$. However, the converse statement is not true in general. We further describe a procedure to construct new almost inner derivations by using finite field extensions. Therefore, we study the so-called 'underlying Lie algebras' $\mathfrak{g}'_k$ and $\mathfrak{g}'_K$.

As we illustrated in Chapter 3, there have been several results of class preserving automorphisms of finite p–groups. In Chapter 7, we describe a way to relate to a finite p–group G a corresponding Lie algebra $L(G)$ over $\mathbb{F}_p$. Hence, in contrast to the rest of the thesis, this chapter focuses on Lie algebras related to finite p–groups. For a certain class of 2–step nilpotent p–groups, there is a nice correspondence between the class preserving automorphisms and the almost inner derivations. However, this is not true in general, as we will illustrate as well.

Part III In the last part, we consider the almost inner derivations for different classes of Lie algebras. We use the observations from the first two parts to study several research questions. Given a Lie algebra $\mathfrak{g}$ (over some field), we determine $\mathrm{AID}(\mathfrak{g})$ over that field and compute which of the inclusions

$$\mathrm{Inn}(\mathfrak{g}) \subseteq \mathrm{CAID}(\mathfrak{g}) \subseteq \mathrm{AID}(\mathfrak{g}) \subseteq \mathrm{Der}(\mathfrak{g}) \tag{1.1}$$

are in fact equalities. Further, we describe how the results for $\mathfrak{g}$ compare to other Lie algebras from the same class.

We show that the only almost inner derivations are the inner ones for many 'standard' Lie algebras, both nilpotent (such as Lie algebras determined by graphs and free nilpotent Lie algebras) and non-nilpotent. These 'negative' results may give the impression that Lie algebras admitting non-trivial almost inner derivations are a rare phenomenon. However, results on low-dimensional Lie algebras prove that even the opposite is true, but it is not easy to collect these examples in a class with a lot of structure. Still, we find infinite families of Lie algebras admitting non-trivial almost inner derivations, such as metabelian filiform Lie algebras. We also specify families of 2–step nilpotent Lie algebras $\{\mathfrak{g}_n\}_n$ having a space $\mathrm{AID}(\mathfrak{g}_n)/\mathrm{Inn}(\mathfrak{g}_n)$ of dimension n, where $n \in \mathbb{N}$. Moreover, we illustrate that many possibilities occur for (1.1). For instance, for all $n \geq 13$, there exists a characteristically nilpotent filiform Lie algebra $\mathfrak{g}_n$ of dimension n with $\mathrm{Inn}(\mathfrak{g}_n) \neq \mathrm{AID}(\mathfrak{g}_n) = \mathrm{Der}(\mathfrak{g}_n)$.

In Chapter 8, we compute the almost inner derivations for different lists of low-dimensional Lie algebras. We first consider Lie algebras over an arbitrary field and obtain a complete result for Lie algebras of dimension at most 3, solvable Lie algebras of dimension 4 and nilpotent Lie algebras of dimension at most 6. Let $\mathfrak{g}$ be an n–dimensional Lie algebra over $\mathbb{F}$. When $\mathrm{char}(\mathbb{F}) \neq 2$ and $n \leq 4$, all almost inner derivations of $\mathfrak{g}$ are inner. However, there exists a 4–dimensional Lie algebra $\mathfrak{g}$ over an infinite field $\mathbb{F}$ of $\mathrm{char}(\mathbb{F}) = 2$ with $\mathrm{Inn}(\mathfrak{g}) \neq \mathrm{AID}(\mathfrak{g}) = \mathrm{Der}(\mathfrak{g})$. We further perform computations for (non-solvable) 4–dimensional Lie algebras over a field of characteristic zero and 5–dimensional Lie algebras over $\mathbb{C}$ and $\mathbb{R}$. In dimension 5, we find a non-nilpotent Lie algebra $\mathfrak{g}$ such that the only almost inner derivations are inner when $\mathfrak{g}$ is considered over $\mathbb{C}$, whereas $\mathrm{Inn}(\mathfrak{g}) \neq \mathrm{AID}(\mathfrak{g}) = \mathrm{Der}(\mathfrak{g})$ holds over $\mathbb{R}$. The different Lie algebras we work with and the corresponding findings from this chapter are listed in several tables in the appendix.

In Chapter 9, we study two-step nilpotent Lie algebras. We show that the only almost inner derivations for Lie algebras determined by graphs are the inner ones. We also consider Lie algebras of genus 1 and 2. Nilpotent Lie algebras of genus 1 only have trivial almost inner derivations. However, for nilpotent Lie algebras $\mathfrak{g}$ with $\dim([\mathfrak{g},\mathfrak{g}]) = 2$, several different possibilities occur. Some Lie algebras do not admit non-inner almost inner derivations at all, while for other, the dimensions of $\mathrm{AID}(\mathfrak{g})/\mathrm{Inn}(\mathfrak{g})$ is rather large in comparison to the dimension of $\mathfrak{g}$ and all intermediate results occur as well. These Lie algebras can be described using 'elementary divisors' and 'minimal indices' of the associated matrix pencils. In this class, we obtain a complete description and classification of all almost inner derivations for Lie algebras over $\mathbb{R}$ or an algebraically closed field of characteristic not two. We further describe nonsingular Lie algebras and give an infinite family of Lie algebras $\{\mathfrak{g}_n\}_n$ such that $\dim(\mathrm{AID}(\mathfrak{g}_n)/\mathrm{Inn}(\mathfrak{g}_n)) = n$.

Further, we consider filiform Lie algebras in Chapter 10. Let $\mathfrak{g}$ be a metabelian filiform Lie algebra over an arbitrary field. If $\mathfrak{g}$ is standard graded, then the only almost inner derivations are the inner ones. However, when $\mathfrak{g}$ is not standard graded, we prove that $\dim(\mathrm{AID}(\mathfrak{g})) = \dim(\mathrm{Inn}(\mathfrak{g})) + 1$. We also perform computations for several other filiform Lie algebras over a field of characteristic zero, such as the Witt Lie algebras. Based on this last type, we construct for all $n \geq 13$ a characteristically nilpotent Lie algebra $\mathfrak{g}_n$ of dimension n for which $\mathrm{Inn}(\mathfrak{g}_n) \neq \mathrm{AID}(\mathfrak{g}_n) = \mathrm{Der}(\mathfrak{g}_n)$ holds.

Chapter 11 is devoted to the study of free nilpotent Lie algebras. We first describe dual Lie algebras as quotients of free 2–step nilpotent Lie algebras. It turns out that there is no correspondence between the almost inner derivations of a Lie algebra and its dual. Further, we show that free 3–step nilpotent Lie algebras (over any field) and free metabelian nilpotent Lie algebras on two generators (over any infinite field) do not admit non-almost inner derivations. The main theorem of this chapter is the following. Let $\mathbb{F}$ be a field of characteristic zero and denote $\mathfrak{f}_{r,c}$ for the free c–step nilpotent Lie algebra over $\mathbb{F}$ on r generators. Then $\mathrm{AID}(\mathfrak{f}_{r,c}) = \mathrm{Inn}(\mathfrak{f}_{r,c})$ holds.

The last chapter contains results for several other classes of Lie algebras. We show that (strictly) upper triangular Lie algebras and almost abelian Lie algebras over an arbitrary field do not admit non-inner almost inner derivations. The same result holds for Lie algebras over an algebraically closed field of characteristic zero for which the solvable radical is abelian. In the last section, we give an overview of the results we obtained for characteristically nilpotent Lie algebras throughout the dissertation.

Part I

Preliminaries

Chapter 2

Lie algebras

The goal of this chapter is to recall the relevant concepts and properties of Lie algebras which are used throughout this thesis. This overview is purely algebraic. The geometric motivation and the correspondence with Lie groups is postponed to Section 3.1. First, we introduce some definitions and examples and describe different linear maps between the Lie algebras. Then, we also provide several types of Lie algebras and explain how they can be classified. Since all facts are standard, we will omit the proofs in most cases. Unless otherwise stated, we work over an arbitrary field $\mathbb{F}$. More information can be found in standard books of Lie algebras, such as [26].

2.1 Basic terminology

This section elaborates on the terminology which will be used in later chapters. A Lie algebra is an algebra for which the bilinear multiplication map satisfies certain extra conditions. These requirements are motivated by the corresponding properties of the Lie bracket for vector fields.

Definition 2.1.1 (Lie algebra)**.** Let $\mathbb{F}$ be a field. An algebra $\mathfrak{g}$ over $\mathbb{F}$ is a Lie algebra when the bilinear multiplication map

$$\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g} : (x, y) \mapsto [x, y]$$

satisfies the following properties:

- The bracket is alternate, so $[x, x] = 0$ holds for all $x \in \mathfrak{g}$,

- For all x, y and $z \in \mathfrak{g}$, the Jacobi identity $[x,[y,z]]+[y,[z,x]]+[z,[x,y]] = 0$ is fulfilled.

The first condition implies skew-symmetry (or anti-commutativity), since

$$0 = [x+y, x+y] = [x,x] + [x,y] + [y,x] + [y,y] = [x,y] + [y,x]$$

holds for all $x, y \in \mathfrak{g}$. When $\mathrm{char}(\mathbb{F}) \neq 2$, both concepts are equivalent. In this thesis, we only consider finite-dimensional Lie algebras, where the dimension of the Lie algebra is its dimension of the vector space over $\mathbb{F}$.

Example 2.1.2. Let $(A, \cdot)$ be an associative algebra over a field $\mathbb{F}$. Then A can be turned into a Lie algebra with Lie bracket

$$[x,y] = x \cdot y - y \cdot x$$

for all $x, y \in A$.

This construction will be used a lot. Let V be an n–dimensional vector space over a field $\mathbb{F}$ and denote $\mathfrak{gl}(V)$ for the set of all linear maps from V to V. Then $\mathfrak{gl}(V)$ is a Lie algebra for which the Lie bracket is defined by

$$[f,g] = f \circ g - g \circ f$$

for all $f, g \in \mathfrak{gl}(V)$. Similarly, the vector space $M_n(\mathbb{F})$ of all $(n \times n)$–matrices becomes a Lie algebra when we define $[A,B] = AB - BA$ for all $A, B \in M_n(\mathbb{F})$. Here, AB denotes the matrix multiplication. When we want to stress the Lie algebra structure of $M_n(\mathbb{F})$, we will denote it with $\mathfrak{gl}_n(\mathbb{F})$.

Definition 2.1.3 (Lie subalgebra)**.** Let $\mathfrak{g}$ be a Lie algebra. A vector subspace $\mathfrak{h} \subseteq \mathfrak{g}$ is a Lie subalgebra if $[X,Y] \in \mathfrak{h}$ for all $X, Y \in \mathfrak{h}$.

Example 2.1.4. Let $\mathbb{F}$ be a field and $n \in \mathbb{N}_0$. We consider different subspaces of $\mathfrak{gl}_n(\mathbb{F})$.

(a) Let $\mathfrak{h}_3(\mathbb{F})$ be the set of all (3×3) strictly upper triangular matrices over $\mathbb{F}$. Since the product of upper triangular matrices is again upper triangular, $\mathfrak{h}_3(\mathbb{F})$ is a Lie subalgebra of $\mathfrak{gl}_3(\mathbb{F})$ which is called the 'Heisenberg Lie algebra'.

(b) Take $\mathfrak{sl}_n(\mathbb{F}) = \{A \in M_n(\mathbb{F}) \mid \mathrm{tr}(A) = 0\}$. For arbitrary $A, B \in \mathfrak{sl}_n(\mathbb{F})$, it follows that

$$\begin{aligned} \mathrm{tr}([A,B]) &= \mathrm{tr}(AB - BA) \\ &= \mathrm{tr}(AB) - \mathrm{tr}(BA) = 0 \end{aligned}$$

by properties of the trace operator. Hence, $\mathfrak{sl}_n(\mathbb{F})$ is a Lie subalgebra of $\mathfrak{gl}_n(\mathbb{F})$.

(c) Consider the set of skew-symmetric matrices

$$\mathfrak{so}_n(\mathbb{F}) = \{A = (a_{ij}) \in M_n(\mathbb{F}) \mid A + A^\top = 0 \text{ and } a_{ii} = 0 \text{ for } 1 \leq i \leq n\}.$$

Take arbitrary $A, B \in \mathfrak{so}_n(\mathbb{F})$. We have

$$\begin{aligned}[A, B]^\top &= (AB - BA)^\top \\ &= B^\top A^\top - A^\top B^\top \\ &= BA - AB = -[B, A].\end{aligned}$$

It can be shown that all diagonal elements of $[A, B]$ are zero as well, so $[A, B] \in \mathfrak{so}_n(\mathbb{F})$. This means that $\mathfrak{so}_n(\mathbb{F})$ is a Lie subalgebra of $\mathfrak{gl}_n(\mathbb{F})$.

The next construction is that of an ideal, a special type of a subalgebra.

Definition 2.1.5 (Ideal)**.** An ideal $\mathfrak{h}$ of a Lie algebra $\mathfrak{g}$ is a subspace of $\mathfrak{g}$ such that $[x, y] \in \mathfrak{h}$ for all $x \in \mathfrak{g}$ and all $y \in \mathfrak{h}$.

There is no distinction between right and left ideals, since the Lie bracket is skew-symmetric. An important example of an ideal is the center of a Lie algebra.

Definition 2.1.6 (Center and centraliser)**.** The center $Z(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ is defined as

$$Z(\mathfrak{g}) = \{x \in \mathfrak{g} \mid [x, y] = 0 \text{ for all } y \in \mathfrak{g}\}.$$

For $x \in \mathfrak{g}$, the centraliser of x is defined as $C_{\mathfrak{g}}(x) := \{y \in \mathfrak{g} \mid [x, y] = 0\}$.

A Lie algebra is a vector space, so we can look at the quotient vector space. Let $\mathfrak{g}$ be a Lie algebra with ideal I, then the cosets $x + I = \{x + y \mid y \in I\}$, with $x \in \mathfrak{g}$, form the quotient vector space

$$\mathfrak{g}/I = \{x + I \mid x \in \mathfrak{g}\}.$$

Definition 2.1.7 (Quotient Lie algebra)**.** Let $\mathfrak{g}$ be a Lie algebra with ideal I. Then $\mathfrak{g}/I$ is a Lie algebra, where for all $x, y \in \mathfrak{g}$, the Lie bracket is defined as

$$[x + I, y + I] := [x, y] + I.$$

Since a Lie algebra is a vector space, it makes sense to consider a basis for it. This is used in the following definition.

Definition 2.1.8 (Structure constants)**.** Let $\mathfrak{g}$ be an n–dimensional Lie algebra over $\mathbb{F}$ with basis $\mathcal{B} = \{e_1, e_2, \ldots, e_n\}$. Then there exist values $c_{ij}^k \in \mathbb{F}$ (with $1 \leq i, j, k \leq n$) such that

$$[e_i, e_j] = \sum_{k=1}^{n} c_{ij}^k e_k.$$

These values c_{ij}^k are the structure constants of $\mathfrak{g}$ with respect to the basis $\mathcal{B}$.

Let $\mathfrak{g}$ be an n–dimensional Lie algebra. By bilinearity, it suffices to specify the Lie brackets only for basis vectors. Moreover, we will only denote the Lie bracket $[e_i, e_j]$ when $1 \leq i < j \leq n$, since $[e_i, e_i] = 0$ and $[e_j, e_i] = -[e_i, e_j]$.

Example 2.1.9. We consider the Lie algebras from Example 2.1.4 where $n = 2$ or $n = 3$.

(a) Define the matrices

$$e_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad e_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad e_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

then $\{e_1, e_2, e_3\}$ is a basis for $\mathfrak{h}_3(\mathbb{F})$. It holds that

$$[e_1, e_2] = e_3 \quad \text{and} \quad [e_1, e_3] = [e_2, e_3] = 0.$$

(b) Consider the matrices

$$e_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad e_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad e_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then $\{e_1, e_2, e_3\}$ is a basis for $\mathfrak{sl}_2(\mathbb{F})$ and we have that

$$[e_1, e_2] = e_3, \qquad [e_1, e_3] = -2e_1 \quad \text{and} \quad [e_2, e_3] = 2e_2.$$

(c) Let $\mathbb{F}$ be a field with $\mathrm{char}(\mathbb{F}) \neq 2$, then $\{e_1, e_2, e_3\}$ is a basis for $\mathfrak{so}_3(\mathbb{F})$, where

$$e_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad e_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

A short computation shows that

$$[e_1, e_2] = e_3, \qquad [e_1, e_3] = -e_2 \quad \text{and} \quad [e_2, e_3] = e_1.$$

For $A \in \mathfrak{so}_3(\mathbb{F})$, the requirement that all diagonal elements of A are zero is redundant if $\mathrm{char}(\mathbb{F}) \neq 2$. However, it is necessary to impose this extra condition when $\mathrm{char}(\mathbb{F}) = 2$. Otherwise, also diagonal matrices in $M_3(\mathbb{F})$ would belong to $\mathfrak{so}_3(\mathbb{F})$.

Usually, only the non-vanishing brackets are mentioned. For instance, we will say that the 3–dimensional Heisenberg Lie algebra $\mathfrak{h}_3(\mathbb{F})$ is given by $[e_1, e_2] = e_3$. The Lie brackets between basis vectors which are not specified are assumed to be zero, so $[e_1, e_3] = [e_2, e_3] = 0$. In many cases, we will present a Lie algebra on the basis of the structure constants, since this is an efficient way for doing computations.

2.2 Maps between Lie algebras

In this section, we introduce the concept of almost inner derivations. Before we give the definition, we first present some important linear maps from one Lie algebra to another. We require that the two Lie algebras are defined over the same field $\mathbb{F}$.

Definition 2.2.1 (Homomorphism)**.** Let $\mathfrak{g}_1$ and $\mathfrak{g}_2$ be Lie algebras over the same field $\mathbb{F}$. A linear map $\varphi : \mathfrak{g}_1 \to \mathfrak{g}_2$ is a homomorphism if

$$\varphi([x, y]) = [\varphi(x), \varphi(y)]$$

holds for all $x, y \in \mathfrak{g}_1$.

The bracket on the left side is taken in $\mathfrak{g}_1$ and the bracket on the right side in $\mathfrak{g}_2$.

Example 2.2.2. Let $\mathfrak{g}$ be a Lie algebra over a field $\mathbb{F}$. For the 'adjoint homomorphism' $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$, an element $x \in \mathfrak{g}$ is mapped to

$$\mathrm{ad}(x) : \mathfrak{g} \to \mathfrak{g} : y \mapsto [x, y].$$

Since the Lie bracket is bilinear, $\mathrm{ad}(x)$ is linear and belongs to $\mathfrak{gl}(\mathfrak{g})$ for all $x \in \mathfrak{g}$. By the same reasoning, ad is a linear map as well. Let $x, y, z \in \mathfrak{g}$ be arbitrary. The Jacobi identity implies that

$$\begin{aligned}
\mathrm{ad}([x, y])(z) &= [[x, y], z] \\
&= [x, [y, z]] - [y, [x, z]] \\
&= (\mathrm{ad}(x) \circ \mathrm{ad}(y))(z) - (\mathrm{ad}(y) \circ \mathrm{ad}(x))(z) \\
&= [\mathrm{ad}(x), \mathrm{ad}(y)](z)
\end{aligned}$$

holds for all $x, y, z \in \mathfrak{g}$. This means that $\mathrm{ad}([x,y]) = [\mathrm{ad}(x), \mathrm{ad}(y)]$, where the first bracket is taken in $\mathfrak{g}$ and the second bracket in $\mathfrak{gl}(\mathfrak{g})$. Hence, the adjoint map is indeed a Lie algebra homomorphism.

Definition 2.2.3 (Isomorphism)**.** Let $\mathfrak{g}_1$ and $\mathfrak{g}_2$ be Lie algebras over the same field $\mathbb{F}$. A linear map $\varphi : \mathfrak{g}_1 \to \mathfrak{g}_2$ is an isomorphism if it is a bijective homomorphism. When there exists an isomorphism between $\mathfrak{g}_1$ and $\mathfrak{g}_2$, the Lie algebras are said to be isomorphic.

Example 2.2.4. Consider $\mathfrak{so}_3(\mathbb{C})$ with basis $\{e_1, e_2, e_3\}$ and given by the Lie brackets

$$[e_1, e_2] = e_3, \qquad [e_1, e_3] = -e_2 \quad \text{and} \quad [e_2, e_3] = e_1.$$

We define

$$x_1 := e_2 + ie_3, \qquad x_2 := -e_2 + ie_3 \quad \text{and} \quad x_3 := 2ie_1$$

and find that

$$\begin{aligned}
[x_1, x_2] &= [e_2 + ie_3, -e_2 + ie_3] = 2ie_1 = x_3, \\
[x_1, x_3] &= [e_2 + ie_3, 2ie_1] = -2e_2 - 2ie_3 = -2x_1, \\
[x_2, x_3] &= [-e_2 + ie_3, 2ie_1] = -2e_2 + 2ie_3 = 2x_2.
\end{aligned}$$

It follows from these computations that $\mathfrak{so}_3(\mathbb{C})$ and $\mathfrak{sl}_2(\mathbb{C})$ are isomorphic. However, it can be shown that $\mathfrak{so}_3(\mathbb{R})$ and $\mathfrak{sl}_2(\mathbb{R})$ are not isomorphic as real Lie algebras.

Two isomorphic Lie algebras are considered to be essentially ‘the same’, since both Lie algebras have the same structure constants after a change of basis. So an interesting problem is to determine how many and which essentially different isomorphism types there are for a given class of Lie algebras. An answer to this question consists of giving an inventory of the structure constants of several Lie algebras (possibly with some parameters) in such a way that all Lie algebras in the given class are isomorphic to at least one Lie algebra of the list. Moreover, one mostly requires that a Lie algebra is isomorphic to exactly one Lie algebra from the inventory. If that is the case, we will speak about a ‘classification’.

Classification up to isomorphism is a difficult problem in the study of finite-dimensional Lie algebras. As the last example illustrates, such a classification depends on the field. A popular and classical choice is to list the low-dimensional Lie algebras. In general, it is hard to obtain a full classification, even over an algebraically closed field of characteristic zero. For example, there are already infinitely many essentially different (so non-isomorphic) 3–dimensional Lie algebras over $\mathbb{C}$. Chapter 8 is devoted to the computation of almost inner derivations of low-dimensional Lie algebras.

Definition 2.2.5 (Derivation)**.** Let $\mathfrak{g}$ be a Lie algebra over a field $\mathbb{F}$. A linear map $D : \mathfrak{g} \to \mathfrak{g}$ is a derivation of $\mathfrak{g}$ if

$$D([x, y]) = [D(x), y] + [x, D(y)]$$

holds for all $x, y \in \mathfrak{g}$.

This identity is called the 'Leibniz' rule'. The set of all derivations of an algebra $\mathfrak{g}$ forms a vector space, which is denoted by $\mathrm{Der}(\mathfrak{g})$. Moreover, it is a subspace of $\mathfrak{gl}(\mathfrak{g})$. We will show that it is a Lie subalgebra as well. Take arbitrary $D, E \in \mathrm{Der}(\mathfrak{g})$ with $x, y \in \mathfrak{g}$. Since derivations are bilinear maps, we obtain that

$$\begin{aligned}
[D, E]([x, y]) &= D(E([x, y])) - E(D([x, y])) \\
&= D([E(x), y] + [x, E(y)]) - E([D(x), y] + [x, D(y)]) \\
&= [D(E(x)), y] + [E(x), D(y)] + [D(x), E(y)] + [x, D(E(y))] \\
&\qquad - [E(D(x)), y] - [D(x), E(y)] - [E(x), D(y)] - [x, E(D(y))] \\
&= [D(E(x)), y] + [x, D(E(y))] - [E(D(x)), y] - [x, E(D(y))] \\
&= [[D, E](x), y] + [x, [D, E](y)].
\end{aligned}$$

This means that $[D, E]$ is a derivation too and $\mathrm{Der}(\mathfrak{g})$ is a Lie subalgebra of $\mathfrak{gl}(\mathfrak{g})$. Choose an arbitrary $x \in \mathfrak{g}$. By the Jacobi identity, we find that

$$\begin{aligned}
\mathrm{ad}(x)([y, z]) &= [x, [y, z]] \\
&= [[x, y], z] + [y, [x, z]] \\
&= [\mathrm{ad}(x)(y), z] + [y, \mathrm{ad}(x)(z)]
\end{aligned}$$

holds for all $y, z \in \mathfrak{g}$. Hence, for every $x \in \mathfrak{g}$, the adjoint homomorphism $\mathrm{ad}(x)$ is a derivation.

Definition 2.2.6 (Inner derivation)**.** Let $\mathfrak{g}$ be a Lie algebra and $x \in \mathfrak{g}$. The map

$$\mathrm{ad}(x) : \mathfrak{g} \to \mathfrak{g} : y \mapsto [x, y]$$

is called an inner derivation of $\mathfrak{g}$.

We denote $\mathrm{Inn}(\mathfrak{g})$ for the set of all inner derivations of $\mathfrak{g}$. By bilinearity, $\mathrm{Inn}(\mathfrak{g})$ can be generated by the maps $\mathrm{ad}(e_i) : \mathfrak{g} \to \mathfrak{g}$, where e_i is a basis vector and $1 \leq i \leq n$. For all $x \in \mathfrak{g}$, the map $\mathrm{ad}(x)$ is the zero-map if and only if $x \in Z(\mathfrak{g})$, which explains the isomorphism

$$\mathrm{Inn}(\mathfrak{g}) \cong \frac{\mathfrak{g}}{Z(\mathfrak{g})}.$$

Hence, $\dim(\mathrm{Inn}(\mathfrak{g}))$ is easy to compute. The following definition is the main topic of this thesis. The motivation to study this notion, which was introduced in [41], is explained in Section 3.1.

Definition 2.2.7 (Almost inner derivation)**.** Let $\mathfrak{g}$ be a Lie algebra. A derivation D is almost inner if $D(x) \in [x, \mathfrak{g}]$ for all $x \in \mathfrak{g}$. The space of all almost inner derivations of $\mathfrak{g}$ is denoted by $\mathrm{AID}(\mathfrak{g})$.

Hence, there exists a map $\varphi_D : \mathfrak{g} \to \mathfrak{g}$ such that $D(x) = [x, \varphi_D(x)]$ for all $x \in \mathfrak{g}$. We will call φ_D a 'determination map' for D. This map is not unique as we may change $\varphi_D(x)$ to $\varphi_D(x) + y$ for any $y \in C_{\mathfrak{g}}(x)$. In most cases, φ_D is not a linear map.

An almost inner derivation is inner when the map φ_D is constant. Moreover, a derivation is almost inner if and only if it coincides on each one-dimensional subspace with an inner derivation. We will say that a linear map D 'satisfies the almost inner condition' when $D(x) \in [x, \mathfrak{g}]$ for all $x \in \mathfrak{g}$, so a derivation is almost inner when it satisfies the almost inner condition. We introduce a new subspace of $\mathrm{AID}(\mathfrak{g})$ as follows.

Definition 2.2.8 (Central almost inner derivation)**.** An almost inner derivation $D \in \mathrm{AID}(\mathfrak{g})$ is called central almost inner if there exists an $x \in \mathfrak{g}$ such that $D - \mathrm{ad}(x)$ maps $\mathfrak{g}$ to the center $Z(\mathfrak{g})$. We denote the space of central almost inner derivations of $\mathfrak{g}$ by $\mathrm{CAID}(\mathfrak{g})$.

The next section is devoted to the introduction of some classes of Lie algebras. Those concepts have an important role in the last part of this thesis, where the almost inner derivations of special classes of Lie algebras will be computed.

2.3 Solvable and nilpotent Lie algebras

As we will explain in the next chapter, the geometric motivation for almost inner derivations is in particular important for nilpotent Lie algebras. This and other related notions are introduced in this section.

Definition 2.3.1 (Product of ideals)**.** Let $\mathfrak{g}$ be a Lie algebra over a field $\mathbb{F}$ with ideals I and J. The product $[I, J]$ of I and J is given by

$$[I, J] = \langle [x, y] \in \mathfrak{g} \mid x \in I \text{ and } y \in J\rangle.$$

This is the smallest ideal containing $[x, y]$ for all $x \in I$ and $y \in J$.

Definition 2.3.2 (Derived series)**.** Let $\mathfrak{g}$ be a Lie algebra over a field $\mathbb{F}$. The derived series of $\mathfrak{g}$ is the series with terms

$$\mathfrak{g}^{(1)} = [\mathfrak{g}, \mathfrak{g}] \qquad \text{and} \qquad \mathfrak{g}^{(k)} = [\mathfrak{g}^{(k-1)}, \mathfrak{g}^{(k-1)}] \quad \text{for } k \geq 2.$$

Here, $\mathfrak{g}^{(1)} = [\mathfrak{g}, \mathfrak{g}]$ is called the 'derived algebra' of $\mathfrak{g}$. This gives rise to a descending series $\mathfrak{g}^{(1)} \supseteq \mathfrak{g}^{(2)} \supseteq \dots$ When the derived series of $\mathfrak{g}$ terminates in the zero subalgebra, $\mathfrak{g}$ is called solvable.

Definition 2.3.3 (Solvable Lie algebra)**.** A non-zero Lie algebra $\mathfrak{g}$ over a field $\mathbb{F}$ is solvable when $\mathfrak{g}^{(k)} = 0$ for some $k \geq 1$.

A Lie algebra for which $[\mathfrak{g}, \mathfrak{g}] = 0$ is called 'abelian'. A non-abelian Lie algebra $\mathfrak{g}$ is 'metabelian' when $\mathfrak{g}^{(2)} = 0$.

Definition 2.3.4 (Solvable radical)**.** Let $\mathfrak{g}$ be a Lie algebra. The (solvable) radical $\mathrm{Rad}(\mathfrak{g})$ of $\mathfrak{g}$ is the maximal solvable ideal of $\mathfrak{g}$.

A notion related to solvability is nilpotency. First, the lower central series is introduced.

Definition 2.3.5 (Lower central series)**.** Let $\mathfrak{g}$ be a Lie algebra over a field $\mathbb{F}$. The lower central series of $\mathfrak{g}$ is the series with terms

$$\gamma_1(\mathfrak{g}) = \mathfrak{g} \quad \text{and} \quad \gamma_k(\mathfrak{g}) = [\mathfrak{g}, \gamma_{k-1}(\mathfrak{g})] \quad \text{for } k \geq 1.$$

We have a descending series $\gamma_1(\mathfrak{g}) \supseteq \gamma_2(\mathfrak{g}) \supseteq \dots$ A Lie algebra is called 'nilpotent' when there exists a natural number $k \in \mathbb{N}_0$ such that every Lie bracket with more than k elements vanishes.

Definition 2.3.6 (Nilpotent Lie algebras)**.** A non-zero Lie algebra $\mathfrak{g}$ over a field $\mathbb{F}$ is nilpotent when $\gamma_k(\mathfrak{g}) = 0$ for some $k \in \mathbb{N}_0$. The smallest possible $c \geq 2$ such that $\gamma_{c+1}(\mathfrak{g}) = 0$ is the nilindex or the nilpotency class of $\mathfrak{g}$. Then $\mathfrak{g}$ is said to be c–step nilpotent.

We will not use the word 'nilindex' for abelian Lie algebras.

Definition 2.3.7 (Type)**.** The type of a nilpotent Lie algebra $\mathfrak{g}$ is a tuple $(p_1, \ldots, p_c)$, where $p_k := \dim(\gamma_k(\mathfrak{g})/\gamma_{k+1}(\mathfrak{g}))$ for all $1 \leq k \leq c$.

Note that c is the nilindex of $\mathfrak{g}$ and that $p_1 + \cdots + p_c = n$. It is easy to see that $\mathfrak{g}^{(k)} \subseteq \gamma_k(\mathfrak{g})$ for all $k \in \mathbb{N}_0$. Hence, any nilpotent Lie algebra is also solvable. The converse is not true: the two-dimensional non-abelian Lie algebra (with basis $\{e_1, e_2\}$ and given by $[e_1, e_2] = e_1$) is solvable, but not nilpotent. Note that the derived algebra $[\mathfrak{g}, \mathfrak{g}]$ of a solvable Lie algebra $\mathfrak{g}$ is nilpotent. Lie algebras of dimension n with nilindex $n-1$ have a special name.

Definition 2.3.8 (Filiform Lie algebra)**.** A Lie algebra $\mathfrak{g}$ of dimension n is called filiform if $\mathfrak{g}$ is nilpotent with nilindex $n-1$.

The name 'filiform Lie algebra' was first introduced by Vergne ([79]). For a nilpotent Lie algebra $\mathfrak{g}$, it is not possible that $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$ or that $[\mathfrak{g}, \mathfrak{g}]$ is of codimension 1 in $\mathfrak{g}$. When $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$, then $\gamma_k(\mathfrak{g}) = \mathfrak{g}$ for all $k \in \mathbb{N}_0$. Suppose that $\dim(\mathfrak{g}) = n$ and $\dim([\mathfrak{g}, \mathfrak{g}]) = n-1$, say $[\mathfrak{g}, \mathfrak{g}] = \langle e_1, \ldots, e_{n-1}\rangle$. We find that

$$\gamma_3(\mathfrak{g}) = [\mathfrak{g}, \langle e_1, \ldots, e_{n-1}\rangle] = [\mathfrak{g}, \mathfrak{g}] = \gamma_2(\mathfrak{g})$$

holds, which is impossible for nilpotent Lie algebras. This implies that for every nilpotent n–dimensional Lie algebra $\mathfrak{g}$, the inequality $\dim([\mathfrak{g}, \mathfrak{g}]) \leq n-2$ is true. Moreover, a similar reasoning by induction shows that

$$\dim(\gamma_k(\mathfrak{g})) \leq n-k$$

holds for all $2 \leq k \leq n$. It follows that for a filiform Lie algebra $\mathfrak{g}$, we have $\dim(\mathfrak{g}/\gamma_2(\mathfrak{g})) = 2$ and

$$\dim\left(\frac{\gamma_k(\mathfrak{g})}{\gamma_{k+1}(\mathfrak{g})}\right) = 1$$

when $2 \leq k \leq n$. This means that the type of $\mathfrak{g}$ is $(2, 1, \ldots, 1)$. Hence, filiform Lie algebras are the nilpotent Lie algebras with maximal possible nilindex. The lower central series still terminates, but there are as many non-zero terms as possible. This explains the name 'filiform', which means threadlike. Filiform Lie algebras will be studied in detail in Chapter 10.

Example 2.3.9. Let $\mathbb{F}$ be an arbitrary field. For the Heisenberg algebra $\mathfrak{g} := \mathfrak{h}_3(\mathbb{F})$, we have that $\mathfrak{g}^{(1)} = \gamma_1(\mathfrak{g}) = \langle e_3\rangle$ and $\mathfrak{g}^{(k)} = \gamma_k(\mathfrak{g}) = \{0\}$ for all $k \geq 2$. Hence, $\mathfrak{g}$ is metabelian with itself as solvable radical. Further, it is filiform with type $(2, 1)$.
For $\mathfrak{g} := \mathfrak{so}_3(\mathbb{F})$, we find that $\mathfrak{g}^{(k)} = \gamma_k(\mathfrak{g}) = \mathfrak{g}$ for all $k \in \mathbb{N}_0$. This means that $\mathfrak{g}$ is not solvable (and not nilpotent). Moreover, $\mathfrak{g}$ does not have proper ideals, so $\mathrm{Rad}(\mathfrak{g}) = \{0\}$.

2.4 Simple and semisimple Lie algebras

In this section, we will introduce simple and semisimple Lie algebras. First, the notion of a direct sum of Lie algebras is explained.

Definition 2.4.1 (Direct sum of Lie algebras)**.** Let $\mathfrak{g}_1$ and $\mathfrak{g}_2$ be two Lie algebras over the same field $\mathbb{F}$. The direct sum $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ of the Lie algebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$ is the vector space direct sum consisting of pairs, where

$$[(x_1, x_2), (y_1, y_2)] = ([x_1, y_1], [x_2, y_2]),$$

where $x_1, y_1 \in \mathfrak{g}_1$ and $x_2, y_2 \in \mathfrak{g}_2$ and such that $[\mathfrak{g}_1, \mathfrak{g}_2] = 0$.

It is clear that both $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are ideals of $\mathfrak{g}$ in this case. This construction is sometimes referred to as the 'external direct sum' of $\mathfrak{g}_1$ and $\mathfrak{g}_2$, since they don't have to be subalgebras of some given Lie algebra.

Definition 2.4.2 (Semidirect product of Lie algebras)**.** Let $\mathfrak{g}_1$ and $\mathfrak{g}_2$ be two Lie algebras over the same field $\mathbb{F}$. Let $\varphi : \mathfrak{g}_2 \to \mathrm{Der}(\mathfrak{g}_1)$ be a Lie algebra homomorphism. The semidirect product $\mathfrak{g}_1 \rtimes_\varphi \mathfrak{g}_2$ of $\mathfrak{g}_1$ and $\mathfrak{g}_2$ is the vector space $\mathfrak{g}_1 \oplus \mathfrak{g}_2$, where the Lie bracket is given by

$$[(x_1, x_2), (y_1, y_2)] = ([x_1, y_1] + \varphi(x_2)(y_1) - \varphi(y_2)(x_1), [x_2, y_2]),$$

where $x_1, y_1 \in \mathfrak{g}_1$ and $x_2, y_2 \in \mathfrak{g}_2$.

Here, $\mathfrak{g}_1$ is an ideal and $\mathfrak{g}_2$ is a subalgebra of $\mathfrak{g}_1 \rtimes_\varphi \mathfrak{g}_2$. When the map φ is clear or not important, it will be omitted.

Definition 2.4.3 (Simple and semisimple Lie algebra)**.** Let $\mathfrak{g}$ be a Lie algebra over a field $\mathbb{F}$. Then $\mathfrak{g}$ is simple if $\mathfrak{g}$ is non-abelian and has no non-zero proper ideals. Further, $\mathfrak{g}$ is called semisimple if it has no non-zero solvable ideals.

Let $\mathfrak{g}$ be a Lie algebra over an arbitrary field $\mathbb{F}$. If $\mathfrak{g}$ is simple, then it is semisimple as well. When $\mathrm{char}(\mathbb{F}) = 0$, then $\mathfrak{g}$ is semisimple if and only if $\mathfrak{g}$ is a direct sum of simple Lie algebras. Note that when a Lie algebra $\mathfrak{g}$ is solvable, it cannot be semisimple, since $\mathfrak{g}$ itself is an ideal of $\mathfrak{g}$.

Example 2.4.4. Let $\mathbb{F}$ be an arbitrary field. Then $\mathfrak{so}_3(\mathbb{F})$ is simple. Further, $\mathfrak{sl}_2(\mathbb{F})$ is simple if and only if $\mathrm{char}(\mathbb{F}) \neq 2$. We find that $\mathfrak{sl}_2(\mathbb{F})$ is isomorphic to $\mathfrak{h}_3(\mathbb{F})$ when $\mathrm{char}(\mathbb{F}) = 2$.

As was conjectured by Killing and Cartan and proven by Levi, a Lie algebra over a field of characteristic zero can be decomposed as a semidirect product of a solvable and semisimple Lie algebra. It is also referred to as the Levi-Mal'cev theorem.

Theorem 2.4.5 (Levi, 1905)**.** *Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over a field $\mathbb{F}$ of characteristic zero. Then $\mathfrak{g} = \mathfrak{r} \rtimes \mathfrak{s}$ is a semidirect product of a semisimple Lie algebra $\mathfrak{s}$ and the solvable radical $\mathfrak{r}$ of $\mathfrak{g}$.*

This is known as the 'Levi decomposition' of a Lie algebra, where the semisimple subalgebra is called the 'Levi subalgebra'.

Chapter 3

Motivation and related results

In the previous chapter, we introduced almost inner derivations of Lie algebras, the main topic of this thesis. The motivation to study this notion finds its origin in spectral theory. The notions of 'class preserving automorphisms of a Lie group' and 'almost inner derivations of a Lie algebra' were defined to build isospectral and non-isometric nilmanifolds. In the first section, we explain this construction and present some definitions concerning Lie groups and the correspondence with Lie algebras. Section 3.2 deals with class preserving automorphisms for finite p–groups. Although there is no direct link with Lie algebras, it turns out that many of the existing results for groups are similar to the properties for almost inner derivations of Lie algebras.

3.1 Spectral theory

A classical question in spectral geometry was whether or not isospectral manifolds are isometric. It turns out that this does not have to be the case and several counterexamples have been given. Gordon and Wilson wanted to have continuous families of isospectral non-isometric manifolds instead of only finitely many. They constructed deformations of nilmanifolds, based on class preserving automorphisms of Lie groups ([41]). To build specific examples, they made use of almost inner derivations of Lie algebras. In this section, we consider this question about isospectral and non-isometric manifolds in more detail and discuss the counterexamples of Gordon and Wilson. Further, we present basic notions about Lie groups and define Lie algebras as the tangent space at the neutral element of a Lie group.

3.1.1 Isospectral manifolds

In spectral geometry, relations between a certain kind of manifolds and spectra of differential operators are studied. One of the fundamental problems is to establish to what extent the eigenvalues of some operator determine the geometry of a given manifold.

Definition 3.1.1 (Riemannian manifold)**.** Let M be a real smooth manifold. A Riemannian metric g on M assigns for every $p \in M$ a positive-definite inner product $g_p : T_pM \times T_pM \to \mathbb{R}$. Together with this metric g, a real smooth manifold M is called a Riemannian manifold (M, g).

The main example of a differential operator is the 'Laplace-Beltrami operator', which is defined as the divergence of the gradient and it is denoted with Δ or with ∇^2.

Definition 3.1.2 (Spectrum)**.** Let (M, g) be a closed Riemannian manifold where the associated Laplace-Beltrami operator Δ acts on smooth p–forms. The spectrum $\operatorname{spec}_p(M, g)$ of the manifold (M, g) is the collection of eigenvalues of Δ, counted with multiplicities.

When two Riemannian manifolds have the same spectrum, they are called 'isospectral'.

Definition 3.1.3 (Isospectral manifolds)**.** Two closed Riemannian manifolds (M, g) and (M', g') are isospectral when $\operatorname{spec}_p(M, g) = \operatorname{spec}_p(M', g')$ for all $0 \leq p \leq \dim(M)$.

Two Riemannian manifolds are called isometric when there exists an isometry from one to another.

Definition 3.1.4 (Isometry)**.** Let $f : M \to M'$ be a diffeomorphism between two Riemannian manifolds (M, g) and (M', g'). Then, f is an isometry if

$$g_p(X, Y) = g'_{f(p)}(f_p(X), f_p(Y))$$

holds for all $p \in M$ and for all $X, Y \in T_pM$.

One of the central research domains in spectral geometry was whether or not isospectral manifolds are necessarily isometric. This question going back to Hermann Weyl can be interpreted as follows: 'Is it possible to determine the whole geometry of the manifold by only looking at its eigenvalues?' It turns out that the answer is negative. In 1964, a counterexample was given by Milnor ([65]), who constructed two isospectral and non-isometric flat tori of dimension 16. A few years later, Mark Kac gave this problem the popular title ‘Can one hear the shape of a drum?’ ([52]). In other words, is it possible to determine the shape of the drum membrane when it is known which sounds it produces? This question remained unsolved until 1992, when Gordon, Webb and Wolpert constructed ‘drums’ with different shapes, but with identical eigenspectra ([39, 40]), see Figure 3.1. To obtain these examples, they used the Sunada method ([77]). Sunada gave a general technique for constructing pairs of isospectral manifolds with a common finite Riemannian covering. Using a similar approach, Gordon and Wilson were in 1984 the first to construct continuous families of isospectral manifolds which are non-isometric. In their paper ([41]), they used class preserving automorphisms of Lie groups. In the following subsections, we will present a concise overview of the different notions and concepts used in their result.

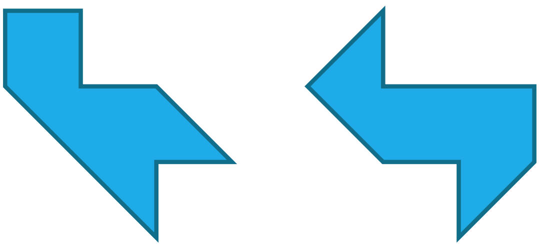

Figure 3.1: Example of different ‘drums’ which sound the same

3.1.2 Lie groups and the correspondence with Lie algebras

In this subsection, we briefly discuss the theory of Lie groups and the correspondence with Lie algebras.

Definition 3.1.5 (Lie group)**.** A (real) Lie group $(G, *)$ is a differentiable manifold with a group structure such that

$$G \times G \to G : (g, h) \mapsto g * h^{-1}$$

is a differentiable map.

We will focus on real Lie groups, although complex Lie groups also exist.

Example 3.1.6. We give some examples of real Lie groups. Take $n \in \mathbb{N}_0$.

(a) The group $(\mathbb{R}^n, +)$ with the natural manifold structure is a Lie group.

(b) The groups $(\mathbb{R}_0, \cdot)$ and $(\mathbb{R}^+, \cdot)$ are Lie groups.

(c) The group $(\mathrm{GL}_n(\mathbb{R}), \cdot)$ is a Lie group when we consider the manifold structure from $\mathbb{R}^{n^2}$.

A subgroup H of a Lie group G is a 'Lie subgroup' if H has a manifold structure such that it is a Lie group itself and such that the inclusion $H \to G$ is an immersion.

Example 3.1.7. We consider some Lie subgroups of $\mathrm{GL}_n(\mathbb{R})$, where $n \in \mathbb{N}_0$.

(a) The 'Heisenberg group' $H_3(\mathbb{R})$ is defined as

$$H_3(\mathbb{R}) = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \mid x, y, z \in \mathbb{R} \right\}.$$

(b) The 'special linear group' $\mathrm{SL}_n(\mathbb{R})$ is defined as

$$\mathrm{SL}_n(\mathbb{R}) = \{A \in \mathrm{GL}_n(\mathbb{R}) \mid \det(A) = 1\}.$$

(c) The 'special orthogonal group' $\mathrm{SO}_n(\mathbb{R})$ is defined as

$$\mathrm{SO}_n(\mathbb{R}) = \{A \in \mathrm{SL}_n(\mathbb{R}) \mid AA^\top = A^\top A = \mathbb{I}_n\}.$$

Definition 3.1.8 (Lie group morphism)**.** For Lie groups G and H, a Lie group morphism from G to H is a continuous map $f : G \to H$ which is also a morphism. If f is bijective and f^{-1} is a Lie group morphism, then f is a Lie group isomorphism.

A Lie group automorphism is a Lie group isomorphism from a group to itself. The set of all automorphisms of a given group is denoted with $\mathrm{Aut}(G)$ and forms a group.

Definition 3.1.9 (Inner automorphism)**.** Let G be a Lie group and $g \in G$, then the inner automorphism for g is given by

$$I_g : G \to G : h \mapsto ghg^{-1}.$$

It is clear that this is a Lie group automorphism. The notation $\mathrm{Inn}(G)$ stands for the set of all inner automorphisms of the group G. Moreover, $\mathrm{Inn}(G)$ is a normal subgroup of $\mathrm{Aut}(G)$. Consider now the group homomorphism

$$G \to \mathrm{Inn}(G) : g \mapsto (I_g : G \to G).$$

Since the kernel of this morphism is equal to the center $Z(G)$ of G, it follows from the isomorphism theorems that $G/Z(G) \cong \mathrm{Inn}(G)$.

There is an interesting correspondence between Lie groups and Lie algebras. Let G be a Lie group with neutral element $1_G \in G$ and define $\mathfrak{g} := T_{1_G}G$ as the tangent space of 1_G. Since $I_g(1_G) = 1_G$, we have that I_g induces an automorphism $(I_g)_* : \mathfrak{g} \to \mathfrak{g}$. Define

$$\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g}) : g \mapsto (I_g)_*.$$

This map is smooth and induces a linear map $\mathrm{Ad}_* : \mathfrak{g} \to T_{\mathrm{Id}}\mathrm{GL}(\mathfrak{g})$ between the tangent spaces. Define $\mathfrak{gl}(\mathfrak{g})$ as the vector space of all linear maps from $\mathfrak{g}$ to $\mathfrak{g}$. Since $\mathrm{GL}(\mathfrak{g})$ is open in $\mathfrak{gl}(\mathfrak{g})$, we can identify $T_{\mathrm{Id}}\mathrm{GL}(\mathfrak{g})$ with $T_{\mathrm{Id}}\mathfrak{gl}(\mathfrak{g})$ and define

$$\mathrm{ad} = \mathrm{Ad}_* : \mathfrak{g} \to T_{\mathrm{Id}}\mathfrak{gl}(\mathfrak{g}).$$

Proposition 3.1.10. *Let G be a Lie group with neutral element 1_G and take $\mathfrak{g} := T_{1_G}G$. Define a Lie bracket*

$$[.,.] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g} : (x, y) \mapsto [x, y] := \mathrm{ad}(x)(y)$$

on $\mathfrak{g}$, then $\mathfrak{g}$ with this Lie bracket defines a Lie algebra.

The Lie algebra $\mathfrak{g}$ in the construction of the previous result is called 'the Lie algebra associated to G'.

Example 3.1.11. We consider the Lie groups from Example 3.1.6 and Example 3.1.7. Take $n \in \mathbb{N}_0$.

- The Lie algebra associated to $(\mathbb{R}^n, +)$ and $\mathrm{GL}_n(\mathbb{R})$ is $\mathbb{R}^n$ respectively $\mathfrak{gl}_n(\mathbb{R})$. Moreover, $(\mathbb{R}_0, \cdot)$ and $(\mathbb{R}^+, \cdot)$ both have $\mathbb{R}$ as associated Lie algebra.
- Further, $H_3(\mathbb{R})$, $\mathrm{SL}_n(\mathbb{R})$ and $\mathrm{SO}_n(\mathbb{R})$ have associated Lie algebras $\mathfrak{h}_3(\mathbb{R})$, $\mathfrak{sl}_n(\mathbb{R})$ respectively $\mathfrak{so}_n(\mathbb{R})$, where we use the notations from Example 2.1.4.

For a Lie group morphism $f : G \to H$, we must have that $f(1_G) = 1_H$. Hence, there is an induced map $f_* : T_{1_G}G \to T_{1_H}H$ between the associated Lie algebras $T_{1_G}G$ and $T_{1_H}H$.

Definition 3.1.12 (Induced Lie algebra morphism)**.** Let G and H be Lie groups with associated Lie algebras $\mathfrak{g}$ respectively $\mathfrak{h}$. A Lie group morphism $f : G \to H$ induces a Lie algebra morphism $f_* : \mathfrak{g} \to \mathfrak{h}$ which is called the Lie algebra morphism induced by f.

Let G be a Lie group with associated Lie algebra $\mathfrak{g}$. For every $X \in \mathfrak{g}$, there is a unique Lie group morphism $f_X : \mathbb{R} \to G$ with $(f_X)_*(1) = X$. This motivates the following definition.

Definition 3.1.13 (Exponential map)**.** Let G be a Lie group with associated Lie algebra $\mathfrak{g}$. The exponential map exp from $\mathfrak{g}$ to G is defined as

$$\exp : \mathfrak{g} \to G : X \mapsto f_X(1).$$

Example 3.1.14. We provide some examples.

(a) The exponential map from the Lie algebra $\mathbb{R}^n$ to the Lie group $(\mathbb{R}^n, +)$ is the identity map.

(b) The exponential map from $\mathfrak{gl}_n(\mathbb{R})$ to $(\mathrm{GL}_n(\mathbb{R}), \cdot)$ is given by

$$\exp : \mathfrak{gl}_n(\mathbb{R}) \to \mathrm{GL}_n(\mathbb{R}) : A \mapsto \sum_{k=0}^{\infty} \frac{A^k}{k!}.$$

When $\mathfrak{g}$ is a nilpotent subalgebra of $\mathfrak{gl}_n(\mathbb{R})$, this expression can be simplified. For instance, the exponential map from $\mathfrak{h}_3(\mathbb{R})$ to $(H_3(\mathbb{R}), \cdot)$ is given by

$$\exp : \mathfrak{h}_3(\mathbb{R}) \to H_3(\mathbb{R}) : A \mapsto \mathbb{I}_n + A + \frac{A^2}{2},$$

since $A^k = 0$ for all $k \geq 3$ and all $A \in \mathfrak{h}_3(\mathbb{R})$.

Theorem 3.1.15. *Let G be a connected and simply connected nilpotent Lie group. Consider the associated Lie algebra $\mathfrak{g}$ with the natural manifold structure. Then the exponential map* $\exp : \mathfrak{g} \to G$ *is a diffeomorphism.*

Hence, the inverse of the exponential map is well-defined when G is a connected and simply connected nilpotent group.

Definition 3.1.16 (Logarithmic map)**.** Let G be a connected and simply connected nilpotent Lie group with associated Lie algebra $\mathfrak{g}$. We define the logarithmic map $\log : G \to \mathfrak{g}$ as the inverse of the exponential map.

Example 3.1.17. We again give some examples.

(a) The logarithmic map from the Lie group $\mathbb{R}^n$ to the Lie algebra $\mathbb{R}^n$ is the identity map.

(b) The logarithmic map from $H_3(\mathbb{R})$ to $\mathfrak{h}_3(\mathbb{R})$ is given by

$$\log : H_3(\mathbb{R}) \to \mathfrak{h}_3(\mathbb{R}) : A \mapsto (A - \mathbb{I}_n) - \frac{(A - \mathbb{I}_n)^2}{2}.$$

Let G be a connected and simply connected Lie group with associated Lie algebra $\mathfrak{g}$. One can show that $\mathrm{Inn}(G)$ and $\mathrm{Aut}(G)$ have $\mathrm{Inn}(\mathfrak{g})$ respectively $\mathrm{Der}(\mathfrak{g})$ as Lie algebra. In particular, $\exp : \mathrm{Der}(\mathfrak{g}) \to \mathrm{Aut}(\mathfrak{g})$ is the matrix exponential. For the rest of this section, a Lie group is assumed to be connected and simply connected.

3.1.3 Isospectral and non-isometric nilmanifolds

In this subsection, we will consider the continuous deformations of nilmanifolds. First, some other notions have to be explained.

Definition 3.1.18 (Left group action)**.** Let $(G, *)$ be a group and X a set. A left group action φ of G on X is a function

$$\varphi : G \times X \to X : (g, x) \mapsto \varphi(g, x) = g \cdot x$$

such that for all $x \in X$ and for all $g, h \in G$, the equations $1_G \cdot x = x$ and $(g * h) \cdot x = g \cdot (h \cdot x)$ are satisfied.

Analogously, a right group action of G on X can be defined. We will focus on left group actions, since every right action can be modified to a left action. Further, when we assume that the set X is a Lie group and that G has a topology, the action φ is required to be continuous. For instance, a Lie group $(G, *)$ acts on itself via the 'left translation' ℓ, which is defined as $\ell_g(x) := \ell(g, x) = g * x$ for all $g, x \in G$.

Definition 3.1.19 (Orbit)**.** Consider a left group action of G on X. The orbit of an element $x \in X$ is given by $G \cdot x = \{g \cdot x \mid g \in G\}$.

The 'quotient of a left action' (or the 'orbit space') $G \backslash X$ is the set of all orbits of X under G. This forms a partition of X, where two elements $x, y \in X$ are equivalent if and only if their orbits coincide. More formally, we have that $x \sim y$ if and only if there exists $g \in G$ with $g \cdot x = y$. Another notion is that of a discrete subgroup of a given group.

Definition 3.1.20 (Discrete subgroup)**.** A subgroup H of a topological group G is called discrete if the subspace topology of H in G is the discrete topology.

This means that there exists an open cover of H such that every open subset contains exactly one element of H. For instance, for $\mathbb{R}$ with the standard topology, $\mathbb{Z}$ is a discrete subgroup, but $\mathbb{Q}$ is not.

Definition 3.1.21 (Cocompact group action)**.** A left action of a group G on a topological space X is cocompact (also called uniform) if the quotient space $G \backslash X$ is a compact space.

Let G be a simply connected nilpotent Lie group and N be a discrete subgroup. If the subgroup N acts uniformly (via left multiplication) on G, then the quotient manifold $N\backslash G$ is a compact nilmanifold.

Definition 3.1.22 (Nilmanifold)**.** Let G be a connected and simply connected nilpotent Lie group. Let N be a discrete, cocompact subgroup of G. The compact quotient space $N\backslash G$ is called a nilmanifold.

A discrete, cocompact subgroup N of G is also called a 'lattice' of G. The notion of a nilmanifold was first used by Anatoly Mal'cev in 1951 ([62]).

Example 3.1.23. Take $k \in \mathbb{N}_0$ and consider the abelian Lie group $(\mathbb{R}^k, +)$ with the discrete cocompact subgroup $(\mathbb{Z}^k, +)$. The resulting nilmanifold is the k–dimensional torus

$$T^k = \frac{\mathbb{R}^k}{\mathbb{Z}^k}.$$

For $k = 1$, this is the circle and for $k = 2$, this is the usual 2–dimensional torus. We have to put a metric on the nilmanifolds to be able to decide when two nilmanifolds are isospectral. Therefore, we want to take the group structure into account.

Definition 3.1.24 (Left invariant metric)**.** A metric g on a Lie group G is left invariant if

$$g_p(u, v) = g_{qp}((\ell_q)_* u, (\ell_q)_* v)$$

for all $p, q \in G$ and all $u, v \in T_pG$.

Let g be a left invariant metric on a Lie group G and N a lattice of G. Then g induces in a unique way a metric on $N\backslash G$ (which we also denote with g) such that the natural projection $(G, g) \to (N\backslash G, g)$ is a Riemannian covering. In this way, we obtain a Riemannian nilmanifold $(N\backslash G, g)$. For the construction of isospectral and non-isometric nilmanifolds, Gordon and Wilson introduced the following notion ([41]).

Definition 3.1.25 (Class preserving automorphism)**.** Let G be a Lie group. An automorphism φ of G is class preserving if and only if for all $x \in G$, there exists $y \in G$ such that $\varphi(x) = yxy^{-1}$.

This notion was introduced differently in [41], but it is in fact equivalent to this one (see [36]). The set of all class preserving automorphisms of G is denoted with $\mathrm{Aut}_c(G)$. Hence, a class preserving automorphism is a generalisation of an inner one, where $y \in G$ can depend on $x \in G$. In [41], it is proven that $\mathrm{Aut}_c(G)$

is a Lie subgroup of $\mathrm{Aut}(G)$. Take $\varphi \in \mathrm{Aut}(G)$ and let g be a left invariant metric on G. Define a metric $\varphi^* g$ by

$$(\varphi^* g)(X, Y) := g(\varphi_* X, \varphi_* Y)$$

for all $X, Y \in \mathfrak{g}$, then $\varphi^* g$ is again left invariant.

Theorem 3.1.26 ([41])**.** *Let $(N\backslash G, g)$ be a compact Riemannian nilmanifold with $\varphi \in \mathrm{Aut}_c(G)$. Then $(N\backslash G, g)$ is isospectral to $(N\backslash G, \varphi^* g)$.*

If $\{\varphi_t\}_t$ is a continuous family of automorphisms in $\mathrm{Aut}_c(G)$ with $\varphi_0 = \mathrm{Id}$, then $(N\backslash G, \varphi_t^* g)$ is called an 'isospectral deformation'. When $\varphi \in \mathrm{Inn}(G)$, the corresponding Riemannian manifolds are isometric as well. However, the manifolds $(N\backslash G, g)$ and $(N\backslash G, \varphi^* g)$ are rarely isometric if $\varphi \in \mathrm{Aut}_c(G)\backslash\mathrm{Inn}(G)$. This description is made more clear in [41] but is rather technical, so we omit it here. Hence, for suitable subsets of $\mathrm{Aut}_c(G)\backslash\mathrm{Inn}(G)$, we obtain a continuous deformation of N which results in isospectral and non-isometric manifolds. Note that φ induces an isometry from $(N\backslash G, \varphi^* g)$ to $(\varphi(N)\backslash G, g)$, so the result can also be interpreted as obtaining isospectral nilmanifolds by fixing the metric and continuously deforming N.

To find concrete examples, it is often easier to consider the analogon of class preserving automorphisms on the Lie algebra level. Therefore, Gordon and Wilson also introduced in [41] the concept of an almost inner derivation (Definition 2.2.7). The relation between the two concepts is expressed in the following proposition.

Proposition 3.1.27 ([41])**.** *Let G be a connected and simply connected nilpotent Lie group with nilindex c. Denote the Lie algebra of G with $\mathfrak{g}$. Then $\mathrm{Aut}_c(G)$ is a simply connected nilpotent Lie group with nilindex $\leq c - 1$ and with Lie algebra $\mathrm{AID}(\mathfrak{g})$.*

There have been several articles concerning continuous deformations of nilmanifolds (see for instance [22, 23, 35, 36, 37, 38, 41, 69]). However, this was always from of differential geometric point of view and the main focus was on constructing examples. The approach was to start from a specific Lie algebra admitting non-inner almost inner derivations and to derive non-inner class preserving automorphisms. Therefore, almost inner derivations of Lie algebras have not been investigated in detail. The aim of this dissertation is to study this concept in a purely algebraic way. Although the motivation from spectral geometry only makes sense for nilpotent Lie algebras over $\mathbb{R}$ or $\mathbb{C}$, there is no need to restrict to this class.

3.2 Class preserving automorphisms of groups

Let G be a group. An automorphism $\varphi : G \to G$ is called 'class preserving' if every element is mapped to a conjugate element. This means that g and $\varphi(g)$ belong to the same conjugacy class for every $g \in G$. By definition, inner automorphisms are class preserving. In 1909, Burnside wanted to know if there exists a finite group with an outer automorphism which is class preserving ([12]). A few years later, he answered his own question by finding groups of order p^6 (for an odd prime p) which contain non-inner class preserving automorphisms ([13]). However, his answer went somewhat unnoticed, so the question was considered to be unsolved. Many years later, Wall ([84]) constructed an infinite number of examples of 2–groups, the smallest of which has order 32. Since then, there have been other results concerning class preserving automorphisms of finite p–groups (see for instance [46, 63, 86, 87, 88, 89, 90]) or groups in general (such as [25, 28, 43, 48, 67, 68, 74, 82, 83]). Since inner automorphisms are class preserving, this last notion is also referred to as 'almost inner', 'pointwise inner' or 'nearly inner' automorphisms.

In this section, we will list some facts concerning class preserving automorphisms. However, this collection is not at all a complete overview of all what is known with respect to this topic. We only mention these statements for which we can compare the results with properties for almost inner derivations of Lie algebras. Although we didn't use the results from this section to prove the statements about almost inner derivations of Lie algebras, it will become clear throughout this thesis that many results are very similar. Hence, this section can be considered as an alternative overview of the rest of this thesis.

3.2.1 Results for general groups

Preliminaries

Let G be a group. An automorphism $\varphi : G \to G$ is called 'class preserving' if every element is mapped to a conjugate element. The set of all class preserving automorphisms is denoted with $\mathrm{Aut}_c(G)$. It is clear that this is a generalisation of the inner automorphisms. When $\varphi \in \mathrm{Aut}_c(G)$, there exists a map $\varphi_D : G \to G$ such that $D(g) = g^{\varphi_D(g)}$ for all $g \in G$. In this case, φ is said to be 'determined' by φ_D. However, this map φ_D is not unique. The following results are well-known.

Proposition 3.2.1. *Let G be a group, then $\mathrm{Aut}_c(G)$ is a normal subgroup of $\mathrm{Aut}(G)$.*

Proof. Note that $\emptyset \neq \mathrm{Inn}(G) \subseteq \mathrm{Aut}_c(G)$, so $\mathrm{Aut}_c(G)$ is not empty. Further, suppose that $D, E \in \mathrm{Aut}_c(G)$ are determined by φ_D respectively φ_E. Then $D \circ E \in \mathrm{Aut}_c(G)$ as well, where

$$\begin{aligned} DE(g) &= D\big(\varphi_E(g)^{-1} g \varphi_E(g)\big) = D(\varphi_E(g))^{-1} D(g) D(\varphi_E(g)) \\ &= D(\varphi_E(g))^{-1} \varphi_D(g)^{-1} g \varphi_D(g) D(\varphi_E(g)) \\ &= (\varphi_D(g) D(\varphi_E(g)))^{-1} g \varphi_D(g) D(\varphi_E(g)) = g^{\varphi_D(g) D(\varphi_E(g))} \end{aligned}$$

holds for all $g \in G$. We also have that $D^{-1} \in \mathrm{Aut}(G)$. Choose an arbitrary $g \in G$ and define $h := D^{-1}(g)$. Since $g = D(h) = \varphi_D(h)^{-1} h \varphi_D(h)$, it turns out that

$$D^{-1}(g) = \varphi_D(h) g \varphi_D(h)^{-1}.$$

Hence, $D^{-1} \in \mathrm{Aut}_c(G)$, where $\varphi_{D^{-1}}(g) = \varphi_D(D^{-1}(g))^{-1}$ for all $g \in G$. This means that $\mathrm{Aut}_c(G)$ is a subgroup of $\mathrm{Aut}(G)$. Let $D \in \mathrm{Aut}_c(G)$ be determined by φ_D and take arbitrary $E \in \mathrm{Aut}(G)$. Choose $g \in G$ and consider $h := E(g)$. Then

$$\begin{aligned} E^{-1} DE(g) &= E^{-1}(\varphi_D(h)^{-1} h \varphi_D(h)) \\ &= E^{-1}(\varphi_D(h))^{-1} g E^{-1}(\varphi_D(h)) \end{aligned}$$

holds, which shows that $\mathrm{Aut}_c(G)$ is a normal subgroup of $\mathrm{Aut}(G)$. □

This proposition shows that the set $\mathrm{Out}_c(G) := \mathrm{Aut}_c(G)/\mathrm{Inn}(G)$ is well-defined. A group G has non-inner class preserving automorphisms if and only if $\mathrm{Out}_c(G) \neq 1$. Moreover, if G is nilpotent, then $\mathrm{Aut}_c(G)$ and $\mathrm{Aut}_c(G)/\mathrm{Inn}(G)$ are nilpotent as well.

Theorem 3.2.2 ([74])**.** *Let G be a nilpotent group of class c. Then $\mathrm{Aut}_c(G)$ is a nilpotent group of class $c-1$ and $\mathrm{Aut}_c(G)/\mathrm{Inn}(G)$ is a nilpotent group of class less than c.*

In Example 3.2.13, we show 2–step nilpotent groups for which $\mathrm{Aut}_c(G)/\mathrm{Inn}(G)$ is non-trivial, but abelian.

Proposition 3.2.3. *Let G_1 and G_2 be groups and consider the direct sum $G := G_1 \oplus G_2$. Suppose that $\varphi_1 \in \mathrm{Aut}_c(G_1)$ and $\varphi_2 \in \mathrm{Aut}_c(G_2)$. For the map $\varphi : G \to G : (g, h) \mapsto (\varphi_1(g), \varphi_2(h))$, it holds that $\varphi \in \mathrm{Aut}_c(G)$.*

Theorem 3.2.4 ([28])**.** *Let G be a finite simple group, then $\mathrm{Aut}_c(G) = \mathrm{Inn}(G)$.*

For most of the statements in this subsection, similar results for almost inner derivations of Lie algebras are proven in Section 4.1.

Other results for general groups

It turns out that for free groups, the only class preserving automorphisms are the inner ones.

Theorem 3.2.5 ([43])**.** *Let G be a free group, then* $\mathrm{Aut}_c(G) = \mathrm{Inn}(G)$*.*

The same result holds for free nilpotent groups.

Theorem 3.2.6 ([25])**.** *Let G be a free nilpotent group, then* $\mathrm{Aut}_c(G) = \mathrm{Inn}(G)$*.*

Chapter 11 is devoted to the study of almost inner derivations for free nilpotent Lie algebras. We also have in that case that the only almost inner derivations are the inner ones.

3.2.2 Results for finite p–groups

Let p be a prime number. We will present some definitions and properties of finite p–groups. For a finite p–group G, the set of class preserving automorphisms $\mathrm{Aut}_c(G)$ is a p–group as well.

Proposition 3.2.7. *Let G be a finite p–group. Then* $\mathrm{Aut}_c(G)$ *is also a p–group.*

Definition 3.2.8 (Elementary abelian group)**.** A p–group G is elementary abelian if G is abelian and every non-trivial element has order p.

An elementary abelian p–group can be considered as a vector space over the field with p elements. An easy example is $(C_2 \oplus C_2, +)$ which has four elements.

Definition 3.2.9 (Frattini subgroup)**.** Let G be a group. The Frattini subgroup $\Phi(G)$ of G is the intersection of all maximal subgroups of G. If there are no maximal subgroups, it is defined as $\Phi(G) := G$.

It can be shown that the Frattini subgroup of G equals the set of non-generators of G, where $g \in G$ is a 'non-generator' if $\langle H, g \rangle \neq G$ for all proper subgroups $H \leq G$.

Definition 3.2.10 (Special and extra special group)**.** A p–group G is called special if G is elementary abelian or if $[G,G] = \Phi(G) = Z(G)$. Further, a group G is called extra special if G is a non-abelian special p–group for which $|Z(G)| = |[G,G]| = |\Phi(G)| = p$.

For an abelian group G, it is clear that $\mathrm{Aut}_c(G) = \mathrm{Inn}(G)$ holds. It turns out that for extra special groups, all class preserving automorphisms are inner as well.

Theorem 3.2.11 ([78]). *Let G be an extra special p–group. Then we have* $\mathrm{Aut}_c(G) = \mathrm{Inn}(G)$.

In Chapter 7, we will consider special p–groups in more detail, as well as the correspondence with almost inner derivations of Lie algebras.

Groups of small order

It can be shown that every non-abelian group of order p^3 is extra special. For groups of order p^4, the only class preserving automorphisms are inner.

Theorem 3.2.12 ([55]). *Let G be a group of order p^4, then* $\mathrm{Aut}_c(G) = \mathrm{Inn}(G)$.

Since all groups of order p and p^2 are abelian, the above observations show that all class preserving automorphisms are inner for a general p–group of order $\leq p^4$. Wall ([84]) found a class of examples of 2–groups G with $\mathrm{Out}_c(G) \neq 1$. Therefore, he used the holomorph of C_{2^m} (with $m \geq 3$), which is a group obtained as a semidirect product, where the multiplicative group of units $U(C_{2^m})$ acts by left multiplication on its additive group C_{2^m}.

Example 3.2.13. Take $m \geq 3$ and consider the groups

$$G_m = \left\{ \begin{pmatrix} b & a \\ 0 & 1 \end{pmatrix} \mid a \in C_{2^m} \text{ and } b \in U(C_{2^m}) \right\}. \tag{3.1}$$

We will represent the matrix

$$\begin{pmatrix} b & a \\ 0 & 1 \end{pmatrix}$$

as the pair (a, b). Note that this group has $2^m \cdot \varphi(2^m) = 2^{2m-1}$ elements, where φ is the Euler function. Wall showed that

$$\mathrm{Aut}_c(G_m) = \mathrm{Inn}(G_m) \oplus \langle D \rangle.$$

holds for all $m \geq 3$, where

$$D : G_m \to G_m : (a, b) \mapsto \begin{cases} (a, b) & \text{if } b \equiv \pm 1 \mod 8, \\ (a + 4, b) & \text{if } b \equiv \pm 3 \mod 8, \end{cases}$$

is a non-inner class preserving automorphism.

There are two non-isomorphic groups of order 32 with non-inner class preserving automorphisms.

Theorem 3.2.14 ([11])**.** *Let G be a group of order 2^5. Then $\mathrm{Aut}_c(G) \neq \mathrm{Inn}(G)$ if and only if G is isomorphic to H_1 or H_2, with presentations*

$$H_1 = \langle a, b, c \mid a^8 = b^2 = c^2 = 1, bab = a^3, cac = a^5, bc = cb\rangle,$$

$$H_2 = \langle a, b, c \mid a^8 = b^2 = c^2 = 1, bab = a^3, cac = a^5, cbc = a^4 b\rangle.$$

Note that H_1 is isomorphic to Wall's example with $m = 3$. When p is an odd prime, there exist groups of order p^5 with $\mathrm{Aut}_c(G) \neq \mathrm{Inn}(G)$ as well ([88]). Further, there are finite groups G with $\mathrm{Aut}_c(G) \neq \mathrm{Inn}(G)$, for which G is not a p–group ([11]). The smallest example has order $96 = 2^5 \cdot 3$. In [11], the set $\mathrm{Aut}_c(G)$ was computed for all groups G of order < 512. The authors found that more than 60 percent of the 56.092 groups of order 256 admit non-inner class preserving automorphisms, illustrating that this is a common property. Chapter 8 is devoted to the study of almost inner derivations of low-dimensional Lie algebras. We will show that for a nilpotent Lie algebra of dimension at most four, all almost inner derivations are inner as well.

Two-step Camina p–groups

The following notion was introduced in [14] and later named after the author Camina.

Definition 3.2.15 (Camina pair and Camina group)**.** Let G be a finite group and N a proper normal subgroup of G. Then (G, N) is a Camina pair if and only if $xN \subset x^G$ for all $x \in G \setminus N$. A group G is called a Camina group if $(G, [G, G])$ is a Camina pair.

Hence, for a Camina group, the coset $g[G, G]$ is contained in a conjugacy class for every $g \in G \setminus [G, G]$. It is well-known that every extra special p–group is a Camina group.

Lemma 3.2.16. *Let G be an extra special p–group. Then G is also a Camina p–group.*

It was shown in [17] that every finite Camina p–group is nilpotent of class at most 3. From now on, we will focus on two-step Camina p–groups. The following results are due to Macdonald.

Theorem 3.2.17 ([60])**.** *Let G be a 2–step nilpotent group with normal subgroup N such that (G, N) is a Camina pair. Then $N = [G, G]$ and G is a special p–group.*

For special groups, there exists a criterion to decide whether or not it is a Camina group.

Theorem 3.2.18 ([60]). *Let $(G, *)$ be a special p–group. Choose elements $a_1, \ldots, a_n \in G$ and $h_1, \ldots, h_m \in [G, G]$ in such a way that*

$$G = \langle g_1, \ldots, g_n \rangle \quad \textit{and} \quad [G, G] = \langle h_1, \ldots, h_m \rangle.$$

Suppose that

$$[g_i, g_j] = (h_1)^{a^1_{ij}} * \cdots * (h_m)^{a^m_{ij}},$$

where $a^k_{ij} \in \mathbb{Z}_p$ for all $1 \leq i, j \leq n$ and $1 \leq k \leq m$. Define the matrices $A_k := (a^k_{ij})_{ij} \in M_n(\mathbb{F}_p)$ and take $\mu_1, \ldots, \mu_m \in \mathbb{F}_p$. Then G is a Camina group if and only if $\det(\mu_1 A_1 + \cdots + \mu_m A_m) = 0$ implies that $\mu_1 = \cdots = \mu_m = 0$.

Two years after he posed the question whether there exist non-inner class preserving automorphisms, Burnside himself found examples of groups of order p^6, were p is an odd prime ([13]). It turns out that when $p \equiv \pm 3 \mod 8$, the group is a 2–step nilpotent Camina p–group.

Example 3.2.19. We look at the example of Burnside. Let p be an odd prime. Consider

$$G = \langle x_1, \ldots, x_6 \mid x_1^p = x_2^p = x_3^p = x_4^p = x_5^p = x_6^p = 1,$$
$$[x_1, x_3] = x_5, [x_1, x_4] = x_6, [x_2, x_3] = x_6, [x_2, x_4] = x_5^2 \rangle$$

Then G is a special Camina group if and only if $p \equiv \pm 3 \mod 8$.

For Camina p–groups, the number of class preserving automorphisms is easy to determine.

Theorem 3.2.20 ([86]). *Let G be a finite Camina p–group of nilpotency class 2. Then $|\mathrm{Aut}_c(G)| = |[G, G]|^n$, where n is the number of elements in a minimal generating set for G.*

It turns out that special Camina p–groups are very closely related to nonsingular Lie algebras. This notion is introduced in Section 4.3 and is also treated in Section 7.2 and Section 9.3.

Finite p–groups for which all automorphisms are class preserving

In 1999, Mann asked whether all p–groups have automorphisms that are not class preserving ([64]). Already before that time, several counterexamples had been given. For instance, Heineken ([46]) introduced for every $p \geq 3$ an infinite class of special Camina p–groups G with $\mathrm{Aut}_c(G) = \mathrm{Aut}(G)$. Malinowska ([63]) found groups G of nilpotency class 3 for which all automorphisms are class preserving. In [11], the authors found two groups of order < 512 for which $\mathrm{Aut}_c(G) = \mathrm{Aut}(G)$ holds, with order 128 respectively 486. Those groups are SmallGroup(128, 932) and SmallGroup(486, 31), where we use the notation from the 'Small Groups library' from the computer algebra system `GAP`.

Example 3.2.21. Consider the group $G := C_4^2 \rtimes \mathcal{D}_4$ of order $2^7 = 128$, which has presentation

$$G = \langle a, b, c, d \mid a^4 = b^4 = c^4 = d^2 = 1, ab = ba, cac^{-1} = a^{-1}b^{-1}, dad = a^{-1}b,$$

$$cbc^{-1} = a^2 b, bd = db, dcd = c^{-1} \rangle.$$

Then G is a 4–step nilpotent group with $\mathrm{Aut}_c(G) = \mathrm{Aut}(G)$.

The following result was proven in [89].

Proposition 3.2.22. *Let G be a finite p–group of nilpotency class 2 such that all automorphisms are class preserving. Then $|G| \geq p^8$.*

The author also gave an example of a group of order 3^8 for which all automorphisms are class preserving.

Example 3.2.23 ([89])**.** Consider the group G with presentation

$$G = \langle x_1, \ldots, x_6 \mid x_1^9 = x_2^9 = x_3^3 = x_4^3 = x_5^3 = x_6^3 = 1,$$

$$[x_1, x_2] = x_1^3, [x_1, x_3] = x_2^3, [x_1, x_4] = x_2^3, [x_1, x_5] = x_2^3,$$

$$[x_1, x_6] = x_2^3, [x_2, x_3] = x_1^3, [x_2, x_4] = x_2^3, [x_2, x_5] = x_1^3,$$

$$[x_2, x_6] = x_2^3, [x_3, x_4] = x_2^3, [x_3, x_5] = x_2^3, [x_3, x_6] = x_1^3,$$

$$[x_4, x_5] = x_1^3, [x_4, x_6] = x_1^3, [x_5, x_6] = x_2^3 \rangle.$$

Then G is a special Camina 3–group with $\mathrm{Aut}_c(G) = \mathrm{Aut}(G)$.

In Section 10.3, we describe for all $n \geq 13$ a filiform Lie algebra $\mathfrak{g}_n$ of dimension n such that $\mathrm{Inn}(\mathfrak{g}_n) \neq \mathrm{AID}(\mathfrak{g}_n) = \mathrm{Der}(\mathfrak{g}_n)$. This phenomenon also exists for non-nilpotent Lie algebras, as we show in Chapter 8.

Other results

The following reasoning comes from [86]. Let G be a finite p–group of order p^t. Let $\{x_1, \ldots, x_n\}$ be a minimal generating set for G. Take $\varphi \in \mathrm{Aut}_c(G)$, then $\varphi(x_i) \in x_i^G$ for all $1 \leq i \leq n$. It follows that

$$|\mathrm{Aut}_c(G)| \leq \prod_{i=1}^{n} |x_i^G|.$$

Let $|[G,G]| = p^m$, then the Burnside basis theorem implies that $n \leq t - m$. It follows from the previous inequality that

$$|\mathrm{Aut}_c(G)| \leq p^{mn} \leq p^{m(t-m)}. \tag{3.2}$$

Theorem 3.2.24. *Let G be a finite p–group. Then the upper bound of* (3.2) *is attained if and only if G is either an abelian p–group, or a non-abelian special Camina p–group.*

In [86], there have been established sharper upper bounds for $|\mathrm{Aut}_c(G)|$.

Theorem 3.2.25 ([87])**.** *Let G be a 2–step nilpotent finite p–group. Let $\{x_1, \ldots, x_n\}$ be a minimal generating set for G. If $[x_i, G]$ is cyclic for all $1 \leq i \leq n$, then* $\mathrm{Aut}_c(G) = \mathrm{Inn}(G)$.

In Chapter 4, we show a similar inequality for $\mathrm{AID}(\mathfrak{g})$ and show that there is an equality for nonsingular Lie algebras. The previous result can be translated to Lie algebras as well, where we don't require that the Lie algebra is 2–step nilpotent. There are also a few results concerning groups with large cyclic subgroups.

Theorem 3.2.26 ([54])**.** *Let G be a finite p–group having a maximal cyclic subgroup. Then* $\mathrm{Aut}_c(G) = \mathrm{Inn}(G)$.

Theorem 3.2.27 ([55])**.** *Let G be a group of order p^n having a cyclic subgroup of order p^{n-2}.*

- *Suppose that p is odd, then* $\mathrm{Aut}_c(G) = \mathrm{Inn}(G)$.
- *Assume that $p = 2$ and $n \geq 5$ and that G does not have any element of order 2^{n-1}. Then* $\mathrm{Aut}_c(G) \neq \mathrm{Inn}(G)$ *if and only if G is isomorphic to H_1*

or H_2*, where*

$$H_1 = \langle x, y, z \mid x^{2^{m-2}} = y^2 = z^2 = 1, yxy = x^{1+2^{m-3}}, zyz = y,$$
$$zxz = x^{-1+2^{m-3}} \rangle$$
$$H_2 = \langle x, y, z \mid x^{2^{m-2}} = y^2 = z^2 = 1, yxy = x^{1+2^{m-3}}, zyz = yx^{2^{m-3}},$$
$$zxz = x^{-1+2^{m-3}} \rangle$$

Theorem 3.2.26 is very similar to the result for almost abelian algebras, see Section 12.2. However, Lie algebras $\mathfrak{g}$ with an ideal of codimension 2 can have non-inner almost inner derivations.

Chapter 4

Background for computations

In the previous chapter, we gave a geometric motivation to study almost inner automorphisms of some nilpotent Lie groups. Due to computational reasons however, it is easier to go to the Lie algebra level, where the analogon is an almost inner derivation of a Lie algebra. In Chapter 2, we already introduced the definitions of (central) almost inner derivations. In this chapter, some properties and examples are given. We also present two techniques which are useful in the computation of $\mathrm{AID}(\mathfrak{g})$. Some parts of this chapter already appeared in [7] and [9]. Unless otherwise specified, we consider Lie algebras over a general field $\mathbb{F}$ (so of any characteristic).

4.1 Properties and examples

Let $\mathfrak{g}$ be a Lie algebra over a field $\mathbb{F}$. Take arbitrary $x, y \in \mathfrak{g}$ and consider $D \in \mathrm{Der}(\mathfrak{g})$. We then have

$$\begin{aligned}
[D, \mathrm{ad}(x)](y) &= D(\mathrm{ad}(x)(y)) - \mathrm{ad}(x)(D(y)) \\
&= D([x, y]) - [x, D(y)] \\
&= [D(x), y] \\
&= \mathrm{ad}(D(x))(y),
\end{aligned}$$

where the one but last equation holds since D is a derivation. This shows that

$$[D, \mathrm{ad}(x)] = \mathrm{ad}(D(x)) \tag{4.1}$$

is inner for all $D \in \mathrm{Der}(\mathfrak{g})$ and all $x \in \mathfrak{g}$. With the aid of this observation, we will show that $\mathrm{Inn}(\mathfrak{g})$, $\mathrm{CAID}(\mathfrak{g})$ and $\mathrm{AID}(\mathfrak{g})$ become Lie subalgebras of $\mathrm{Der}(\mathfrak{g})$ when we use the Lie bracket $[D, D'] = DD' - D'D$.

Proposition 4.1.1. *We have the following inclusions of Lie subalgebras*

$$\mathrm{Inn}(\mathfrak{g}) \ \subseteq\ \mathrm{CAID}(\mathfrak{g}) \ \subseteq\ \mathrm{AID}(\mathfrak{g}) \ \subseteq\ \mathrm{Der}(\mathfrak{g}).$$

Proof. Let $\mathfrak{g}$ be a Lie algebra and choose arbitrary $x, y, z \in \mathfrak{g}$. The proof goes in different steps.

- We first show that $\mathrm{Inn}(\mathfrak{g})$ is a subalgebra of $\mathrm{Der}(\mathfrak{g})$. Since $\mathrm{ad}(x)$ and $\mathrm{ad}(y)$ are derivations, (4.1) implies that $[\mathrm{ad}(x), \mathrm{ad}(y)] = \mathrm{ad}([x, y])$.

- Further, we prove that $\mathrm{AID}(\mathfrak{g})$ is a subalgebra of $\mathrm{Der}(\mathfrak{g})$. Take almost inner derivations $D_1, D_2 \in \mathrm{AID}(\mathfrak{g})$, then there exist x_1 and x_2 in $\mathfrak{g}$ such that $D_1(x) = [x, x_1]$ respectively $D_2(x) = [x, x_2]$. Then the Lie bracket of D_1 and D_2 is given by

$$\begin{aligned}
[D_1, D_2](x) &= D_1(D_2(x)) - D_2(D_1(x)) \\
&= D_1([x, x_2]) - D_2([x, x_1]) \\
&= [D_1(x), x_2] + [x, D_1(x_2)] - [D_2(x), x_1] - [x, D_2(x_1)] \\
&= [[x, x_1], x_2] + [x, D_1(x_2)] - [[x, x_2], x_1] - [x, D_2(x_1)] \\
&= [[x, x_1], x_2] - [[x, x_2], x_1] + [x, D_1(x_2) - D_2(x_1)] \\
&= [x, [x_1, x_2]] + [x, D_1(x_2) - D_2(x_1)].
\end{aligned}$$

  The third equality holds because D_1 and D_2 are derivations. In the last step, the Jacobi identity is used. Hence,

$$[D_1, D_2](x) = [x, [x_1, x_2] + D_1(x_2) - D_2(x_1)] \ \in [x, \mathfrak{g}]$$

  holds, which means that $[D_1, D_2] \in \mathrm{AID}(\mathfrak{g})$ is an almost inner derivation. Therefore, $\mathrm{AID}(\mathfrak{g})$ is a subalgebra of $\mathrm{Der}(\mathfrak{g})$.

- Finally, we show $\mathrm{CAID}(\mathfrak{g})$ is a subalgebra of $\mathrm{Der}(\mathfrak{g})$ as well. Take elements $C_1, C_2 \in \mathrm{CAID}(\mathfrak{g})$, then there exist $y_1, y_2 \in \mathfrak{g}$ such that $C_1 - \mathrm{ad}(y_1)$ and $C_2 - \mathrm{ad}(y_2)$ map $\mathfrak{g}$ to $Z(\mathfrak{g})$. Note that $[C_1, C_2] \in \mathrm{AID}(\mathfrak{g})$ by the previous point. By using (4.1) several times, we obtain

$$\begin{aligned}
&[C_1 - \mathrm{ad}(y_1), C_2 - \mathrm{ad}(y_2)] \\
&\qquad = [C_1, C_2] - [C_1, \mathrm{ad}(y_2)] - [\mathrm{ad}(y_1), C_2] + [\mathrm{ad}(y_1), \mathrm{ad}(y_2)] \\
&\qquad = [C_1, C_2] - \mathrm{ad}(C_1(y_2)) + \mathrm{ad}(C_2(y_1)) + \mathrm{ad}([y_1, y_2]).
\end{aligned}$$

Hence, $[C_1, C_2] - \mathrm{ad}(C_1(y_2) - C_2(y_1) - [y_1, y_2]) = [C_1 - \mathrm{ad}(y_1), C_2 - \mathrm{ad}(y_2)]$ maps $\mathfrak{g}$ to $Z(\mathfrak{g})$, and hence $[C_1, C_2] \in \mathrm{CAID}(\mathfrak{g})$. $\square$

We also have the following results.

Proposition 4.1.2. *The subalgebra* $\mathrm{Inn}(\mathfrak{g})$ *is a Lie ideal in all subalgebras of* $\mathrm{Der}(\mathfrak{g})$ *containing it. Further,* $\mathrm{CAID}(\mathfrak{g})$ *is a Lie ideal in* $\mathrm{AID}(\mathfrak{g})$*.*

Proof. The first statement immediately follows from (4.1). Let $C \in \mathrm{CAID}(\mathfrak{g})$ and $D \in \mathrm{AID}(\mathfrak{g})$. We need to show that $[D, C] \in \mathrm{CAID}(\mathfrak{g})$. We already know from the previous proposition that $[D, C] \in \mathrm{AID}(\mathfrak{g})$. Fix an element $x \in \mathfrak{g}$ such that $C' := C - \mathrm{ad}(x)$ maps $\mathfrak{g}$ to $Z(\mathfrak{g})$. Define $D' := [D, C] - \mathrm{ad}(D(x))$. We can again apply (4.1) to find that $\mathrm{ad}(D(x)) = [D, \mathrm{ad}(x)]$, which means that $D' = [D, C']$. This leads to

$$\begin{aligned}
D'(y) &= [D, C'](y) \\
&= D(C'(y)) - C'(D(y))
\end{aligned}$$

for all $y \in \mathfrak{g}$. Since C' maps $\mathfrak{g}$ to $Z(\mathfrak{g})$ and D maps $Z(\mathfrak{g})$ to 0, we have that D' maps $\mathfrak{g}$ to $Z(\mathfrak{g})$. $\square$

Remark 4.1.3. We conjecture that $\mathrm{AID}(\mathfrak{g})$ is always a Lie ideal in $\mathrm{Der}(\mathfrak{g})$, for Lie algebras $\mathfrak{g}$ over an arbitrary field $\mathbb{F}$. However, there is no obvious algebraic argument for this statement. Moreover, the result seems not to be known. At least, we didn't find a counterexample among the checks we did for low-dimensional Lie algebras.

For a given Lie algebra $\mathfrak{g}$ over a field $\mathbb{F}$, we want to know which of the inclusions in the chain $\mathrm{Inn}(\mathfrak{g}) \subseteq \mathrm{CAID}(\mathfrak{g}) \subseteq \mathrm{AID}(\mathfrak{g}) \subseteq \mathrm{Der}(\mathfrak{g})$ are actually equalities. We will establish some examples to show different possibilities. It is clear that $\mathrm{Inn}(\mathfrak{g}) = \mathrm{CAID}(\mathfrak{g}) = \mathrm{AID}(\mathfrak{g}) = 0$ for abelian Lie algebras. Recall that a Lie algebra $\mathfrak{g}$ is called 'complete', if $Z(\mathfrak{g}) = 0$ and $\mathrm{Der}(\mathfrak{g}) = \mathrm{Inn}(\mathfrak{g})$. Of course we have $\mathrm{Inn}(\mathfrak{g}) = \mathrm{CAID}(\mathfrak{g}) = \mathrm{AID}(\mathfrak{g})$ in this case. In particular semisimple Lie algebras over a field of characteristic zero, and parabolic subalgebras of semisimple Lie algebras are complete.

Consider again an arbitrary field $\mathbb{F}$ (of any characteristic). First, we will compute in detail the almost inner derivations of the Heisenberg Lie algebra $\mathfrak{h}_3(\mathbb{F})$, given by the Lie brackets $[e_1, e_2] = e_3$.

Example 4.1.4. For $\mathfrak{g} = \mathfrak{h}_3(\mathbb{F})$ we have $\mathrm{AID}(\mathfrak{g}) = \mathrm{Inn}(\mathfrak{g})$.
Take an arbitrary inner derivation $\mathrm{ad}(x) \in \mathrm{Inn}(\mathfrak{g})$. Then there exist values $x_1, x_2, x_3 \in \mathbb{F}$ such that $x = x_1e_1 + x_2e_2 + x_3e_3$. By bilinearity and the fact that $\mathrm{ad}(e_3) = 0$, we have that

$$\mathrm{ad}(x) = x_1\mathrm{ad}(e_1) + x_2\mathrm{ad}(e_2),$$

so $\mathrm{Inn}(\mathfrak{g})$ is 2–dimensional. Since every derivation is also a linear map, we can represent it as a matrix. For instance, the inner derivation $\mathrm{ad}(e_1)$ can be represented as

$$\mathrm{ad}(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

where in the j–th column, we write the image of e_j with respect to the basis. This means that $\mathrm{ad}(e_1)$ maps e_1 and e_3 to zero and e_2 to e_3. Note that we abuse the notation by denoting $\mathrm{ad}(e_1)$ for both the derivation and the matrix representing the derivation. Similarly, we can consider an arbitrary derivation D of $\mathfrak{h}_3(\mathbb{F})$ in matrix form. Then D can be written as a linear combination of derivations, namely

$$D := a_1\mathrm{ad}(e_1) + a_2\mathrm{ad}(e_2) + d_1D_1 + \cdots + d_4D_4,$$

where $a_i, d_j \in \mathbb{F}$ (with $1 \leq i \leq 2$ and $1 \leq j \leq 4$). The corresponding matrix is

$$D = \begin{pmatrix} d_1 & d_3 & 0 \\ d_2 & d_4 & 0 \\ -a_2 & a_1 & d_1 + d_4 \end{pmatrix}.$$

For example, the derivation D_1 corresponds to the matrix

$$D_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

This shows that $\mathrm{Der}(\mathfrak{g}) = \langle \mathrm{ad}(e_1), \mathrm{ad}(e_2), D_1, D_2, D_3, D_4 \rangle$. Assume that D is almost inner. Then $D(e_1) \in [e_1, \mathfrak{g}] = \langle e_3 \rangle$, so that $d_1 = d_2 = 0$. In the same way, $D(e_2) \in [e_2, \mathfrak{g}] = \langle e_3 \rangle$ implies that $d_3 = d_4 = 0$. It follows that $D \in \mathrm{Inn}(\mathfrak{g})$.

In this example, it was easy to prove that for instance D_1 is not almost inner, since the condition $D_1(x) \in [x, \mathfrak{g}]$ is not satisfied for $x = e_1$.

Definition 4.1.5 (Almost inner for a basis)**.** Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{F}$ with basis $\mathcal{B}$. Then a linear map $D : \mathfrak{g} \to \mathfrak{g}$ is $\mathcal{B}$–almost inner if $D(x) \in [x, \mathfrak{g}]$ holds for all basis elements $x \in \mathcal{B}$.

Of course, this definition depends on the chosen basis. However, it gives a necessary condition for a derivation to be almost inner which is, given the structure constants of the Lie algebra, very easy to check.

The Lie algebra in the next example has more interesting (i.e. non-inner) almost inner derivations. In what follows, we will denote $E_{i,j}$ for the matrix with entry 1 at position (i, j) and 0 otherwise. When we interpret $E_{i,j}$ as a linear map, then $E_{i,j}(e_k) = 0$ for $k \neq j$, and $E_{i,j}(e_j) = e_i$.

Example 4.1.6. Let $\mathfrak{g}$ be the filiform nilpotent Lie algebra which has a basis $\mathcal{B} = \{e_1, \dots, e_5\}$ and non-zero Lie brackets

$$\begin{aligned} &[e_1, e_2] = e_3, \quad [e_1, e_3] = e_4, \\ &[e_1, e_4] = e_5, \quad [e_2, e_3] = e_5. \end{aligned}$$

Then we have $\mathrm{AID}(\mathfrak{g}) = \mathrm{Inn}(\mathfrak{g}) \oplus \langle E_{5,2} \rangle$.
The proof again follows by a direct computation. An arbitrary derivation is of the form

$$D = a_1 \mathrm{ad}(e_1) + \cdots + a_4 \mathrm{ad}(e_4) + d_1 D_1 + d_2 D_2 + d_3 D_3 + e_{5,2} E_{5,2}$$

(with coefficients in $\mathbb{F}$) and can be described as a matrix

$$D = \begin{pmatrix} d_1 & 0 & 0 & 0 & 0 \\ d_2 & 2d_1 & 0 & 0 & 0 \\ -a_2 & a_1 & 3d_1 & 0 & 0 \\ -a_3 & d_3 & a_1 & 4d_1 & 0 \\ -a_4 & -a_3 + e_{5,2} & a_2 + d_3 & a_1 + d_2 & 5d_1 \end{pmatrix},$$

so $\mathrm{Der}(\mathfrak{g}) = \langle \mathrm{ad}(e_1), \dots, \mathrm{ad}(e_4), D_1, D_2, D_3, E_{5,2} \rangle$. Further, $E_{5,2}$ is almost inner with determination map $\varphi_{E_{5,2}} : \mathfrak{g} \to \mathfrak{g}$, where the image of $x = x_1 e_1 + \cdots + x_5 e_5$ is given by

$$\varphi_{E_{5,2}}(x) = \begin{cases} \frac{x_2}{x_1} e_4 & \text{if } x_1 \neq 0, \\ e_3 & \text{if } x_1 = 0. \end{cases}$$

The derivation $E_{5,2}$ is not inner. However, we have $E_{5,2}(x) = [x, \varphi_{E_{5,2}}(x)]$ for all $x \in \mathfrak{g}$, so $E_{5,2}$ is almost inner (in fact, central almost inner). Consider the derivation $D := d_1 D_1 + d_2 D_2 + d_3 D_3$ (with $d_1, d_2, d_3 \in \mathbb{F}$). Then D is $\mathcal{B}$–almost inner if and only if $d_1 = d_2 = d_3 = 0$. Hence, we have that $\mathrm{CAID}(\mathfrak{g}) = \mathrm{AID}(\mathfrak{g}) = \langle \mathrm{ad}(e_1), \dots, \mathrm{ad}(e_4), E_{5,2} \rangle$.

Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{F}$ with basis $\mathcal{B} = \{e_1, \dots, e_n\}$. Since an almost inner derivation has to be $\mathcal{B}$–almost inner, it is clear that

$$\dim(\mathrm{AID}(\mathfrak{g})) \leq \sum_{i=1}^{n} \dim[e_i, \mathfrak{g}], \tag{4.2}$$

so this gives an upper bound for $\mathrm{AID}(\mathfrak{g})$ which is easy to determine. In general however, it is not enough to check the condition $D(x) \in [x, \mathfrak{g}]$ only for basis vectors of $\mathfrak{g}$.

Example 4.1.7. Let $\mathfrak{g}$ be the Lie algebra over a field $\mathbb{F}$ which has basis $\mathcal{B} := \{e_1, e_2, e_3\}$ and is given by $[e_1, e_2] = e_2$ and $[e_1, e_3] = e_3$. A short computation shows that an arbitrary derivation for $\mathfrak{g}$ looks like

$$D = a_1\mathrm{ad}(e_1) + a_2\mathrm{ad}(e_2) + a_3\mathrm{ad}(e_3) + e_{2,3}E_{2,3} + e_{3,2}E_{3,2} + e_{3,3}E_{3,3}$$

(with coefficients in $\mathbb{F}$) and with matrix form

$$D = \begin{pmatrix} 0 & 0 & 0 \\ -a_2 & a_1 & e_{2,3} \\ a_3 & e_{3,2} & a_1 + e_{3,3} \end{pmatrix}.$$

This means that $\mathrm{Der}(\mathfrak{g}) = \langle \mathrm{ad}(e_1), \mathrm{ad}(e_2), \mathrm{ad}(e_3), E_{2,3}, E_{3,2}, E_{3,3} \rangle$. Consider the derivation $D := E_{3,3}$. Then D is $\mathcal{B}$–almost inner since $D(e_i) = 0 = [e_i, 0]$ for $1 \leq i \leq 2$ and $D(e_3) = e_3 = [e_3, -e_1]$. However, D is not an almost inner derivation. Indeed, choose an arbitrary $x = x_1e_1 + x_2e_2 + x_3e_3 \in \mathfrak{g}$, then $[e_2+e_3, x] = -x_1(e_2+e_3)$ holds. This gives a contradiction, since $D(e_2+e_3) = e_3$.

This example shows that a derivation which is $\mathcal{B}$–almost inner does not have to be an almost inner derivation. Hence, the condition is necessary, but not sufficient. The next proposition contains a few more easy facts on almost inner derivations.

Proposition 4.1.8. *Let $\mathfrak{g}$ and $\mathfrak{g}'$ be Lie algebras over the same field $\mathbb{F}$. Then the following statements hold.*

(a) Let $D \in \mathrm{AID}(\mathfrak{g})$. Then $D(\mathfrak{g}) \subseteq [\mathfrak{g}, \mathfrak{g}]$ and $D(Z(\mathfrak{g})) = 0$.

(b) Let I be an ideal of $\mathfrak{g}$, then $D(I) \subseteq I$ and D induces $\bar{D} \in \mathrm{AID}(\mathfrak{g}/I)$.

(c) For $C \in \mathrm{CAID}(\mathfrak{g})$ there exists an $x \in \mathfrak{g}$ such that $C_{|[\mathfrak{g},\mathfrak{g}]} = \mathrm{ad}(x)_{|[\mathfrak{g},\mathfrak{g}]}$.

(d) If $\mathfrak{g}$ is 2–step nilpotent, then $\mathrm{CAID}(\mathfrak{g}) = \mathrm{AID}(\mathfrak{g})$.

(e) If $Z(\mathfrak{g}) = 0$, then $\mathrm{CAID}(\mathfrak{g}) = \mathrm{Inn}(\mathfrak{g})$.

(f) *If* $\mathfrak{g}$ *is nilpotent, then* $\mathrm{AID}(\mathfrak{g})$ *is nilpotent and all* $D \in \mathrm{AID}(\mathfrak{g})$ *are nilpotent.*

(g) *We have* $\mathrm{AID}(\mathfrak{g} \oplus \mathfrak{g}') = \mathrm{AID}(\mathfrak{g}) \oplus \mathrm{AID}(\mathfrak{g}')$ *for the direct sum of two Lie algebras.*

Proof. We will prove each of the statements. Take arbitrary $C \in \mathrm{CAID}(\mathfrak{g})$ and $D \in \mathrm{AID}(\mathfrak{g})$.

(a) By definition, an almost inner derivation maps $\mathfrak{g}$ into $[\mathfrak{g}, \mathfrak{g}]$. Further, for $x \in Z(\mathfrak{g})$, we have $D(x) \in [x, \mathfrak{g}] = 0$.

(b) Let $y \in I$, then we have $D(y) \in [y, \mathfrak{g}] \subseteq [I, \mathfrak{g}] \subseteq I$. Define

$$\bar{D} : \mathfrak{g}/I \to \mathfrak{g}/I : x + I \mapsto D(x) + I.$$

This is a well-defined linear map due to the first statement. It is a routine check to find that $\bar{D}$ is a derivation and that it satisfies the almost inner condition with determination map $\varphi_{\bar{D}} : \mathfrak{g}/I \to \mathfrak{g}/I : x + I \mapsto \varphi_D(x) + I$. Hence, $\bar{D} \in \mathrm{AID}(\mathfrak{g}/I)$.

(c) Since $C \in \mathrm{CAID}(\mathfrak{g})$, there exists an $x \in \mathfrak{g}$ such that $C' := C - \mathrm{ad}(x)$ satisfies $C'(\mathfrak{g}) \subseteq Z(\mathfrak{g})$. We further have

$$C'([y, z]) = [C'(y), z] + [y, C'(z)]$$

for all $y, z \in \mathfrak{g}$, because C' is a derivation. This shows that $C'([\mathfrak{g}, \mathfrak{g}]) = 0$, so $C(y) = \mathrm{ad}(x)(y)$ for all elements $y \in [\mathfrak{g}, \mathfrak{g}]$.

(d) If $\mathfrak{g}$ is 2–step nilpotent, then $D(\mathfrak{g}) \subseteq [\mathfrak{g}, \mathfrak{g}] \subseteq Z(\mathfrak{g})$ for any $D \in \mathrm{AID}(\mathfrak{g})$. Hence we have $\mathrm{AID}(\mathfrak{g}) \subseteq \mathrm{CAID}(\mathfrak{g})$, which gives equality.

(e) Suppose that $Z(\mathfrak{g}) = 0$ and $C \in \mathrm{CAID}(\mathfrak{g})$. Then there exists an $x \in \mathfrak{g}$ such that $C(x) - \mathrm{ad}(x) = 0$. Hence C is inner.

(f) Let $D \in \mathrm{AID}(\mathfrak{g})$ and $x \in \mathfrak{g}$, then $D^k(x) \in [x, [\mathfrak{g}, [\ldots, [\mathfrak{g}, \mathfrak{g}] \cdots]]]$, where we have k times $\mathfrak{g}$. If k is higher than the nilpotence class of $\mathfrak{g}$, it follows that $D^k(x) = 0$, hence D is nilpotent. By Engel's theorem, $\mathrm{AID}(\mathfrak{g})$ is nilpotent.

(g) For the last statement, consider $E \in \mathrm{AID}(\mathfrak{g} \oplus \mathfrak{g}')$. Then the restrictions $E_{|\mathfrak{g}} \in \mathrm{AID}(\mathfrak{g})$ and $E_{|\mathfrak{g}'} \in \mathrm{AID}(\mathfrak{g}')$ are almost inner derivations. It is easy to see that the map $E \mapsto E_{|\mathfrak{g}} \oplus E_{|\mathfrak{g}'}$ gives a one-to-one correspondence between the Lie algebras $\mathrm{AID}(\mathfrak{g} \oplus \mathfrak{g}')$ and $\mathrm{AID}(\mathfrak{g}) \oplus \mathrm{AID}(\mathfrak{g}')$. $\square$

Let $\mathfrak{g}$ be a Lie algebra over a field $\mathbb{F}$. In this dissertation, we study several research questions.

- What is $\mathrm{AID}(\mathfrak{g})$?
  By performing similar computations as for the examples above, we are able to compute all almost inner derivations of $\mathfrak{g}$, just by determining which derivations satisfy the almost inner condition. However, this is not very efficient. Therefore, we will elaborate on some techniques to have more general results. Section 4.2 gives a method to decide that a given derivation cannot be almost inner. This is in particular useful for Lie algebras for which the only almost inner derivations are the inner ones. In Section 4.3, we give a way to restrict the derivations for which we have to check the almost inner condition.

- Which of the inclusions $\mathrm{Inn}(\mathfrak{g}) \subseteq \mathrm{CAID}(\mathfrak{g}) \subseteq \mathrm{AID}(\mathfrak{g}) \subseteq \mathrm{Der}(\mathfrak{g})$ are equalities?
  We already saw that for complete Lie algebras, all these sets are equal. Further, we showed that all almost inner derivations are inner for the Heisenberg Lie algebra. In Example 4.1.6, we also treated a Lie algebra $\mathfrak{g}$ for which $\mathrm{AID}(\mathfrak{g})$ and $\mathrm{Inn}(\mathfrak{g})$ are different. Throughout this thesis, we will study more examples showing that all possibilities occur.

- What is the importance of the field $\mathbb{F}$?
  In the second part, we will show that the field over which $\mathfrak{g}$ is defined, has an impact on the set of all almost inner derivations.

- How do the results for $\mathfrak{g}$ compare to other Lie algebras from the same class?
  This is the main question for the last part. We will consider different classes of (nilpotent) Lie algebras and study whether similar Lie algebras behave in an analogous way or not.

4.2 Fixed basis vectors

In the previous section, we computed the almost inner derivations of a given Lie algebra by checking the almost inner condition for all derivations. However, one does not always need to know the derivation algebra explicitly. Instead one can use a concept which we will call 'fixed basis vectors'. This notion is very useful, also for proving several results on almost inner derivations. Unfortunately, although the definition is elementary, it is not particularly clear, so we will need to explain it with some examples. The results of this section also appeared in [7].

For the rest of this section, $\mathfrak{g}$ is an n–dimensional Lie algebra over an arbitrary field $\mathbb{F}$ and with chosen basis $\{e_1, \ldots, e_n\}$. If $x = \sum_{j=1}^{n} x_j e_j$, then we denote by $t_i(x) = x_i$ the i–th coordinate of x with respect to the given basis.

Definition 4.2.1 (Fixed basis vector)**.** Let D be an almost inner derivation of $\mathfrak{g}$ determined by a map $\varphi_D : \mathfrak{g} \to \mathfrak{g}$. A basis vector e_i is a fixed basis vector for D with fixed value $\alpha \in \mathbb{F}$ if and only if for all $1 \leq j \leq n$, the following statement is satisfied:

$$\text{if } e_j \notin C_{\mathfrak{g}}(e_i), \text{ then } t_i(\varphi_D(e_j)) = \alpha.$$

Note that $\alpha \in \mathbb{F}$ must be the same for all j where this condition applies. As an example, we look at the Heisenberg Lie algebra $\mathfrak{h}_3(\mathbb{F})$ with basis $\{e_1, e_2, e_3\}$ and Lie bracket $[e_1, e_2] = e_3$.

Example 4.2.2. Consider $\mathfrak{g} = \mathfrak{h}_3(\mathbb{F})$ and let D be an almost inner derivation of $\mathfrak{g}$ with determination map $\varphi_D \colon \mathfrak{g} \to \mathfrak{g}$. Then every basis vector e_i is fixed. For $i = 1$ we have $C_{\mathfrak{g}}(e_1) = \langle e_1, e_3 \rangle$ and the condition just applies for $j = 2$: since $e_2 \notin C_{\mathfrak{g}}(e_1)$, we must have $t_1(\varphi_D(e_2)) = \alpha$. Certainly this is true, with α given by the map φ_D. The same holds for $i = 2$, where we have $C_{\mathfrak{g}}(e_2) = \langle e_2, e_3 \rangle$. For $i = 3$, we have $C_{\mathfrak{g}}(e_3) = \mathfrak{g}$, so that the condition is trivially satisfied.

The importance of finding fixed basis vectors comes from the following fact. If each basis vector for every almost inner derivation is fixed, then we have $\mathrm{AID}(\mathfrak{g}) = \mathrm{Inn}(\mathfrak{g})$. We already saw that this is the case for $\mathfrak{h}_3(\mathbb{F})$ in Example 4.1.4. We will prove this more general result in Corollary 4.2.6. Often we can show that every basis vector is fixed without knowing the structure of $\mathrm{Der}(\mathfrak{g})$. A trivial example is the following lemma.

Lemma 4.2.3. *Let $\mathfrak{g}$ be a Lie algebra with basis $\{e_1, \ldots, e_n\}$, such that for given $1 \leq i \leq n$, the number of basis vectors in $C_{\mathfrak{g}}(e_i)$ is equal to $\dim(\mathfrak{g})$ or $\dim(\mathfrak{g}) - 1$. Then the basis vector e_i is fixed.*

Proof. In this case the condition for a fixed basis vector is vacuously true, or can be satisfied uniquely by the α given by the map φ_D. ☐

We used this argument for $1 \leq i \leq 3$ in the example of $\mathfrak{g} = \mathfrak{h}_3(\mathbb{F})$ above. We also want to present an example, where not every basis vector is fixed. For the Lie algebra $\mathfrak{g}$ of Example 4.1.6, we will show that there is an almost inner derivation D determined by a map φ_D such that not every basis vector is fixed.

Example 4.2.4. For the Lie algebra $\mathfrak{g}$ of Example 4.1.6 and the almost inner derivation $D := E_{5,2}$, the basis vector e_3 is not fixed.
We already saw that D is determined by $\varphi_D : \mathfrak{g} \to \mathfrak{g}$, where an arbitrary $x = x_1 e_1 + \cdots + x_5 e_5 \in \mathfrak{g}$ is mapped to

$$\varphi_D(x) = \begin{cases} \frac{x_2}{x_1} e_4 & \text{if } x_1 \neq 0, \\ e_3 & \text{if } x_1 = 0. \end{cases}$$

Definition 4.2.1 for this φ_D and $i = 3$ says the following: for all $1 \leq j \leq 5$, if $e_j \notin C_{\mathfrak{g}}(e_3) = \langle e_3, e_4, e_5 \rangle$, then $t_3(\varphi_D(e_j)) = \alpha$, each time for the same fixed $\alpha \in \mathbb{F}$. We have $\varphi_D(e_1) = 0$ and $\varphi_D(e_2) = e_3$, so that

$$t_3(\varphi_D(e_2)) = t_3(e_3) = 1,$$

$$t_3(\varphi_D(e_1)) = t_3(0) = 0.$$

This means that there is no fixed α and thus e_3 is not fixed.

Lemma 4.2.5. *Let $D : \mathfrak{g} \to \mathfrak{g}$ be an almost inner derivation determined by a map $\varphi_D : \mathfrak{g} \to \mathfrak{g}$. If e_i is a fixed basis vector with fixed value α, then $D' = D + \mathrm{ad}(\alpha e_i)$ is an almost inner derivation which is determined by a map $\varphi_{D'} : \mathfrak{g} \to \mathfrak{g}$ such that for all $1 \leq j, k \leq n$:*

$$t_j(\varphi_{D'}(e_k)) = t_j(\varphi_D(e_k)) \qquad \textit{for } i \neq j,$$

$$t_i(\varphi_{D'}(e_k)) = 0.$$

Proof. Clearly D' is an almost inner derivation, and we have that

$$(D + \mathrm{ad}(\alpha e_i))(x) = [x, \varphi_D(x)] + [\alpha e_i, x] = [x, \varphi_D(x) - \alpha e_i].$$

So D' is determined by the map $\tilde{\varphi}_{D'} : \mathfrak{g} \to \mathfrak{g} : x \mapsto \varphi_D(x) - \alpha e_i$. Define the map

$$\varphi_{D'} : \mathfrak{g} \to \mathfrak{g} : x \mapsto \begin{cases} \varphi_D(x) - \alpha e_i & \text{if } x \notin \{e_1, e_2, \ldots, e_n\}, \\ \varphi_D(x) - t_i(\varphi_D(x)) e_i & \text{if } x \in \{e_1, e_2, \ldots, e_n\}. \end{cases}$$

We claim that D' is also determined by this new map $\varphi_{D'}$. Indeed, for all non basis vectors we have $\varphi_{D'}(x) = \tilde{\varphi}_{D'}(x)$, so we only have to consider basis vectors. Let e_j be a basis vector. Then there are two possibilities.

Case 1: For $e_j \in C_{\mathfrak{g}}(e_i)$, we have

$$D'(e_j) = D(e_j) = [e_j, \varphi_D(e_j)] = [e_j, \varphi_D(e_j) - t_i(\varphi_D(x)) e_i] = [e_j, \varphi_{D'}(e_j)].$$

Case 2: When $e_j \notin C_{\mathfrak{g}}(e_i)$, then $t_i(\varphi_D(e_j)) = \alpha$, from which it follows that $\tilde{\varphi}_{D'}(e_j) = \varphi_{D'}(e_j)$.

In both cases, we see that D' is determined by $\varphi_{D'}$. By definition of $\varphi_{D'}$ it is also easy to see that the requirements $t_j(\varphi_{D'}(e_k)) = t_j(\varphi_D(e_k))$, for $j \neq i$, and $t_i(\varphi_{D'}(e_k)) = 0$ hold. $\square$

As an immediate consequence, we obtain the following result.

Corollary 4.2.6. *Let $D \in \mathrm{AID}(\mathfrak{g})$ be determined by a map φ_D. If each basis vector is fixed, then $D \in \mathrm{Inn}(\mathfrak{g})$.*

Proof. Let α_i denote the fixed value of e_i. Define

$$D' := D + \mathrm{ad}(\alpha_1 e_1) + \mathrm{ad}(\alpha_2 e_2) + \cdots + \mathrm{ad}(\alpha_n e_n) = D + \mathrm{ad}(v),$$

where $v := \sum_{i=1}^n \alpha_i e_i$. Then by iteratively applying Lemma 4.2.5, we find that D' is an almost inner derivation D', determined by a map $\varphi_{D'}$. Further, $\varphi_{D'}(e_i) = 0$ for all $1 \le i \le n$. This implies that $D'(e_i) = 0$ for all basis vectors e_i and hence $D' = 0$ or $D = -\mathrm{ad}(v) \in \mathrm{Inn}(\mathfrak{g})$. □

The next results are two technical lemmas, providing a way to find fixed basis vectors. We will use the following notation: Let $1 \le i_1, i_2, \ldots, i_r \le n$, then

$$\mathfrak{g}_{i_1,i_2,\ldots,i_r} = \langle e_i \mid i \notin \{i_1, i_2, \ldots, i_r\} \rangle$$

denotes the vector space spanned by all basis vectors not in the set $\{e_{i_1}, e_{i_2}, \ldots, e_{i_r}\}$.

Lemma 4.2.7. *Assume that $1 \le i, j, k, l, m \le n$ and $l \ne m$. Moreover assume that there exist nonzero scalars $\alpha, \beta \in \mathbb{F}$ such that*

$$\begin{aligned}
&[e_j, e_i] - \alpha e_l \in \mathfrak{g}_{l,m}\\
&[e_k, e_i] - \beta e_m \in \mathfrak{g}_{l,m}\\
&[e_j, \mathfrak{g}_i] \subseteq \mathfrak{g}_{l,m}\\
&[e_k, \mathfrak{g}_i] \subseteq \mathfrak{g}_{l,m}.
\end{aligned}$$

Then for any almost inner derivation $D \in \mathrm{AID}(\mathfrak{g})$ determined by a map φ_D, we have that $t_i(\varphi_D(e_j)) = t_i(\varphi_D(e_k))$.

Proof. Let $a := t_i(\varphi_D(e_j))$, $b := t_i(\varphi_D(e_k))$ and $c := t_i(\varphi_D(e_j + e_k))$. Then there exist vectors $v, v', v'' \in \mathfrak{g}_i$ such that

$$\varphi_D(e_j) = a e_i + v,$$

$$\varphi_D(e_k) = b e_i + v',$$

$$\varphi_D(e_j + e_k) = c e_i + v''.$$

Using these notations, we find that

$$D(e_j + e_k) = [e_j + e_k, c e_i + v''] = c\alpha e_l + c\beta e_m + w'' \tag{4.3}$$

for some $w'' \in \mathfrak{g}_{l,m}$. On the other hand, we have that

$$D(e_j) + D(e_k) = [e_j, ae_i + v] + [e_k, be_i + v'] = a\alpha e_l + w + b\beta e_m + w' \quad (4.4)$$

for some $w, w' \in \mathfrak{g}_{l,m}$. Now, as D is a linear map, the two expressions (4.3) and (4.4) must be equal, and so by comparing the l–th and m–th coordinate, we find that

$$c\alpha = a\alpha \quad \text{and} \quad c\beta = b\beta.$$

As both α and β are nonzero, this implies that $a = b$. Hence, we have that $t_i(\varphi_D(e_j)) = t_i(\varphi_D(e_k))$. □

Suppose that $\mathfrak{g}$ is a Lie algebra which satisfies the conditions of the lemma. Let $D \in \mathrm{AID}(\mathfrak{g})$ be an arbitrary almost inner derivation determined by φ_D. As shown in the proof, we must have that $t_i(\varphi_D(e_j)) = t_i(\varphi_D(e_k))$ to ensure that the almost inner condition is satisfied for $e_j + e_k$. We will show this procedure with the same Lie algebra as in Example 4.1.7.

Example 4.2.8. Consider the Lie algebra $\mathfrak{g}$ over a field $\mathbb{F}$ which has basis $\mathcal{B} = \{e_1, e_2, e_3\}$ and is given by $[e_1, e_2] = e_2$ and $[e_1, e_3] = e_3$. Since we have

$$\begin{aligned}
&[e_2, e_1] + e_2 \in \mathfrak{g}_{2,3} \\
&[e_3, e_1] + e_3 \in \mathfrak{g}_{2,3} \\
&[e_2, \mathfrak{g}_1] \subseteq \mathfrak{g}_{2,3} \\
&[e_3, \mathfrak{g}_1] \subseteq \mathfrak{g}_{2,3},
\end{aligned}$$

we can apply Lemma 4.2.7 with $(i, j, k, l, m) = (1, 2, 3, 2, 3)$. Let $D \in \mathrm{AID}(\mathfrak{g})$ be determined by φ_D. It follows that $t_1(\varphi_D(e_2)) = t_1(\varphi_D(e_3))$, which means that e_1 is a fixed basis vector. Further, e_2 and e_3 are fixed basis vectors as well by Lemma 4.2.3. Hence, we can conclude from Corollary 4.2.6 that D is an inner derivation, so $\mathrm{AID}(\mathfrak{g}) = \mathrm{Inn}(\mathfrak{g})$.

The second lemma is similar, but the conditions are slightly changed.

Lemma 4.2.9. *Assume that $1 \leq i, j, k, l \leq n$. Moreover, assume that there exist nonzero scalars $\alpha, \beta \in \mathbb{F}$ such that*

$$\begin{aligned}
&[e_j, e_i] - \alpha e_l \in \mathfrak{g}_l \\
&[e_k, e_i] - \beta e_l \in \mathfrak{g}_l \\
&[e_j, \mathfrak{g}_i] \subseteq \mathfrak{g}_l \\
&[e_k, \mathfrak{g}_i] \subseteq \mathfrak{g}_l.
\end{aligned}$$

Then for any almost inner derivation $D \in \mathrm{AID}(\mathfrak{g})$ determined by a map φ_D, we have that $t_i(\varphi_D(e_j)) = t_i(\varphi_D(e_k))$.

Proof. Let $a := t_i(\varphi_D(e_j))$, $b := t_i(\varphi_D(e_k))$ and $c := t_i(\varphi_D(\beta e_j - \alpha e_k))$. Let $v, v', v'' \in \mathfrak{g}_i$ be such that

$$\varphi_D(e_j) = ae_i + v,$$

$$\varphi_D(e_k) = be_i + v',$$

$$\varphi_D(\beta e_j - \alpha e_k) = ce_i + v''.$$

Then we have that

$$D(\beta e_j - \alpha e_k) = [\beta e_j - \alpha e_k, ce_i + v''] = \beta c\alpha e_l - \alpha c\beta e_l + w'' \tag{4.5}$$

for some $w'' \in \mathfrak{g}_l$. On the other hand, we have that

$$\beta D(e_j) - \alpha D(e_k) = \beta[e_j, ae_i + v] - \alpha[e_k, be_i + v'] = \beta a\alpha e_l + w - \alpha b\beta e_l + w' \tag{4.6}$$

for some $w, w' \in \mathfrak{g}_l$. By comparing the l–th coordinate of (4.5) and (4.6), we find that

$$\alpha\beta(a - b) = 0.$$

Since α and β are non-zero, we find that $a = b$ and $t_i(\varphi_D(e_j)) = t_i(\varphi_D(e_k))$. $\square$

Example 4.2.10. Let $\mathfrak{g}$ be the Lie algebra over a field $\mathbb{F}$ which has a basis $\mathcal{B} = \{e_1, e_2, e_3, e_4\}$ and is given by $[e_1, e_2] = e_4$ and $[e_1, e_3] = e_4$. We have

$$\begin{aligned} &[e_2, e_1] + e_4 \in \mathfrak{g}_4 \\ &[e_3, e_1] + e_4 \in \mathfrak{g}_4 \\ &[e_2, \mathfrak{g}_1] \subseteq \mathfrak{g}_4 \\ &[e_3, \mathfrak{g}_1] \subseteq \mathfrak{g}_4. \end{aligned}$$

Let $D \in \mathrm{AID}(\mathfrak{g})$ be an almost inner derivation determined by φ_D. It follows from Lemma 4.2.9 with $(i, j, k, l) = (1, 2, 3, 4)$ that $t_1(\varphi_D(e_2)) = t_1(\varphi_D(e_3))$. Hence, e_1 is a fixed basis vector. Further, Lemma 4.2.3 implies that the other basis vectors are fixed as well. This means that D is an inner derivation, so $\mathrm{AID}(\mathfrak{g}) = \mathrm{Inn}(\mathfrak{g})$.

With these technical lemmas, it is possible to obtain results on the almost inner derivations without having to explicitly know the derivation algebra.

4.3 Skew matrix pencils

In the previous section, we introduced fixed basis vectors as a useful tool in the computation of the almost inner derivations of a Lie algebra. In this section, we link Lie algebras with matrix pencils. The results of this section also appeared in [9]. We work in this section with an arbitrary field $\mathbb{F}$.

For the moment, we only consider 2–step nilpotent Lie algebras. Let $\mathfrak{g}$ be a 2–step nilpotent Lie algebra over a field $\mathbb{F}$. We fix a basis $\mathcal{B} := \{x_1, \dots, x_n, y_1, \dots, y_m\}$ of $\mathfrak{g}$, where $\{y_1, \dots, y_m\}$ is a basis of $[\mathfrak{g}, \mathfrak{g}]$. For any $1 \leq i, j \leq n$, we have that

$$[x_i, x_j] = c_{ij}^1 y_1 + \cdots + c_{ij}^m y_m,$$

where $c_{ij}^k \in \mathbb{F}$ are the structure constants of $\mathfrak{g}$. Define for every $1 \leq k \leq m$ the skew-symmetric matrix $A_k = (c_{ij}^k)_{1 \leq i,j \leq n}$. Let $\mu_1, \dots, \mu_m$ be algebraically independent variables over $\mathbb{F}$.

Definition 4.3.1 (Associated pencil)**.** Let $\mathfrak{g}$ be a 2–step nilpotent Lie algebra over an arbitrary field $\mathbb{F}$. Then $\mu_1 A_1 + \cdots + \mu_m A_m$ $(\in M_n(\mathbb{F}[\mu_1, \dots, \mu_m]))$ is called the pencil associated to $\mathfrak{g}$ with respect to the basis $\mathcal{B}$.

This pencil depends on the choice of basis of $\mathfrak{g}$. As for all $1 \leq k \leq m$ the matrix A_k is skew-symmetric, $\mu_1 A_1 + \cdots + \mu_m A_m$ is called a skew matrix pencil. In Section 9.2, we will elaborate on more theory concerning skew matrix pencils for the specific case where $m = 2$. To reduce the number of subscripts, we will denote the matrix pencil $\mu_1 A_1 + \mu_2 A_2$ as $\mu A + \lambda B$.

If $D : \mathfrak{g} \to \mathfrak{g}$ is an almost inner derivation, then D maps $\mathfrak{g}$ to $[\mathfrak{g}, \mathfrak{g}]$ and the center $Z(\mathfrak{g})$ to zero. Hence, the space $\mathcal{C}(\mathfrak{g})$ below contains $\mathrm{AID}(\mathfrak{g})$.

Definition 4.3.2 (Central derivation)**.** Let $\mathfrak{g}$ be a 2–step nilpotent Lie algebra over a field $\mathbb{F}$. Then

$$\mathcal{C}(\mathfrak{g}) = \{D \in \mathrm{End}(\mathfrak{g}) \mid D(\mathfrak{g}) \subseteq [\mathfrak{g}, \mathfrak{g}] \text{ and } D(Z(\mathfrak{g})) = 0\}$$

is a subalgebra of $\mathrm{Der}(\mathfrak{g})$ with $\mathrm{AID}(\mathfrak{g}) \subseteq \mathcal{C}(\mathfrak{g})$. It is called the algebra of central derivations.

Since $[\mathfrak{g}, \mathfrak{g}] \subseteq Z(\mathfrak{g})$ for 2–step nilpotent Lie algebras, it is easy to see that any $D \in \mathcal{C}(\mathfrak{g})$ is indeed a derivation of $\mathfrak{g}$.

We will assume that $\mathfrak{g}$ is a 2–step nilpotent Lie algebra and fix a basis

$$\{x_1, \dots, x_n, y_1, \dots, y_m\},$$

where $y_1, \dots, y_m$ is a basis of $[\mathfrak{g}, \mathfrak{g}]$. Let $\mu_1 A_1 + \cdots + \mu_m A_m$ be the associated pencil. Every element in $v \in \mathfrak{g}$ can be written as $v = x + y$ in this basis, where $x = a_1 x_1 + \cdots + a_n x_n$ and $y = b_1 y_1 + \cdots + b_m y_m$ for all $a_i, b_i \in \mathbb{F}$, with $1 \leq i \leq n$. For every $D \in \mathcal{C}(\mathfrak{g})$, we have $D(y) = 0$ and $D(x) = d_1(x) y_1 + \cdots + d_m(x) y_m$

holds for some $d_1, \ldots, d_m \in \mathrm{Hom}(U, \mathbb{F})$, where $U = \langle x_1, x_2, \ldots, x_n \rangle$. Recall that $D \in \mathrm{AID}(\mathfrak{g})$ if and only if there exists a map $\varphi_D \colon \mathfrak{g} \to \mathfrak{g}$ such that

$$D(v) = [v, \varphi_D(v)]$$

for all $v \in \mathfrak{g}$. We may assume that $\varphi_D(v) = \varphi_D(x) \in \langle x_1, \ldots, x_n \rangle$ for $v = x + y$ as above. This means that we suppose that $v = x$ and

$$\varphi_D(x) = c_1(x)x_1 + \cdots + c_n(x)x_n$$

for some $c_i(x) \in \mathbb{F}$, where $1 \leq i \leq n$. We denote

$$c(x) = (c_1(x), \ldots, c_n(x))^\top,$$

$$a(x) = (a_1, \ldots, a_n)^\top,$$

$$L(x) = \begin{pmatrix} a(x)^\top A_1 \\ \vdots \\ a(x)^\top A_m \end{pmatrix} \in M_{m,n}(\mathbb{F}),$$

$$d(x) = \begin{pmatrix} d_1(x) \\ \vdots \\ d_m(x) \end{pmatrix}.$$

Lemma 4.3.3. *Let $\mathfrak{g}$ be a 2–step nilpotent Lie algebra over $\mathbb{F}$ with the notations as above. A given map $D \in \mathcal{C}(\mathfrak{g})$ is in $\mathrm{AID}(\mathfrak{g})$ if and only if $L(x)c(x) = d(x)$ has a solution in the unknowns $c_i(x)$ (with $1 \leq i \leq n$) for all $x = a_1x_1 + \cdots + a_nx_n$ in $\mathfrak{g}$.*

Proof. We have $D \in \mathrm{AID}(\mathfrak{g})$ if and only if there exists a map $\varphi_D \colon \mathfrak{g} \to \mathfrak{g}$ such that for all $x = a_1x_1 + \cdots + a_nx_n$ (or, equivalently, for all $a_1, \ldots, a_n \in \mathbb{F}$), we have that $D(x) = [x, \varphi_D(x)]$. This is the case if and only if we can find $c_i(x)$ for $1 \leq i \leq n$ with

$$D(x) = \left[\sum_{i=1}^n a_i x_i, \sum_{j=1}^n c_j(x) x_j \right] = \sum_{i=1}^n \sum_{j=1}^n a_i c_j(x)(c_{ij}^1 y_1 + \cdots + c_{ij}^m y_m).$$

This is equivalent to the system of linear equations $L(x)c(x) = d(x)$ for all $x = a_1x_1 + \cdots + a_nx_n$. $\square$

We obtain the following result.

Proposition 4.3.4. *Let $\mathfrak{g}$ be a 2–step nilpotent Lie algebra over $\mathbb{F}$ with associated pencil $\mu_1 A_1 + \cdots + \mu_m A_m$ and the notations as above. Then a given map $D \in \mathcal{C}(\mathfrak{g})$ is in $\mathrm{AID}(\mathfrak{g})$ if and only if for all $x = a_1 x_1 + \cdots + a_n x_n$ in $\mathfrak{g}$ and all $\mu_1, \ldots, \mu_m \in \mathbb{F}$, we have the condition*

$$(\mu_1 A_1 + \cdots + \mu_m A_m) a(x) = 0 \Longrightarrow \mu_1 d_1(x) + \cdots + \mu_m d_m(x) = 0. \tag{4.7}$$

Proof. Suppose that $D \in \mathrm{AID}(\mathfrak{g})$ and $(\mu_1 A_1 + \cdots + \mu_m A_m) a(x) = 0$. Since the matrices A_k are skew-symmetric for all $1 \leq k \leq m$, we have

$$0 = (\mu_1 A_1 + \cdots + \mu_m A_m) a(x) = -(\mu_1 a(x)^\top A_1 + \cdots + \mu_m a(x)^\top A_m)^\top,$$

so that $\mu_1 a(x)^\top A_1 + \cdots + \mu_m a(x)^\top A_m = 0$. Note that $a(x)^\top A_k$ is the k–th row of $L(x)$ for each $1 \leq k \leq m$. Since $D \in \mathrm{AID}(\mathfrak{g})$, both $L(x)$ and the extended matrix $(L(x) \mid d(x))$ have the same rank by Lemma 4.3.3. Hence for any linear combination of rows of $L(x)$ which equals zero, the same linear combination of rows of $(L(x) \mid d(x))$ equals zero. It follows that $\mu_1 d_1(x) + \cdots + \mu_m d_m(x) = 0$. The converse direction can be proven in a similar way. $\square$

We illustrate this result with an example.

Example 4.3.5. Let $\mathfrak{g}$ be the 7–dimensional real Lie algebra with basis $\{x_1, \ldots, x_5, y_1, y_2\}$ and Lie brackets defined by

$$\begin{aligned} [x_1, x_3] &= y_1, \quad [x_1, x_4] = y_2, \\ [x_2, x_4] &= y_1, \quad [x_2, x_5] = y_2. \end{aligned}$$

The associated matrix pencil, which we denote as $\mu A + \lambda B$, is given by

$$\mu A + \lambda B = \begin{pmatrix} 0 & 0 & \mu & \lambda & 0 \\ 0 & 0 & 0 & \mu & \lambda \\ -\mu & 0 & 0 & 0 & 0 \\ -\lambda & -\mu & 0 & 0 & 0 \\ 0 & -\lambda & 0 & 0 & 0 \end{pmatrix}.$$

It is easy to see that $\dim(\mathrm{Inn}(\mathfrak{g})) = 5$. Let $D \in \mathcal{C}(\mathfrak{g})$ and $x = a_1 x_1 + \cdots + a_5 x_5$. Then the matrix of D is given by

$$D = \left(\begin{array}{ccccc|c} \multicolumn{5}{c|}{0} & 0 \\ \hline r_1 & r_2 & r_3 & r_4 & r_5 & \multirow{2}{*}{0} \\ s_1 & s_2 & s_3 & s_4 & s_5 & \end{array}\right).$$

We thus have $d_1(x) = a_1 r_1 + \cdots + a_5 r_5$ and $d_2(x) = a_1 s_1 + \cdots + a_5 s_5$. For all $(\mu, \lambda) \neq (0, 0)$, the kernel of $\mu A + \lambda B$ for this Lie algebra is 1–dimensional, generated by $a(x) = (0, 0, \lambda^2, -\mu\lambda, \mu^2)^\top$. Therefore condition (4.7) applied to this vector yields

$$\begin{aligned} 0 &= \mu d_1(x) + \lambda d_2(x) \\ &= \mu(a_1r_1 + \cdots + a_5r_5) + \lambda(a_1s_1 + \cdots + a_5s_5) \\ &= \mu^3r_5 + \mu^2\lambda(s_5 - r_4) + \mu\lambda^2(r_3 - s_4) + \lambda^3s_3 \end{aligned}$$

for all $\mu, \lambda \in \mathbb{F}$. Hence $D \in \mathrm{AID}(\mathfrak{g})$ if and only if $r_5 = s_3 = 0$ and $s_5 = r_4$ and $s_4 = r_3$. Thus we have $\dim(\mathrm{AID}(\mathfrak{g})) = 6$.

Remark 4.3.6. Consider values $\mu_1, \ldots, \mu_m \in \mathbb{F}$ with $\det(\mu_1A_1 + \cdots + \mu_mA_m) \neq 0$, then

$$(\mu_1A_1 + \cdots + \mu_mA_m)a(x)^\top = 0$$

is only satisfied for $a_1 = \cdots = a_n = 0$. This means that $d_1(x) = \cdots = d_m(x) = 0$, so condition (4.7) is always valid in that case. Hence, it suffices to consider those values $\mu_1, \ldots, \mu_m \in \mathbb{F}$ for which $\det(\mu_1A_1 + \cdots + \mu_mA_m) = 0$.

When n is odd, we have that $\det(\mu_1A_1 + \cdots + \mu_mA_m) = 0$, since A_k is skew-symmetric for all $1 \leq k \leq m$. In practice, previous remark is only useful when n is even.

Example 4.3.7. Consider the Lie algebra $\mathfrak{g}$ over a field $\mathbb{F}$ which has a basis $\{x_1, \ldots, x_4, y_1, y_2\}$ and non-vanishing Lie brackets

$$[x_1, x_2] = y_1, \qquad [x_1, x_4] = y_2 \quad \text{and} \quad [x_2, x_3] = y_2.$$

It is easy to see that $\dim(\mathrm{Inn}(\mathfrak{g})) = 4$. Let $D \in \mathcal{C}(\mathfrak{g})$ and $x = a_1x_1 + \cdots + a_4x_4$. Then the matrix of D is given by

$$D = \left(\begin{array}{cccc|c} \multicolumn{4}{c|}{0} & 0 \\ \hline r_1 & r_2 & r_3 & r_4 & \multirow{2}{*}{0} \\ s_1 & s_2 & s_3 & s_4 & \end{array}\right).$$

We also have $d_1(x) = a_1r_1 + \cdots + a_4r_4$ and $d_2(x) = a_1s_1 + \cdots + a_4s_4$. For $\mathfrak{g}$, we have that

$$\det(\mu A + \lambda B) = \lambda^4.$$

It follows from Remark 4.3.6 that we may assume without loss of generality that $\lambda = 0$. A general solution for

$$\begin{pmatrix} 0 & \mu & 0 & 0 \\ -\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

is given by $a(x) = (0, 0, k_3, k_4)^\top$, where $k_3, k_4 \in \mathbb{F}$. For these vectors, we find that $d_1(a(x)) = 0$ if and only if $r_3 = r_4 = 0$. Hence, we have that $\dim(\mathrm{AID}(\mathfrak{g})) = \dim(\mathrm{Inn}(\mathfrak{g})) + 2 = 6$.

Definition 4.3.8 (Nonsingular Lie algebra)**.** Let $\mathfrak{g}$ be a 2–step nilpotent Lie algebra over a field $\mathbb{F}$. When the associated pencil $\mu_1 A_1 + \cdots + \mu_m A_m$ satisfies $\det(\mu_1 A_1 + \cdots + \mu_m A_m) = 0$ if and only if $\mu_1 = \cdots = \mu_m = 0$, the Lie algebra $\mathfrak{g}$ is called nonsingular.

For nonsingular Lie algebras, it is very easy to compute the almost inner derivations.

Corollary 4.3.9. *Let $\mathfrak{g}$ be a nonsingular Lie algebra over a field $\mathbb{F}$ of type (n, m). We then have that $\mathrm{AID}(\mathfrak{g}) = \mathcal{C}(\mathfrak{g})$ and $\dim(\mathrm{AID}(\mathfrak{g})) = m \dim(\mathrm{Inn}(\mathfrak{g})) = mn$.*

Proof. By assumption, the system $(\mu_1 A_1 + \cdots + \mu_m A_m)a(x) = 0$ only has the trivial solution $x = 0$. Hence, condition (4.7) is satisfied and it follows from Proposition 4.3.4 that $\mathrm{AID}(\mathfrak{g}) = \mathcal{C}(\mathfrak{g})$. Since the suppositions imply that $[\mathfrak{g}, \mathfrak{g}] = Z(\mathfrak{g})$, we have $\dim(\mathrm{Inn}(\mathfrak{g})) = n$, spanned by $\mathrm{ad}(x_1), \ldots, \mathrm{ad}(x_n)$ and $\dim(\mathcal{C}(\mathfrak{g})) = mn$. It follows that $\dim(\mathrm{AID}(\mathfrak{g})) = mn$. □

In Section 9.3, we will study nonsingular Lie algebras in more detail. Several examples will be given in other sections as well.

When $\mathfrak{g}$ is not 2–step nilpotent, we can associate a matrix pencil to $\mathfrak{g}$ as well. Consider a nilpotent Lie algebra $\mathfrak{g}$ over a field $\mathbb{F}$. Fix a basis

$$\{e_1, \ldots, e_n, e_{n+1}, \ldots, e_{n+m}, e_{n+m+1}, \ldots, e_{n+m+l}\}$$

such that $[\mathfrak{g}, \mathfrak{g}] \subseteq \langle e_{n+1}, \ldots, e_{n+m+l} \rangle$ and $Z(\mathfrak{g}) = \langle e_{n+m+1}, \ldots, e_{n+m+l} \rangle$. Then we have for all $1 \leq i, j \leq n + m$ that

$$[e_i, e_j] = c_{ij}^{n+1} e_{n+1} + \cdots + c_{ij}^{n+m+l} e_{n+m+l},$$

where $c_{ij}^k \in \mathbb{F}$ are the structure constants of $\mathfrak{g}$. For every $n + 1 \leq k \leq n + m + l$, we define the skew-symmetric matrix $A_k := (c_{ij}^k)_{1 \leq i,j \leq n+m}$. The associated matrix pencil for $\mathfrak{g}$ is

$$\mu_{n+1} A_{n+1} + \cdots + \mu_{n+m+l} A_{n+m+l}.$$

With the same technique as before, we can check whether or not a linear map (not necessarily a central derivation) satisfies the almost inner condition. However, since $\mathfrak{g}$ is not 2–step nilpotent, we still have to check if it is a derivation. We will clarify this observation with the aid of an example.

Example 4.3.10. Let $\mathfrak{g}$ be the Lie algebra over a field $\mathbb{F}$ which has a basis $\{e_1, e_2, e_3, e_4, e_5\}$ and non-vanishing Lie brackets

$$[e_1, e_2] = e_4, \qquad [e_1, e_4] = e_5 \quad \text{and} \quad [e_2, e_3] = e_5,$$

so $n = 3$ and $m = l = 1$. When we denote $\mu := \mu_4$ and $\lambda := \mu_5$, the associated pencil

$$\mu A_4 + \lambda A_5 = \begin{pmatrix} 0 & \mu & 0 & \lambda \\ -\mu & 0 & \lambda & 0 \\ 0 & -\lambda & 0 & 0 \\ -\lambda & 0 & 0 & 0 \end{pmatrix}$$

for $\mathfrak{g}$ is the same matrix pencil as in Example 4.3.7. We can conclude from Remark 4.3.6 that for instance $E_{5,4} : \mathfrak{g} \to \mathfrak{g}$ satisfies the almost inner condition. However, it is not a derivation, since

$$E_{5,4}([e_1, e_2]) = E_{5,4}(e_4) = e_5,$$

but $[E_{5,4}(e_1), e_2] + [e_1, E_{5,4}(e_2)] = 0$. To determine $\mathrm{AID}(\mathfrak{g})$, we also need the derivation algebra. An arbitrary derivation for $\mathfrak{g}$ is given by

$$D = a_1\mathrm{ad}(e_1) + \cdots + a_4\mathrm{ad}(e_4) + d_1 D_1 + \cdots + d_5 D_5 + e_{5,3}E_{5,3}$$

and has matrix form

$$D = \begin{pmatrix} d_1 & 0 & 0 & 0 & 0 \\ d_2 & d_4 & 0 & 0 & 0 \\ d_3 & d_5 & 2d_1 & 0 & 0 \\ -a_2 & a_1 & -d_2 & d_1 + d_4 & 0 \\ -a_4 & -a_3 & a_2 + e_{5,3} & a_1 - d_3 & 2d_1 + d_4 \end{pmatrix},$$

which means that $\mathrm{Der}(\mathfrak{g}) = \langle \mathrm{ad}(e_1), \ldots, \mathrm{ad}(e_4), D_1, \ldots, D_5, E_{5,3} \rangle$. It is clear that no non-zero linear combination of $D_1, \ldots, D_5$ satisfies the almost inner condition. Proposition 4.3.4 implies that $E_{5,3}$ is almost inner. We can conclude that $\mathrm{AID}(\mathfrak{g}) = \langle \mathrm{ad}(e_1), \ldots, \mathrm{ad}(e_4), E_{5,3} \rangle$.

As a conclusion, skew matrix pencils are particularly important for 2–step nilpotent Lie algebras. However, it can also be used for other Lie algebras in testing which linear maps satisfy the almost inner condition, but other techniques also play a role in determining the derivation algebra. In the next chapter, we will see another example of a matrix pencil associated to a 3–step nilpotent Lie algebra.

Part II

Lie algebras over different fields

Chapter 5

Importance of the field

A Lie algebra $\mathfrak{g}$ over a field $\mathbb{F}$ can be specified by giving the structure constants with respect to a given basis. When these structure constants are integers, we can interpret these structure constants and the associated Lie algebra over any other field. In this case, one might expect that the field itself plays a minor role in the computation of $\mathrm{Der}(\mathfrak{g})$ or $\mathrm{AID}(\mathfrak{g})$. However, as we will show, this is not the case. In Section 5.1, we will explain how the characteristic of the field has an impact on the derivation algebra. Section 5.2 deals with how the field itself influences which derivations are almost inner and which are not. For this, we will use the associated matrix pencil from Section 4.3 and in particular the zeros of the determinant.

The observations from this chapter will be shown on the basis of a famous Lie algebra. Jacobson proved ([50]) that over a field of characteristic zero, a Lie algebra with a nonsingular derivation is nilpotent. Dixmier and Lister showed ([24]) that the converse of this result is not valid. The example they gave was the 8–dimensional Lie algebra $\mathfrak{g}$ over a field $\mathbb{F}$ of characteristic zero with basis $\{e_1, \ldots, e_8\}$, given by

$$\begin{array}{llll} [e_1, e_2] = e_5, & [e_1, e_3] = e_6, & [e_1, e_4] = e_7, & [e_1, e_5] = -e_8, \\ [e_2, e_3] = e_8, & [e_2, e_4] = e_6, & [e_2, e_6] = -e_7, & \\ [e_3, e_4] = -e_5, & [e_3, e_5] = -e_7, & [e_4, e_6] = -e_8. & \end{array} \tag{5.1}$$

and having only nilpotent derivations. Later, nilpotent Lie algebras whose derivations are all nilpotent were called ‘characteristically nilpotent’, or in short CNLAs. In Section 12.4, we will give more information about almost inner derivations for characteristically nilpotent Lie algebra. Although CNLAs are mainly studied in characteristic zero, it is in fact not necessary to impose this condition. Indeed, the Lie brackets from (5.1) define a Lie algebra for every field and a CNLA for all fields $\mathbb{F}$ with $\mathrm{char}(\mathbb{F}) \notin \{2, 3\}$.

5.1 Derivations

Let $\mathbb{F}$ be an arbitrary field and $\mathfrak{g}$ a Lie algebra over $\mathbb{F}$ with basis $\mathcal{B} = \{e_1, \dots, e_n\}$. We denote c_{ij}^k for the structure constants (so $1 \leq i, j, k \leq n$). Let $D : \mathfrak{g} \to \mathfrak{g}$ be a derivation. Since D is a linear map, it can be described as a matrix, which we will denote by $D = (d_{ij})$. By bilinearity of the Lie bracket, it suffices to check Leibniz' rule for the basis vectors. Let e_i and e_j be two arbitrary basis vectors, then

$$D([e_i, e_j]) = D\left(\sum_{l=1}^{n} c_{ij}^l e_l\right) = \sum_{l=1}^{n} c_{ij}^l D(e_l) \; = \; \sum_{l=1}^{n}\sum_{k=1}^{n} c_{ij}^l d_{kl} e_k \tag{5.2}$$

holds. Analogously, we find that

$$\begin{aligned}
[D(e_i), e_j] + [e_i, D(e_j)] &= \left[\sum_{l=1}^{n} d_{li} e_l, e_j\right] + \left[e_i, \sum_{l=1}^{n} d_{lj} e_l\right] \\
&= \sum_{l=1}^{n} d_{li}[e_l, e_j] + \sum_{l=1}^{n} d_{lj}[e_i, e_l] \\
&= \sum_{l=1}^{n}\sum_{k=1}^{n} d_{li} c_{lj}^k e_k + \sum_{l=1}^{n}\sum_{k=1}^{n} d_{lj} c_{il}^k e_k \\
&= \sum_{l=1}^{n}\sum_{k=1}^{n} (d_{li} c_{lj}^k + d_{lj} c_{il}^k) e_k. \qquad (5.3)
\end{aligned}$$

Since D is a derivation, equations (5.2) and (5.3) have to be the same, which means that

$$\sum_{l=1}^{n} c_{ij}^l d_{kl} = \sum_{l=1}^{n} (d_{li} c_{lj}^k + d_{lj} c_{il}^k) \tag{5.4}$$

has to hold for all $1 \leq i, j, k \leq n$. We can impose without loss of generality that $1 \leq i < j \leq n$, so there are in total $n^2(n-1)/2$ equations which have to be satisfied at the same time. However, it is possible that some of them are meaningless or redundant. These equations give relations on the different matrix entries of the derivation and a solution depends on the characteristic of the field.

Consider a field $\mathbb{F}$ of $\mathrm{char}(\mathbb{F}) \notin \{2,3\}$. Let $\mathfrak{g}$ be the Dixmier-Lister Lie algebra over $\mathbb{F}$. A direct computation shows that an arbitrary derivation D of $\mathfrak{g}$ is a linear combination

$$D = a_1 \mathrm{ad}(e_1) + \cdots + a_6 \mathrm{ad}(e_6) + d_1 D_1 + \cdots + d_6 D_6$$

and has matrix form

$$\left(\begin{array}{cccccc|c}
\multicolumn{6}{c|}{0} & 0 \\ \hline
-a_2 & a_1 & a_4 & -a_3 & 0 & 0 & \\
-a_3 & -a_4 & a_1 & a_2 & 0 & 0 & 0 \\
-a_4 + d_1 + d_2 & a_6 + d_4 & a_5 + d_5 & a_1 + d_3 & -a_3 & -a_2 & \\
a_5 - d_2 & a_3 + d_3 - d_4 & a_2 + d_1 & a_6 + d_6 & -a_1 & -a_4 &
\end{array}\right),$$

which means that $\mathrm{Der}(\mathfrak{g}) = \langle \mathrm{ad}(e_1), \ldots, \mathrm{ad}(e_6), D_1, \ldots, D_6 \rangle$. This was also computed in [24] and resulted in the same derivation algebra. Note that every derivation is indeed nilpotent, meaning that $\mathfrak{g}$ is a CNLA.

When $\mathrm{char}(\mathbb{F}) = 2$, we have that

$$\mathrm{Der}(\mathfrak{g}) = \langle \mathrm{ad}(e_1), \ldots, \mathrm{ad}(e_6), D_1, \ldots, D_6, D_1^{(2)}, D_2^{(2)} \rangle.$$

For arbitrary $x = \sum_{i=1}^{8} x_i e_i$ the additional derivations $D_1^{(2)}, D_2^{(2)} : \mathfrak{g} \to \mathfrak{g}$ are given by

$$D_1^{(2)}(x) = x_2 e_2 + x_4 e_4 + x_5 e_5 + x_7 e_7 + x_8 e_8,$$

$$D_2^{(2)}(x) = x_2 e_3 + x_4 e_1 + x_5 e_6.$$

We have for instance that $D_1^{(2)}([e_2, e_4]) = D_1^{(2)}(e_6) = 0$, but

$$[D_1^{(2)}(e_2), e_4] + [e_2, D_1^{(2)}(e_4)] = 2e_6,$$

so $D_1^{(2)} \notin \mathrm{Der}(\mathfrak{g})$ when $\mathrm{char}(\mathbb{F}) \neq 2$. For $\mathrm{char}(\mathbb{F}) = 3$, we have

$$\mathrm{Der}(\mathfrak{g}) = \langle \mathrm{ad}(e_1), \ldots, \mathrm{ad}(e_6), D_1, \ldots, D_6, D_1^{(3)}, \ldots, D_4^{(3)} \rangle.$$

The image of $x = \sum_{i=1}^{8} x_i e_i$ for the maps $D_1^{(3)}, \ldots, D_4^{(3)} : \mathfrak{g} \to \mathfrak{g}$ is

$$D_1^{(3)}(x) = x_1 e_1 - x_2 e_2 - x_3 e_3 + x_4 e_4 + x_7 e_7 + x_8 e_8,$$

$$D_2^{(3)}(x) = x_1 e_3 - x_3 e_6 + x_4 e_2 - x_5 e_8 + x_8 e_7,$$

$$D_3^{(3)}(x) = x_1 e_4 - x_2 e_3 + x_3 e_2 - x_4 e_1 + x_5 e_6 - x_6 e_5,$$

$$D_4^{(3)}(x) = x_2 e_4 + x_3 e_1 - x_4 e_5 + x_5 e_7 + x_7 e_8.$$

When $\mathrm{char}(\mathbb{F}) \in \{2,3\}$, it is clear that $\mathfrak{g}$ is not a CNLA anymore. This illustrates that the derivation algebra depends on the characteristic of the field.

5.2 Almost inner derivations

In this section, we will compute the almost inner derivations for the Dixmier-Lister Lie algebra over an arbitrary field $\mathbb{F}$. It turns out that the number of different solutions of $X^3 - 1$ over $\mathbb{F}$ determines the dimension of $\mathrm{AID}(\mathfrak{g})$. The matrix pencil $P := \mu_5 A_5 + \mu_6 A_6 + \mu_7 A_7 + \mu_8 A_8$ associated to $\mathfrak{g}$ is given by

$$P = \begin{pmatrix} 0 & \mu_5 & \mu_6 & \mu_7 & -\mu_8 & 0 \\ -\mu_5 & 0 & \mu_8 & \mu_6 & 0 & -\mu_7 \\ -\mu_6 & -\mu_8 & 0 & -\mu_5 & -\mu_7 & 0 \\ -\mu_7 & -\mu_6 & \mu_5 & 0 & 0 & -\mu_8 \\ \mu_8 & 0 & \mu_7 & 0 & 0 & 0 \\ 0 & \mu_7 & 0 & \mu_8 & 0 & 0 \end{pmatrix}$$

and has determinant $\det(P) = (\mu_7^3 - \mu_8^3)^2$. Denote $\mu := \frac{\mu_7}{\mu_8}$, then we have that $\det(P) = 0$ if $\mu^3 - 1 = (\mu - 1)(\mu^2 + \mu + 1) = 0$. Hence, the importance and relevance of the polynomial $X^3 - 1$ is already indicated by the determinant of the matrix pencil.

Proposition 5.2.1. *Let $\mathfrak{g}$ be the Dixmier-Lister Lie algebra over an arbitrary field $\mathbb{F}$.*

- *If $X^3 - 1$ has three different roots over $\mathbb{F}$, then $\mathrm{AID}(\mathfrak{g}) = \mathrm{Inn}(\mathfrak{g})$.*
- *If $X^3 - 1$ only has one (not necessarily simple) root over $\mathbb{F}$, then we have $\mathrm{AID}(\mathfrak{g}) = \mathrm{Inn}(\mathfrak{g}) \oplus \langle D_1, \ldots, D_4 \rangle$.*

Proof. Suppose that $\mathrm{char}(\mathbb{F}) \in \{2, 3\}$ and take $d_1, \ldots, d_6 \in \mathbb{F}$ arbitrarily. Let $\sum_{i=1}^{6} d_i D_i + E$ be an almost inner derivation, where we take

$$E \in \langle D_i^{(2)} : \mathfrak{g} \to \mathfrak{g} \mid 1 \le i \le 2 \rangle$$

when $\mathrm{char}(\mathbb{F}) = 2$ and $E \in \langle D_i^{(3)} : \mathfrak{g} \to \mathfrak{g} \mid 1 \le i \le 4 \rangle$ for $\mathrm{char}(\mathbb{F}) = 3$. Since $E(\mathfrak{g}) \not\subseteq [\mathfrak{g}, \mathfrak{g}]$ when $E \neq 0$, this implies that $E = 0$. Hence, it suffices for the rest of the proof to only look at the space $\langle D_1, \ldots, D_6 \rangle$ to determine $\mathrm{AID}(\mathfrak{g})$.

Let $z \in \mathbb{F}$ be a solution of $X^3 - 1 = 0$, so $z^3 = 1$. This exists over any field $\mathbb{F}$, since we can take $z = 1$. Consider $D := \sum_{i=1}^{6} d_i D_i$, where $d_1, \ldots, d_6 \in \mathbb{F}$. Take an arbitrary $x = \sum_{i=1}^{8} x_i e_i \in \mathfrak{g}$ and suppose that D is almost inner. By definition of D, we have

$$D(x) = ((d_1 + d_2)x_1 + d_3 x_4 + d_4 x_2 + d_5 x_3)e_7 + (d_1 x_3 - d_2 x_1 + (d_3 - d_4)x_2 + d_6 x_4)e_8.$$

This means that

$$D(e_1 - ze_3 - z^2e_6) = (d_1 + d_2 - zd_5)e_7 + (-d_2 - zd_1)e_8. \tag{5.5}$$

We further see that

$$\begin{aligned}&[e_1 - ze_3 - z^2e_6, x] \\ &\qquad = (x_2 + zx_4)e_5 + (zx_1 + x_3)e_6 + (zx_2 - z^2x_4 - x_5)(-ze_7 + e_8),\end{aligned}$$

so the coordinates of e_7 and e_8 are the same up to a factor $-z$ for every almost inner derivation. Applying this observation in (5.5) yields

$$d_1 + d_2 - zd_5 = zd_2 + z^2d_1. \tag{5.6}$$

For $z = 1$, this gives $d_5 = 0$. A similar computation shows that

$$D(e_2 - ze_4 - z^2e_5) = (-zd_3 + d_4)e_7 + (d_3 - d_4 - zd_6)e_8. \tag{5.7}$$

We also find that

$$\begin{aligned}&[e_2 + ze_4 - z^2e_5, x] \\ &\qquad = (-x_1 - zx_3)e_5 + (-zx_2 + x_4)e_6 + (zx_1 - z^2x_3 - x_6)(e_7 - ze_8).\end{aligned}$$

Since the coordinates for e_7 and e_8 are the same up to a factor $-z$, equation (5.7) implies that

$$d_3 - d_4 - zd_6 = z^2d_3 - zd_4. \tag{5.8}$$

For $z = 1$, it follows that $d_6 = 0$. For the rest of the proof, we will consider different cases.

- Suppose that $X^3 - 1 = (X - 1)(X^2 + X + 1) = 0$ has three different solutions over $\mathbb{F}$, so there exists $\beta \in \mathbb{F}$ such that the set of all solutions is given by $\{\beta, \beta^2 = -\beta - 1, \beta^3 = 1\}$. Note that $\operatorname{char}(\mathbb{F}) \neq 3$, since otherwise $X^3 - 1 = (X - 1)^3$ only has one solution (of multiplicity 3). We can insert $z = \beta$ respectively $z = \beta^2$ in (5.6) and obtain

$$\begin{aligned} d_1 + d_2 &= \beta d_2 + \beta^2 d_1 \\ d_1 + d_2 &= \beta^2 d_2 + \beta d_1. \end{aligned} \tag{5.9}$$

  From these equations, we find that $\beta d_2 + \beta^2 d_1 = \beta^2 d_2 + \beta d_1$ and this means $(\beta^2 - \beta)(d_1 - d_2) = 0$. By using in (5.9) that $d_1 = d_2$, we obtain $2d_1 = (\beta^2 + \beta)d_1 = -d_1$. We conclude that $d_1 = d_2 = 0$, since $\operatorname{char}(\mathbb{F}) \neq 3$.

Similarly, by applying $z = \beta$ respectively $z = \beta^2$ in (5.8), we obtain

$$d_3 - d_4 = \beta^2 d_3 - \beta d_4 \tag{5.10}$$

$$d_3 - d_4 = \beta d_3 - \beta^2 d_4.$$

The last two equations imply that $\beta^2 d_3 - \beta d_4 = \beta d_3 - \beta^2 d_4$ and thus $(\beta^2 - \beta)(d_3 + d_4) = 0$. We insert in (5.10) the fact that $d_4 = -d_3$ and find that $2d_3 = (\beta^2 + \beta)d_3 = -d_3 = 0$. Since $\mathrm{char}(\mathbb{F}) \neq 3$, we have that $d_3 = d_4 = 0$.

As a conclusion, we see that, when $X^3 - 1 = 0$ has three different solutions over $\mathbb{F}$, then the derivation $D = \sum_{i=1}^{6} d_i D_i$ is almost inner if and only if $d_1 = \cdots = d_6 = 0$.

- Suppose that $X^3 - 1 = 0$ only has one root over $\mathbb{F}$. On the one hand, it is possible that $X^3 - 1 = 0$ has a multiple root 1, which occurs if and only if $\mathrm{char}(\mathbb{F}) = 3$. On the other hand, when $\mathrm{char}(\mathbb{F}) \neq 3$, this means that $X^2 + X + 1 = 0$ is irreducible over $\mathbb{F}$. In both cases, it turns out that there are non-inner almost inner derivations. We will show for each $1 \leq i \leq 4$ that D_i is almost inner by giving a map $\varphi_{D_i} : \mathfrak{g} \to \mathfrak{g}$ such that $D_i(x) = [x, \varphi_{D_i}(x)]$ for every $x \in \mathfrak{g}$.

  Consider the map $\varphi_{D_1} : \mathfrak{g} \to \mathfrak{g}$, where the image of $x = \sum_{i=1}^{8} x_i e_i$ is given by

$$\varphi_{D_1}(x) = \begin{cases} e_5 & \text{if } x_1 + x_3 = 0, \\ \psi_1(x) & \text{if } x_3 \neq -x_1 \text{ and } x_1x_2 \neq x_3x_4, \\ \psi_2(x) & \text{otherwise,} \end{cases}$$

  where $\psi_1(x) = \frac{1}{x_1x_2 - x_3x_4}((x_1x_4 - x_2x_3)e_5 + (x_3^2 - x_1^2)e_6)$ and where $\psi_2(x)$ is defined as

$$\frac{1}{x_1^2 - x_1x_3 + x_3^2}(x_3(x_1 - x_3)e_2 + x_1(x_1 - x_3)e_4 + (x_1(x_6 - x_3) - x_3x_6)e_5).$$

  Note that ψ_2 is well-defined when $X^2 + X + 1 = 0$ has no solutions over $\mathbb{F}$. For $\mathrm{char}(\mathbb{F}) = 3$, we have $x_1^2 - x_1x_3 + x_3^2 = (x_1 + x_3)^2$, but $x_3 = -x_1$ is treated in another case. We will show that $D_1(x) = [x, \varphi_{D_1}(x)]$ for all $x \in \mathfrak{g}$. Consider an arbitrary $x = \sum_{i=1}^{8} x_i e_i$ in $\mathfrak{g}$. Therefore, we distinguish different cases. If $x_1 + x_3 = 0$, then

$$\begin{aligned}[x, \varphi_{D_1}(x)] &= x_1[e_1, e_5] + x_3[e_3, e_5] \\ &= -x_1e_8 - x_3e_7 \\ &= x_1e_7 + x_3e_8.\end{aligned}$$

When $x_3 \neq -x_1$ and $x_1x_2 \neq x_3x_4$, we have

$$\begin{aligned}
[x, \varphi_{D_1}(x)] = \frac{1}{x_1x_2 - x_3x_4}&(x_1(x_1x_4 - x_2x_3)[e_1, e_5] + x_2(x_3^2 - x_1^2)[e_2, e_6] \\
&+ x_3(x_1x_4 - x_2x_3)[e_3, e_5] + x_4(x_3^2 - x_1^2)[e_4, e_6]) \\
= \frac{1}{x_1x_2 - x_3x_4}&(-x_1(x_1x_4 - x_2x_3)e_8 - x_2(x_3^2 - x_1^2)e_7 \\
&- x_3(x_1x_4 - x_2x_3)e_7 - x_4(x_3^2 - x_1^2)e_8) \\
= \frac{1}{x_1x_2 - x_3x_4}&((-x_2x_3^2 + x_1^2x_2 - x_1x_3x_4 + x_2x_3^2)e_7 \\
&+ (-x_1^2x_4 + x_1x_2x_3 - x_3^2x_4 + x_1^2x_4)e_8) \\
= \frac{1}{x_1x_2 - x_3x_4}&(x_1x_2 - x_3x_4)(x_1e_7 + x_3e_8) \\
= x_1e_7 + x_3e_8&.
\end{aligned}$$

If $x_3 \neq -x_1$ and $x_1x_2 = x_3x_4$, then $[x, \varphi_{D_1}(x)] = [x, \psi_2(x)]$ is given by

$$\begin{aligned}
[x, \psi_2(x)] = \frac{1}{x_1^2 - x_1x_3 + x_3^2}&(x_1x_3(x_1 - x_3)[e_1, e_2] + x_1^2(x_1 - x_3)[e_1, e_4] \\
&+ x_1(x_1x_6 - x_1x_3 - x_3x_6)[e_1, e_5] + x_1x_2(x_1 - x_3)[e_2, e_4] \\
&+ x_3^2(x_1 - x_3)[e_3, e_2] + x_1x_3(x_1 - x_3)[e_3, e_4] \\
&+ x_3(x_1x_6 - x_1x_3 - x_3x_6)[e_3, e_5] + x_3x_4(x_1 - x_3)[e_4, e_2] \\
&+ x_3x_6(x_1 - x_3)[e_6, e_2] + x_1x_6(x_1 - x_3)[e_6, e_4]) \\
= \frac{1}{x_1^2 - x_1x_3 + x_3^2}&(x_1x_3(x_1 - x_3)e_5 + x_1^2(x_1 - x_3)e_7 \\
&- x_1(x_1x_6 - x_1x_3 - x_3x_6)e_8 + x_1x_2(x_1 - x_3)e_6 \\
&- x_3^2(x_1 - x_3)e_8 - x_1x_3(x_1 - x_3)e_5 \\
&- x_3(x_1x_6 - x_1x_3 - x_3x_6)e_7 - x_3x_4(x_1 - x_3)e_6 \\
&+ x_3x_6(x_1 - x_3)e_7 + x_1x_6(x_1 - x_3)e_8)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{x_1^2 - x_1x_3 + x_3^2}((x_1x_3 - x_1x_3)(x_1 - x_3)e_5 \\
&\quad + (x_1x_2 - x_3x_4)(x_1 - x_3)e_6 \\
&\quad + (x_1^3 - x_1^2x_3 - x_1x_3x_6 + x_1x_3^2 + x_3^2x_6 + x_1x_3x_6 - x_3^2x_6)e_7 \\
&\quad + (-x_1^2x_6 + x_1^2x_3 + x_1x_3x_6 - x_1x_3^2 + x_3^3 + x_1^2x_6 - x_1x_3x_6)e_8) \\
&= \frac{1}{x_1^2 - x_1x_3 + x_3^2}(x_1^2 - x_1x_3 + x_3^2)(x_1e_7 + x_3e_8) \\
&= x_1e_7 + x_3e_8.
\end{aligned}
$$

Consider further the map

$$
\varphi_{D_2}(x) = \begin{cases} e_5 & \text{if } x_1 + x_3 = 0, \\ \psi_3(x) & \text{if } x_3 \neq -x_1 \text{ and } x_1x_2 \neq x_3x_4, \\ \psi_4(x) & \text{otherwise,} \end{cases}
$$

where $\psi_3(x)$ and $\psi_4(x)$ are defined as

$$
\psi_3(x) = \frac{x_1}{x_1x_2 - x_3x_4}((x_2 + x_4)e_5 - (x_1 + x_3)e_6),
$$

$$
\psi_4(x) = \frac{x_1}{x_1^2 - x_1x_3 + x_3^2}(x_3e_2 + x_1e_4 + (x_1 - x_3 + x_6)e_5).
$$

Let $x = \sum_{i=1}^{8} x_ie_i$ be arbitrary. When $x_1 + x_3 = 0$, then

$$
\begin{aligned}
[x, \varphi_{D_2}(x)] &= x_1[e_1, e_5] + x_3[e_3, e_5] \\
&= -x_1e_8 - x_3e_7 \\
&= x_1(e_7 - e_8).
\end{aligned}
$$

If $x_3 \neq -x_1$ and $x_1x_2 - x_3x_4 \neq 0$, then we have

$$
\begin{aligned}
[x, \varphi_{D_2}(x)] &= \frac{x_1}{x_1x_2 - x_3x_4}(x_1(x_2 + x_4)[e_1, e_5] - x_2(x_1 + x_3)[e_2, e_6] \\
&\quad + x_3(x_2 + x_4)[e_3, e_5] - x_4(x_1 + x_3)[e_4, e_6]) \\
&= \frac{x_1}{x_1x_2 - x_3x_4}(-x_1(x_2 + x_4)e_8 + x_2(x_1 + x_3)e_7 \\
&\quad - x_3(x_2 + x_4)e_7 + x_4(x_1 + x_3)e_8).
\end{aligned}
$$

By rearranging the different terms, we find that

$$
\begin{aligned}
[x, \varphi_{D_2}(x)] &= \frac{x_1}{x_1x_2 - x_3x_4}((x_1x_2 + x_2x_3 - x_2x_3 - x_3x_4)e_7 \\
&\qquad + (-x_1x_2 - x_1x_4 + x_1x_4 + x_3x_4)e_8) \\
&= \frac{x_1}{x_1x_2 - x_3x_4}(x_1x_2 - x_3x_4)(e_7 - e_8) \\
&= x_1(e_7 - e_8). \qquad (5.11)
\end{aligned}
$$

Otherwise, for $x_3 \neq -x_1$ and $x_1x_2 = x_3x_4$, it follows that

$$
\begin{aligned}
[x, \varphi_{D_2}(x)] &= \frac{x_1}{x_1^2 - x_1x_3 + x_3^2}(x_1x_3[e_1, e_2] + x_1^2[e_1, e_4] \\
&\qquad + (x_1^2 - x_1x_3 + x_1x_6)[e_1, e_5] + x_1x_2[e_2, e_4] + x_3^2[e_3, e_2] \\
&\qquad + x_1x_3[e_3, e_4] + (x_1x_3 - x_3^2 + x_3x_6)[e_3, e_5] \\
&\qquad + x_3x_4[e_4, e_2] + x_3x_6[e_6, e_2] + x_1x_6[e_6, e_4]) \\
&= \frac{x_1}{x_1^2 - x_1x_3 + x_3^2}(x_1x_3e_5 + x_1^2e_7 - (x_1^2 - x_1x_3 + x_1x_6)e_8 \\
&\qquad + x_1x_2e_6 - x_3^2e_8 - x_1x_3e_5 - (x_1x_3 - x_3^2 + x_3x_6)e_7 \\
&\qquad - x_3x_4e_6 + x_3x_6e_7 + x_1x_6e_8) \\
&= \frac{x_1}{x_1^2 - x_1x_3 + x_3^2}((x_1x_3 - x_1x_3)e_5 + (x_1x_2 - x_3x_4)e_6 \\
&\qquad + (x_1^2 - x_1x_3 + x_3^2 - x_3x_6 + x_3x_6)e_7 \\
&\qquad + (-x_1^2 + x_1x_3 - x_1x_6 - x_3^2 + x_1x_6)e_8) \\
&= \frac{x_1}{x_1^2 - x_1x_3 + x_3^2}(x_1^2 - x_1x_3 + x_3^2)(e_7 - e_8) \\
&= x_1(e_7 - x_8).
\end{aligned}
$$

This shows that $D_2(x) = [x, \varphi_{D_2}(x)]$ for all $x \in \mathfrak{g}$ and $D_2 : \mathfrak{g} \to \mathfrak{g}$ is an almost inner derivation.

Consider the map $\varphi_{D_3} : \mathfrak{g} \to \mathfrak{g} : x = \sum_{i=1}^{8} x_i e_i \mapsto \varphi_{D_3}(x)$, with

$$\varphi_{D_3}(x) = \begin{cases} e_6 & \text{if } x_2 + x_4 = 0, \\ \psi_5(x) & \text{if } x_4 \neq -x_2 \text{ and } x_1x_2 \neq x_3x_4, \\ \psi_6(x) & \text{otherwise,} \end{cases}$$

where we have

$$\psi_5(x) = \frac{1}{x_1x_2 - x_3x_4}((x_4^2 - x_2^2)e_5 + (x_2x_3 - x_1x_4)e_6).$$

Further, $\psi_6(x)$ is defined as

$$\frac{1}{x_2^2 - x_2x_4 + x_4^2}(x_4(x_2 - x_4)e_1 + x_2(x_2 - x_4)e_3 + (x_2(x_5 - x_4) - x_4x_5)e_6).$$

Let $x = \sum_{i=1}^{8} x_i e_i$ be arbitrary. When $x_2 + x_4 = 0$, it follows that

$$\begin{aligned}[x, \varphi_{D_3}(x)] &= x_2[e_2, e_6] + x_4[e_4, e_6] \\ &= -x_2e_7 - x_4e_8 \\ &= x_4e_7 + x_2e_8.\end{aligned}$$

If $x_4 \neq -x_2$ and $x_1x_2 - x_3x_4 \neq 0$, then we have

$$\begin{aligned}[x, \varphi_{D_3}(x)] = &\frac{1}{x_1x_2 - x_3x_4}(x_1(x_4^2 - x_2^2)[e_1, e_5] + x_2(x_2x_3 - x_1x_4)[e_2, e_6] \\ &+ x_3(x_4^2 - x_2^2)[e_3, e_5] + x_4(x_2x_3 - x_1x_4)[e_4, e_6]) \\ = &\frac{1}{x_1x_2 - x_3x_4}(-x_1(x_4^2 - x_2^2)e_8 - x_2(x_2x_3 - x_1x_4)e_7 \\ &- x_3(x_4^2 - x_2^2)e_7 - x_4(x_2x_3 - x_1x_4)e_8) \\ = &\frac{1}{x_1x_2 - x_3x_4}((-x_2^2x_3 + x_1x_2x_4 - x_3x_4^2 + x_2^2x_3)e_7 \\ &+ (-x_1x_4^2 + x_1x_2^2 - x_2x_3x_4 + x_1x_4^2)e_8) \\ = &\frac{1}{x_1x_2 - x_3x_4}(x_1x_2 - x_3x_4)(x_4e_7 + x_2e_8) \\ = &x_4e_7 + x_2e_8.\end{aligned}$$

For $x_4 \neq -x_2$ and $x_1x_2 = x_3x_4$, we have that $[x, \varphi_{D_3}(x)] = [x, \psi_6(x)]$ is given by

$$\begin{aligned}
[x, \psi_6(x)] = \frac{1}{x_2^2 - x_2x_4 + x_4^2}&(x_1x_2(x_2-x_4)[e_1,e_3] + x_2x_4(x_2-x_4)[e_2,e_1] \\
&+ x_2^2(x_2-x_4)[e_2,e_3] + x_2(-x_2x_4 + x_2x_5 - x_4x_5)[e_2,e_6] \\
&+ x_3x_4(x_2-x_4)[e_3,e_1] + x_4^2(x_2-x_4)[e_4,e_1] \\
&+ x_2x_4(x_2-x_4)[e_4,e_3] + x_4(x_2x_5 - x_2x_4 - x_4x_5)[e_4,e_6] \\
&+ x_4x_5(x_2-x_4)[e_5,e_1] + x_2x_5(x_2-x_4)[e_5,e_3]) \\
= \frac{1}{x_2^2 - x_2x_4 + x_4^2}&(x_1x_2(x_2-x_4)e_6 - x_2x_4(x_2-x_4)e_5 \\
&+ x_2^2(x_2-x_4)e_8 - x_2(-x_2x_4 + x_2x_5 - x_4x_5)e_7 \\
&- x_3x_4(x_2-x_4)e_6 - x_4^2(x_2-x_4)e_7 + x_2x_4(x_2-x_4)e_5 \\
&- x_4(-x_2x_4 + x_2x_5 - x_4x_5)e_8 + x_4x_5(x_2-x_4)e_8 \\
&+ x_2x_5(x_2-x_4)e_7) \\
= \frac{1}{x_2^2 - x_2x_4 + x_4^2}&((-x_2x_4 + x_2x_4)(x_2-x_4)e_5 \\
&+ (x_1x_2 - x_3x_4)(x_2-x_4)e_6 \\
&+ (x_2^2x_4 - x_2^2x_5 + x_2x_4x_5 - x_2x_4^2 + x_4^3 + x_2^2x_5 - x_2x_4x_5)e_7 \\
&+ (x_2^3 - x_2^2x_4 + x_2x_4^2 - x_2x_4x_5 + x_4^2x_5 + x_2x_4x_5 - x_4^2x_5)e_8) \\
= \frac{1}{x_2^2 - x_2x_4 + x_4^2}&(x_2^2 - x_2x_4 + x_4^2)(x_4e_7 + x_2e_8) \\
= x_4e_7 + x_2e_8&.
\end{aligned}$$

This shows that $D_3 : \mathfrak{g} \to \mathfrak{g}$ is almost inner.

Consider the map $\varphi_{D_4} : \mathfrak{g} \to \mathfrak{g}$, where the image of $x = \sum_{i=1}^{8} x_ie_i$ is given by

$$\varphi_{D_4}(x) = \begin{cases} -e_6 & \text{if } x_2 + x_4 = 0, \\ \psi_7(x) & \text{if } x_4 \neq -x_2 \text{ and } x_1x_2 \neq x_3x_4, \\ \psi_8(x) & \text{otherwise,} \end{cases}$$

where $\psi_7(x)$ and $\psi_8(x)$ are defined as

$$\psi_7(x) = \frac{x_2}{x_1x_2 - x_3x_4}((x_2 + x_4)e_5 - (x_1 + x_3)e_6),$$

$$\psi_8(x) = \frac{x_2}{x_2^2 - x_2x_4 + x_4^2}(-x_4e_1 - x_2e_3 + (-x_2 + x_4 - x_5)e_6).$$

Take an arbitrary $x = \sum_{i=1}^8 x_ie_i$. If $x_4 = -x_2$, then

$$\begin{aligned}[x, \varphi_{D_4}(x)] &= -x_2[e_2, e_6] - x_4[e_4, e_6] \\ &= x_2e_7 + x_4e_8 = x_2(e_7 - e_8).\end{aligned}$$

When $x_4 \neq -x_2$ and $x_1x_2 - x_3x_4 \neq 0$, it follows from a similar computation as in (5.11) that

$$\begin{aligned}[x, \varphi_{D_4}(x)] &= \frac{x_2}{x_1x_2 - x_3x_4}[x, (x_2 + x_4)e_5 - (x_1 + x_3)e_6] \\ &= x_2(e_7 - e_8).\end{aligned}$$

For $x_4 \neq -x_2$ and $x_1x_2 = x_3x_4$, we find that

$$\begin{aligned}[x, \varphi_{D_4}(x)] &= \frac{x_2}{x_2^2 - x_2x_4 + x_4^2}(-x_1x_2[e_1, e_3] - x_2x_4[e_2, e_1] - x_2^2[e_2, e_3] \\ &\quad + x_2(-x_2 + x_4 - x_5)[e_2, e_6] - x_3x_4[e_3, e_1] - x_4^2[e_4, e_1] \\ &\quad - x_2x_4[e_4, e_3] + x_4(-x_2 + x_4 - x_5)[e_4, e_6] - x_4x_5[e_5, e_1] \\ &\quad - x_2x_5[e_5, e_3]) \\ &= \frac{x_2}{x_2^2 - x_2x_4 + x_4^2}(-x_1x_2e_6 + x_2x_4e_5 - x_2^2e_8 \\ &\quad - x_2(-x_2 + x_4 - x_5)e_7 + x_3x_4e_6 + x_4^2e_7 - x_2x_4e_5 \\ &\quad - x_4(-x_2 + x_4 - x_5)e_8 - x_4x_5e_8 - x_2x_5e_7) \\ &= \frac{x_2}{x_2^2 - x_2x_4 + x_4^2}((x_2x_4 - x_2x_4)e_5 + (-x_1x_2 + x_3x_4)e_6 \\ &\quad + (x_2^2 - x_2x_4 + x_2x_5 + x_4^2 - x_2x_5)e_7 \\ &\quad + (-x_2^2 + x_2x_4 - x_4^2 + x_4x_5 - x_4x_5)e_8) \\ &= \frac{x_2}{x_2^2 - x_2x_4 + x_4^2}(x_2^2 - x_2x_4 + x_4^2)(e_7 - e_8).\end{aligned}$$

Since $D_4(x) = [x, \varphi_{D_4}(x)]$ holds for all $x \in \mathfrak{g}$ that, we can conclude that $D_4 : \mathfrak{g} \to \mathfrak{g}$ is an almost inner derivation as well. $\square$

Let $\mathfrak{g}$ be the Dixmier-Lister Lie algebra over $\mathbb{F}$. Then all almost inner derivations are inner when $\mathbb{F} = \mathbb{C}$, whereas we have $\dim(\mathrm{AID}(\mathfrak{g})) = \dim(\mathrm{Inn}(\mathfrak{g})) + 4$ for $\mathbb{F} = \mathbb{R}$. Hence, the dimension of $\mathrm{AID}(\mathfrak{g})$ depends on the field $\mathbb{F}$, in particular on the number of different solutions of the polynomial $X^3 - 1 = 0$ over $\mathbb{F}$. Intuitively, this dissimilarity between the different fields can be explained as follows. The dimension of $\mathrm{AID}(\mathfrak{g})/\mathrm{Inn}(\mathfrak{g})$ is connected with the factorisation of $\det(P) = (\mu^3 - 1)^2$ over $\mathbb{F}$. Denote l for the number of different linear factors of $\det(P)$. It seems in our example that

$$\dim(\mathrm{AID}(\mathfrak{g})/\mathrm{Inn}(\mathfrak{g})) = \deg(\det(P)) - 2l = 2 \cdot (3 - l).$$

We will see this phenomenon again in Section 9.2 and Section 9.3 while studying matrix pencils in more detail.

Chapter 6

Finite field extensions

In this chapter, we will consider finite field extensions $K : k$. Let $\mathfrak{g}$ be a Lie algebra defined over k, then it can be considered as a Lie algebra over K as well. We will use the notations $\mathfrak{g}_k$ respectively $\mathfrak{g}_K$ to specify the Lie algebra we are working with. In the first section, we investigate the connection between $\mathrm{AID}(\mathfrak{g}_k)$ and $\mathrm{AID}(\mathfrak{g}_K)$ and introduce the so-called 'underlying Lie algebras' $\mathfrak{g}'_k$ and $\mathfrak{g}'_K$. In Section 6.2, we give a way to find new almost inner derivations of $\mathfrak{g}'_k$ on the basis of $\mathrm{AID}(\mathfrak{g}_K)$. Most of this chapter also appeared in [8]. Since we work with field extensions, it is convenient to use the notation $K : k$. This is in contrast to other chapters, where a field is denoted with $\mathbb{F}$.

6.1 Change of base field

Consider a separable field extension of finite degree $[K : k] = n$. Let $\mathfrak{g}_k$ be a Lie algebra defined over k, then we obtain a Lie algebra

$$\mathfrak{g}_K = K \otimes_k \mathfrak{g}_k$$

over K by linearly extending the Lie bracket. The primitive element theorem ensures that $K = k(s)$ for some $s \in K$. Then $\mathcal{B} = \{1, s, s^2, \dots, s^{n-1}\}$ is a vector space basis of K over k. It follows that

$$\mathfrak{g}_K = \mathfrak{g}_k \oplus s\mathfrak{g}_k \oplus \cdots \oplus s^{n-1}\mathfrak{g}_k \tag{6.1}$$

holds as vector spaces over k. The typical example is $K = \mathbb{C}$ and $k = \mathbb{R}$, where $\{1, s\} = \{1, i\}$ and $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g}_{\mathbb{R}} \oplus i\mathfrak{g}_{\mathbb{R}}$.

We can also consider $\mathfrak{g}_K$ as a Lie algebra over k, which we will denote with $\mathfrak{g}'_k$. Note that, as sets, we have $\mathfrak{g}'_k = \mathfrak{g}_K$, but $\dim_k(\mathfrak{g}'_k) = n \cdot \dim_K(\mathfrak{g}_K)$. Finally, $\mathfrak{g}'_K := K \otimes_k \mathfrak{g}'_k$ is again a Lie algebra over K, with $\dim_K(\mathfrak{g}'_K) = \dim_k(\mathfrak{g}'_k)$. The Lie algebra $\mathfrak{g}'_K$ has the same basis and structure constants as $\mathfrak{g}'_k$, but is a Lie algebra over K and not over k. The following property, which follows from the results of [21], relates the Lie algebras $\mathfrak{g}'_K$ and $\mathfrak{g}_K$.

Proposition 6.1.1. *Consider a finite Galois extension $k \subseteq K$ with $[K : k] = n$. Let $\mathfrak{g}_k$ be a Lie algebra over k. Using the notations introduced above, we have that $\mathfrak{g}'_K \cong \bigoplus_n \mathfrak{g}_K$.*

We will explain the constructions of these different Lie algebras for the special situation when $[K : k] = 2$, where we require that the fields do not have characteristic two. In that case, we have $K = k(s)$ for some $s \in K \setminus k$ with $d := s^2 \in k$. Hence, we can write $K = k(\sqrt{d})$. Let $\mathfrak{g}_k$ be a Lie algebra over k of dimension r, then there exists a basis $\{e_1, e_2, \ldots, e_r\}$ and there are structure constants c_{ij}^p such that $[e_i, e_j] = \sum_{p=1}^r c_{ij}^p e_p$. The Lie algebra $\mathfrak{g}_K = K \otimes_k \mathfrak{g}_k$ has the same structure constants and the same basis. Further, $\mathfrak{g}'_k$ has basis $\{e_1, e_2, \ldots, e_r, se_1, se_2, \ldots, se_r\}$. Denote $f_i := se_i$ for all $1 \leq i \leq r$, then the structure constants are

$$[e_i, e_j] = \sum_{p=1}^r c_{ij}^p e_p, \qquad [e_i, f_j] = \sum_{p=1}^r c_{ij}^p f_p, \qquad [f_i, f_j] = d \sum_{p=1}^r c_{ij}^p e_p = d[e_i, e_j].$$

In general, it is cumbersome to construct an explicit isomorphism between $\mathfrak{g}'_K$ and $\bigoplus_n \mathfrak{g}_K$, but for $n = 2$, it is straightforward. The Lie algebra $\mathfrak{g}'_K$ has basis $\{e_1, e_2, \ldots, e_r, f_1, f_2, \ldots, f_r\}$ and structure constants as above. We take for $\mathfrak{g}_K \oplus \mathfrak{g}_K$ a basis $\{a_1, a_2, \ldots, a_r, b_1, b_2, \ldots, b_r\}$ with

$$[a_i, a_j] = \sum_{p=1}^r c_{ij}^p a_p, \qquad [b_i, b_j] = \sum_{p=1}^r c_{ij}^p b_p, \qquad [a_i, b_j] = 0.$$

Let $\varphi : \mathfrak{g}'_K \to \mathfrak{g}_K \oplus \mathfrak{g}_K$ be the linear map with $\varphi(e_i) = a_i + b_i$ and $\varphi(f_i) = sa_i - sb_i$ for all $1 \leq i \leq r$. Then φ is an isomorphism of vector spaces. Moreover, φ is a Lie algebra morphism. Indeed, take $1 \leq i, j \leq r$ arbitrarily, then we have that

$$\begin{aligned}
[\varphi(e_i), \varphi(e_j)] &= [a_i + b_i, a_j + b_j] \\
&= [a_i, a_j] + [b_i, b_j] \\
&= \sum_{p=1}^r c_{ij}^p (a_p + b_p) \\
&= \varphi([e_i, e_j]).
\end{aligned} \tag{6.2}$$

Furthermore, also

$$
\begin{aligned}
[\varphi(e_i), \varphi(f_j)] &= [a_i + b_i, sa_j - sb_j] \\
&= s \sum_{p=1}^{r} c_{ij}^p (a_p - b_p) \\
&= \varphi\left(\sum_{p=1}^{r} c_{ij}^p f_p\right) \\
&= \varphi([e_i, f_j])
\end{aligned}
$$

is satisfied. Using (6.2), we find that

$$
\begin{aligned}
[\varphi(f_i), \varphi(f_j)] &= [s(a_i - b_i), s(a_j - b_j)] = s^2[a_i, a_j] + s^2[b_i, b_j] \\
&= d\left([a_i, a_j] + [b_i, b_j]\right) = \varphi(d[e_i, e_j]) \\
&= \varphi([f_i, f_j])
\end{aligned}
$$

and this finalises the proof. This will be illustrated in the next example for $k = \mathbb{R}$ and $K = \mathbb{C}$. Then we can take $s = i$ with $i^2 = -1$.

Example 6.1.2. Consider the real Heisenberg Lie algebra $\mathfrak{g}_{\mathbb{R}}$, then $\mathfrak{g}_{\mathbb{C}}$ is the complex Heisenberg Lie algebra. Both of these algebras can be described via a basis $\{e_1, e_2, e_3\}$, where the non-zero Lie brackets are given by $[e_1, e_2] = e_3$. The Lie algebras $\mathfrak{g}'_{\mathbb{R}}$ and $\mathfrak{g}'_{\mathbb{C}}$ have a basis $\{e_1, e_2, e_3, f_1 = ie_1, f_2 = ie_2, f_3 = ie_3\}$ and non-zero brackets

$$
\begin{aligned}
&[e_1, e_2] = e_3, \quad [e_1, f_2] = f_3, \\
&[f_1, e_2] = f_3, \quad [f_1, f_2] = -e_3.
\end{aligned}
$$

Further, we have that $\mathfrak{g}'_{\mathbb{C}} \cong \mathfrak{g}_{\mathbb{C}} \oplus \mathfrak{g}_{\mathbb{C}}$ by Proposition 6.1.1. However, as we will show later on, $\mathfrak{g}'_{\mathbb{R}}$ and $\mathfrak{g}_{\mathbb{R}} \oplus \mathfrak{g}_{\mathbb{R}}$ are not isomorphic.

Now, we return to the general situation where $[K : k] = n \geq 2$. Suppose that $D \in \mathrm{Der}(\mathfrak{g}_k)$, then we can consider $D_K = 1_K \otimes_k D$ where $D_K : \mathfrak{g}_K \to \mathfrak{g}_K$ is the K–linear map such that $(D_K)_{|\mathfrak{g}_k} = D$. This means that $D_K \in \mathrm{Der}(\mathfrak{g}_K)$. Conversely, if $D \in \mathrm{Der}(\mathfrak{g}_K)$ and $D(\mathfrak{g}_k) \subseteq \mathfrak{g}_k$, then $D_{|\mathfrak{g}_k} \in \mathrm{Der}(\mathfrak{g}_k)$.

Remark 6.1.3. These two "procedures" are inverses of each other. Indeed, for $D \in \mathrm{Der}(\mathfrak{g}_k)$, we have that $(D_K)_{|\mathfrak{g}_k} = D$. Moreover, for $D \in \mathrm{Der}(\mathfrak{g}_K)$ with $D(\mathfrak{g}_k) \subseteq \mathfrak{g}_k$ we have $\left(D_{|\mathfrak{g}_k}\right)_K = D$.

Lemma 6.1.4. *We have* $\mathrm{Der}(\mathfrak{g}_K) = K \otimes_k \mathrm{Der}(\mathfrak{g}_k)$.

Proof. We already mentioned that any derivation $D \in \mathrm{Der}(\mathfrak{g}_k)$ can be viewed as a derivation $D_K \in \mathrm{Der}(\mathfrak{g}_K)$. From this, the conclusion $K \otimes_k \mathrm{Der}(\mathfrak{g}_k) \subseteq \mathrm{Der}(\mathfrak{g}_K)$ is clear. We now show the other inclusion. Let $D \in \mathrm{Der}(\mathfrak{g}_K)$. We can write

$$D_{|\mathfrak{g}_k} = D_1 + sD_2 + \cdots + s^{n-1}D_n,$$

where $D_i : \mathfrak{g}_k \to \mathfrak{g}_k$ is a derivation for all $1 \leq i \leq n$. Take

$$D' = 1_K \otimes_k D_1 + s \otimes_k D_2 + \cdots + s^{n-1} \otimes_k D_n,$$

then we find that $D' \in K \otimes_k \mathrm{Der}(\mathfrak{g}_k)$. Since also $D_{|\mathfrak{g}_k} = D'_{|\mathfrak{g}_k}$ holds, this implies that $D = D' \in K \otimes_k \mathrm{Der}(\mathfrak{g}_k)$. $\square$

Hence, there is a nice correspondence between the derivations of $\mathfrak{g}_k$ and $\mathfrak{g}_K$. For almost inner derivations, there is a partial result.

Lemma 6.1.5. *Let $D \in \mathrm{Der}(\mathfrak{g}_k)$. If $D_K \in \mathrm{AID}(\mathfrak{g}_K)$, then also $D \in \mathrm{AID}(\mathfrak{g}_k)$.*

Proof. Let $\mathcal{B} = \{1, s, \dots, s^{n-1}\}$ be a basis of K over k and $D : \mathfrak{g}_k \to \mathfrak{g}_k$ be a derivation. Assume that $D_K \in \mathrm{AID}(\mathfrak{g}_K)$, then there exists a map $\varphi : \mathfrak{g}_K \to \mathfrak{g}_K$ such that $D_K(x) = [x, \varphi(x)]$ holds for all $x \in \mathfrak{g}_K$. We can write

$$\varphi(x) := \varphi_1(x) + s\varphi_2(x) + \cdots + s^{n-1}\varphi_n(x),$$

where $\varphi_i : \mathfrak{g}_K \to \mathfrak{g}_k$ for all $1 \leq i \leq n$. Take an arbitrary $x \in \mathfrak{g}_k$. Then we obtain

$$D(x) = [x, \varphi_1(x)] + s[x, \varphi_2(x)] + \cdots + s^{n-1}[x, \varphi_n(x)].$$

Since $D(x) \in \mathfrak{g}_k$, it follows from equation (6.1) that for all $x \in \mathfrak{g}_k$, we have

$$D(x) = [x, \varphi_1(x)] \in \mathfrak{g}_k$$

and $[x, \varphi_i(x)] = 0$ for all $2 \leq i \leq n$. Hence, this means that $D \in \mathrm{AID}(\mathfrak{g}_k)$. $\square$

Note that the converse of this result does not hold in general, since there exist examples for which $D \in \mathrm{AID}(\mathfrak{g}_k)$ holds, but $D_K \notin \mathrm{AID}(\mathfrak{g}_K)$. This phenomenon will be illustrated in Example 6.2.4.

Proposition 6.1.6. *If $\mathrm{AID}(\mathfrak{g}_K) \neq \mathrm{Inn}(\mathfrak{g}_K)$, then also $\mathrm{AID}(\mathfrak{g}_k) \neq \mathrm{Inn}(\mathfrak{g}_k)$.*

Proof. Denote as before $\mathcal{B} = \{1, s, \dots, s^{n-1}\}$ for a basis of K over k. Let $D \in \mathrm{AID}(\mathfrak{g}_K)$ with $D \notin \mathrm{Inn}(\mathfrak{g}_K)$, then there exists a map $\varphi : \mathfrak{g}_K \to \mathfrak{g}_K$ such that $D(x) = [x, \varphi(x)]$ for all $x \in \mathfrak{g}_K$. Furthermore, there are maps $\varphi_i : \mathfrak{g}_K \to \mathfrak{g}_k$ (for all $1 \leq i \leq n$) such that

$$\varphi = \varphi_1 + s\varphi_2 + \cdots + s^{n-1}\varphi_n.$$

Define for each $1 \le i \le n$ the map

$$D_i : \mathfrak{g}_k \to \mathfrak{g}_k : x \mapsto [x, \varphi_i(x)].$$

We claim that each D_i is a derivation (and thus an almost inner derivation). Let $x, y \in \mathfrak{g}_k$, then

$$\begin{aligned} D([x,y]) &= [[x,y], \varphi_1([x,y]) + s\varphi_2([x,y]) + \cdots + s^{n-1}\varphi_n([x,y])] \\ &= D_1([x,y]) + sD_2([x,y]) + \cdots + s^{n-1}D_n([x,y]). \end{aligned}$$

On the other hand, we have

$$\begin{aligned} &[D(x), y] + [x, D(y)] \\ &\quad = [[x, \varphi_1(x) + \cdots + s^{n-1}\varphi_n(x)], y] + [x, [y, \varphi_1(y) + \cdots + s^{n-1}\varphi_n(y)]] \\ &\quad = [D_1(x) + \cdots + s^{n-1}D_n(x), y] + [x, D_1(y) + \cdots + s^{n-1}D_n(y)] \\ &\quad = \big([D_1(x), y] + [x, D_1(y)]\big) + \cdots + s^{n-1}\big([D_n(x), y] + [x, D_n(y)]\big). \end{aligned}$$

The above equations imply that $D_i([x,y]) = [D_i(x), y] + [x, D_i(y)]$, since D is a derivation. Hence, we have $D_i \in \mathrm{Der}(\mathfrak{g}_k)$ for all $1 \le i \le n$.

Moreover, we claim that there is at least one $1 \le i \le n$ for which $D_i \notin \mathrm{Inn}(\mathfrak{g}_k)$. Suppose on the contrary that $D_i \in \mathrm{Inn}(\mathfrak{g}_k)$ for all $1 \le i \le n$. Then there exist elements $\alpha_i \in \mathfrak{g}_k$ such that $D_i(x) = [x, \alpha_i]$ holds for all $x \in \mathfrak{g}_k$. Denote $\alpha := \alpha_1 + s\alpha_2 + \cdots + s^{n-1}\alpha_n \in K$. This means that

$$\begin{aligned} D(x) &= [x, \alpha_1] + s[x, \alpha_2] + \cdots + s^{n-1}[x, \alpha_n] \\ &= [x, \alpha_1 + s\alpha_2 + \cdots + s^{n-1}\alpha_n] \\ &= [x, \alpha] \end{aligned}$$

for all $x \in \mathfrak{g}_k$. Now consider $-\mathrm{ad}(\alpha) \in \mathrm{Der}(\mathfrak{g}_K)$, then $D_{|\mathfrak{g}_k} = -\mathrm{ad}(\alpha)_{|\mathfrak{g}_k}$. Since two derivations of $\mathfrak{g}_K$ are equal when they agree on $\mathfrak{g}_k$, this implies that $D = -\mathrm{ad}(\alpha)$ is inner. This contradiction shows that for at least one $1 \le i \le n$, we have $D_i \in \mathrm{AID}(\mathfrak{g}_k) \setminus \mathrm{Inn}(\mathfrak{g}_k)$. □

This proposition means that if the Lie algebra over the bigger field $\mathfrak{g}_K$ admits a non-trivial almost inner derivation, then also the Lie algebra over the smaller field $\mathfrak{g}_k$. The converse does not hold in general, see again Example 6.2.4.

6.2 Constructing new almost inner derivations

We keep using the same notations as in the previous section. We will show how to find new almost inner derivations of the Lie algebra $\mathfrak{g}'_k$, determined by $\mathrm{AID}(\mathfrak{g}_K)$. Remember that $\mathfrak{g}'_k$ and $\mathfrak{g}_K$ denote the same set, but they are Lie algebras over a different field (namely over k respectively over K).
Define the set

$$\mathcal{E}(\mathfrak{g}_K) := \{D \in \mathrm{AID}(\mathfrak{g}_K) \mid D(\mathfrak{g}_K) \text{ is one-dimensional and } D(\mathfrak{g}_K) \subseteq Z(\mathfrak{g}_K)\}.$$

We will construct, starting from a fixed element $D \in \mathcal{E}(\mathfrak{g}_K)$, a collection of almost inner derivations of $\mathfrak{g}'_k$ which are not inner, even when D itself is an inner derivation of $\mathfrak{g}_K$. Take an arbitrary $D \in \mathcal{E}(\mathfrak{g}_K)$. Since $D(\mathfrak{g}_K)$ is one-dimensional, $\ker(D)$ is of codimension 1. Hence, $\mathfrak{g}_K = \langle y \rangle + \ker(D)$ for some $y \in \mathfrak{g}_K \setminus \ker(D)$. We also fix a choice of y and let $0 \neq z = D(y)$. Any element of $\mathfrak{g}_K$ can be written as $ay + c$, where $a \in K$ and $c \in \ker(D)$. Denote again $\mathcal{B} = \{1, s, \dots, s^{n-1}\}$ for a basis of K over k, then any element $a \in K$ can be uniquely written as $a = a_1 + a_2 s + \cdots + a_n s^{n-1}$ with $a_i \in k$ for all $1 \leq i \leq n$. We use the notation $t_i(a) := a_i$ to denote the i–th coordinate of a with respect to the basis $\mathcal{B}$. We now have that $D : \mathfrak{g}_K \to \mathfrak{g}_K : ay + c \mapsto az$. Since $D \in \mathrm{AID}(\mathfrak{g}_K)$, there exists a map $\varphi_D : \mathfrak{g}_K \to \mathfrak{g}_K$ such that

$$D(ay + c) = [ay + c, \varphi_D(ay + c)].$$

Associated to D, we introduce n new k–linear maps of $\mathfrak{g}'_k$, namely

$$D_i : \mathfrak{g}'_k \to \mathfrak{g}'_k : ay + c \mapsto t_i(a) s^{i-1} z,$$

where $1 \leq i \leq n$. Note that $D = D_1 + D_2 + \cdots + D_n$. We remark here that the maps D_i do depend on the choice of y. In what follows, we always assume that for a given D, a fixed y outside of $\ker(D)$ has been chosen. We can now multiply the above maps with powers of s to get a total of n^2 different k–linear maps $s^{j-1} D_i : \mathfrak{g}'_k \to \mathfrak{g}'_k$, with $1 \leq i, j \leq n$.

Lemma 6.2.1. *For any $D \in \mathcal{E}(\mathfrak{g}_K)$, we have that $s^{j-1} D_i \in \mathrm{AID}(\mathfrak{g}'_k)$ for all $1 \leq i, j \leq n$.*

Proof. First note that $[\mathfrak{g}_K, \mathfrak{g}_K] \subseteq \ker(D)$. Indeed, let $x, y \in \mathfrak{g}_K$. Because $D(\mathfrak{g}_K) \subseteq Z(\mathfrak{g}_K)$ by definition, we have that $[D(x), y] + [x, D(y)] = 0$ and

$$D([x, y]) = [D(x), y] + [x, D(y)] = 0.$$

This last equation implies that $D_i([x, y]) = 0$ and hence, D_i is a derivation for all $1 \leq i \leq n$. Since D is almost inner, it is determined by a map $\varphi_D : \mathfrak{g}_K \to \mathfrak{g}_K$.

Define

$$\varphi_{D_i} : \mathfrak{g}'_k \to \mathfrak{g}'_k : ay + c \mapsto \begin{cases} 0 & \text{if } a = 0, \\ \frac{t_i(a)s^{i-1}}{a}\varphi_D(ay+c) & \text{if } a \neq 0. \end{cases}$$

For $a = 0$, we have that

$$D_i(ay + c) = 0 = [ay + c, \varphi_{D_i}(ay + c)].$$

When $a \neq 0$, it follows that

$$\begin{aligned} [ay + c, \varphi_{D_i}(ay + c)] &= \frac{t_i(a)s^{i-1}}{a}[ay + c, \varphi_D(ay + c)] \\ &= \frac{t_i(a)s^{i-1}}{a}D(ay + c) \\ &= t_i(a)s^{i-1}z \\ &= D_i(ay + c). \end{aligned}$$

This shows that $D_i \in \mathrm{AID}(\mathfrak{g}'_k)$ for all $1 \leq i \leq n$. Moreover, for each $1 \leq j \leq n$, the map $s^{j-1}D_i : \mathfrak{g}'_k \to \mathfrak{g}'_k$ is an almost inner derivation, determined by the map $\varphi_{s^{j-1}D_i} := s^{j-1}\varphi_{D_i}$. □

In this way, each $D \in \mathcal{E}(\mathfrak{g}_K)$ gives rise to n^2 almost inner derivations $s^{j-1}D_i$ of $\mathfrak{g}'_k$, where $1 \leq i, j \leq n$.

Proposition 6.2.2. *Suppose that $D \in \mathcal{E}(\mathfrak{g}_K)$ and define the k–vector space $A := \langle s^{j-1}D_i : \mathfrak{g}'_k \to \mathfrak{g}'_k \mid 1 \leq i, j \leq n\rangle$, spanned by the maps $s^{j-1}D_i$. Then $\dim(A) = n^2$ and*

$$A \cap \mathrm{Inn}(\mathfrak{g}'_k) = \begin{cases} \langle s^{j-1}D : \mathfrak{g}'_k \to \mathfrak{g}'_k \mid 1 \leq j \leq n\rangle & \textit{if } D \in \mathrm{Inn}(\mathfrak{g}_K), \\ \{0\} & \textit{if } D \notin \mathrm{Inn}(\mathfrak{g}_K). \end{cases}$$

Hence we have $\dim(A \cap \mathrm{Inn}(\mathfrak{g}'_k)) = n$ or $\dim(A \cap \mathrm{Inn}(\mathfrak{g}'_k)) = 0$.

Proof. We will first show that the maps $s^{j-1}D_i$ are k–linearly independent. So assume that $\alpha_{i,j} \in k$ and that

$$\sum_{i=1}^{n}\sum_{j=1}^{n} \alpha_{i,j}s^{j-1}D_i = 0.$$

Take $1 \leq l \leq n$ and apply the above to $s^{l-1}y$, then $\left(\sum_{j=1}^{n} \alpha_{l,j} s^{j-1}\right) s^{l-1}z = 0$.

As $s^{l-1}z \neq 0$, it follows that $\sum_{j=1}^{n} \alpha_{l,j} s^{j-1} = 0$. Since $\mathcal{B} = \{1, s, s^2, \dots, s^{n-1}\}$ is a basis of K over k, this implies that all coefficients $\alpha_{l,j} = 0$, showing that the maps $s^{j-i}D_i$ are k–linearly independent and that $\dim(A) = n^2$.

For the second part of the proof, take a map $E \in A \cap \mathrm{Inn}(\mathfrak{g}_k')$. Then there exist values $\alpha_{i,j}$ for all $1 \leq i, j \leq n$ such that $E = \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_{i,j} s^{j-1} D_i$. We further have that $E = \mathrm{ad}(x)$ for some $x \in \mathfrak{g}_k'$. Note that E is K–linear as well, since E can be seen as an inner derivation of $\mathfrak{g}_K$. Mimicking the observations from before, we have for every $1 \leq l \leq n$ that

$$E(s^{l-1}y) = \left(\sum_{j=1}^{n} \alpha_{l,j} s^{j-1}\right) s^{l-1}z.$$

Since E is also K–linear, it must hold that $E(s^{l-1}y) = s^{l-1}E(y)$ and therefore, we get the equality

$$\sum_{j=1}^{n} \alpha_{l,j} s^{j-1} = \sum_{j=1}^{n} \alpha_{1,j} s^{j-1}.$$

Take an arbitrary $1 \leq j \leq n$. It follows that $\alpha_{1,j} = \alpha_{2,j} = \dots = \alpha_{n,j}$ and we let $\beta_j = \alpha_{1,j}$ be this common value. This implies that

$$E = \sum_{i=1}^{n} (\sum_{j=1}^{n} \beta_j s^{j-1}) D_i = \beta(D_1 + D_2 + \dots + D_n) = \beta D,$$

where $\beta = \sum_{j=1}^{n} \beta_j s^{j-1} \in K$. So $E = \beta D$ for some $\beta \in K$ and therefore $E \in \mathrm{Inn}(\mathfrak{g}_K)$ if and only if $D \in \mathrm{Inn}(\mathfrak{g}_K)$. When this is the case, the above shows that $E \in \langle s^{j-1}D \mid 1 \leq j \leq n \rangle$. This finishes the proof, since $\mathrm{Inn}(\mathfrak{g}_K) = \mathrm{Inn}(\mathfrak{g}_k')$ holds as sets. $\square$

In many cases, the above proposition allows us to construct Lie algebras over a non algebraically closed field k with many almost inner derivations which are not inner. As an example of this, we have the following result.

Corollary 6.2.3. *Let K be a field extension of a field k, with $[K : k] = n \geq 2$. Using the notation from above, assume that $\mathfrak{g}_k$ is a c–step nilpotent Lie algebra for $c \geq 2$ with $\dim(\gamma_c(\mathfrak{g}_k)) = 1$, then $\dim(\mathrm{AID}(\mathfrak{g}_k')) - \dim(\mathrm{Inn}(\mathfrak{g}_k')) \geq n^2 - n > 0$.*

Proof. Let $v \in \gamma_{c-1}(\mathfrak{g}_k)$ be an element with $[v, \mathfrak{g}_k] \neq 0$. Such a v exists, since we assume that $\mathfrak{g}_k$ is c–step nilpotent. Moreover, $\gamma_c(\mathfrak{g}_k) = \mathrm{im}(\mathrm{ad}(v)) \subseteq Z(\mathfrak{g}_k)$

holds, which means that $D = \mathrm{ad}(v) \in \mathcal{E}(\mathfrak{g}_K)$. It follows from Proposition 6.2.2 that $A = \langle s^{j-1}D_i \mid 1 \leq i, j \leq n\rangle$ is an n^2–dimensional subspace of $\mathrm{AID}(\mathfrak{g}'_k)$ intersecting $\mathrm{Inn}(\mathfrak{g}'_k)$ in a n–dimensional space, which proves the corollary. $\square$

The above corollary can for example be applied when $\mathfrak{g}_k$ is a filiform Lie algebra. Note that the lower bound of Corollary 6.2.3 is not always very tight, since we can apply Proposition 6.2.2 for linearly independent maps of $\mathcal{E}(\mathfrak{g}_K)$. However, it is not easy to prove general statements about the exact value of $\dim(\mathrm{AID}(\mathfrak{g}'_k))$. As in Example 6.1.2, we will illustrate the concepts of this section for the Heisenberg Lie algebras.

Example 6.2.4. Denote $\mathfrak{g}_\mathbb{R}$ and $\mathfrak{g}_\mathbb{C}$ for the real respectively complex Heisenberg Lie algebra, then we have $\mathrm{AID}(\mathfrak{g}_\mathbb{R}) = \mathrm{Inn}(\mathfrak{g}_\mathbb{R})$ and $\mathrm{AID}(\mathfrak{g}_\mathbb{C}) = \mathrm{Inn}(\mathfrak{g}_\mathbb{C})$. It follows from Proposition 6.1.1 that

$$\mathrm{AID}(\mathfrak{g}'_\mathbb{C}) \cong \mathrm{AID}(\mathfrak{g}_\mathbb{C} \oplus \mathfrak{g}_\mathbb{C}) = \mathrm{AID}(\mathfrak{g}_\mathbb{C}) \oplus \mathrm{AID}(\mathfrak{g}_\mathbb{C}) = \mathrm{Inn}(\mathfrak{g}_\mathbb{C}) \oplus \mathrm{Inn}(\mathfrak{g}_\mathbb{C}) \cong \mathrm{Inn}(\mathfrak{g}'_\mathbb{C}),$$

where we also use Proposition 4.1.8. Let $D = \mathrm{ad}(e_1)$ and $E = \mathrm{ad}(e_2)$ be inner derivations of $\mathfrak{g}_\mathbb{C}$, then both $D, E \in \mathcal{E}(\mathfrak{g}_\mathbb{C})$.

Take an arbitrary $x = a_1e_1 + a_2e_2 + a_3e_3 + b_1f_1 + b_2f_2 + b_3f_3 \in \mathfrak{g}'_\mathbb{R}$. We can now consider the $\mathbb{R}$–linear maps $D_1,\ iD_1,\ E_1,\ iE_1$ as defined above and these satisfy

$$D_1(x) = a_2e_3,$$

$$iD_1(x) = a_2f_3,$$

$$E_1(x) = -a_1e_3,$$

$$iE_1(x) = -a_1f_3.$$

Lemma 6.2.1 implies that $D_1, iD_1, E_1, iE_1 \in \mathrm{AID}(\mathfrak{g}'_\mathbb{R})$. Further, it is also easy to see that $\langle D_1, iD_1, E_1, iE_1\rangle \cap \mathrm{Inn}(\mathfrak{g}'_\mathbb{R}) = 0$, so that we obtain

$$\dim(\mathrm{AID}(\mathfrak{g}'_\mathbb{R})) \geq 4 + \dim(\mathrm{Inn}(\mathfrak{g}'_\mathbb{R})) = 8,$$

$$\dim(\mathrm{AID}(\mathfrak{g}'_\mathbb{C})) = \dim(\mathrm{Inn}(\mathfrak{g}'_\mathbb{C})) = \dim(\mathrm{Inn}(\mathfrak{g}_\mathbb{C} \oplus \mathfrak{g}_\mathbb{C})) = 4.$$

The map D_1 does, like any other derivation, extend to a derivation of $\mathfrak{g}'_\mathbb{C}$. However, it will no longer be almost inner, since $D_1(e_2 + if_2) = e_3$, but

$$[e_2 + if_2, x] = (-a_1 + b_1)e_3 - (a_1 + b_1)f_3.$$

for all $x = a_1e_1 + a_2e_2 + a_3e_3 + b_1f_1 + b_2f_2 + b_3f_3 \in \mathfrak{g}$. A similar reasoning holds for the maps iD_1, E_1 and iE_1.

In fact, we have $\dim(\mathrm{AID}(\mathfrak{g}'_{\mathbb{R}}) = 8$ in this example. By renaming the basis vectors as

$$x_1 = e_1, \quad x_2 = e_2, \quad x_3 = f_1, \quad x_4 = f_2, \quad y_1 = e_3 \text{ and } y_2 = f_3,$$

we find that the associated matrix pencil

$$\mu A + \lambda B = \begin{pmatrix} 0 & \mu & 0 & \lambda \\ -\mu & 0 & -\lambda & 0 \\ 0 & \lambda & 0 & -\mu \\ -\lambda & 0 & \mu & 0 \end{pmatrix}$$

has determinant $(\mu^2 + \lambda^2)^2$, which means that $\mathfrak{g}'_{\mathbb{R}}$ is a nonsingular Lie algebra. Corollary 4.3.9 implies that $\dim(\mathrm{AID}(\mathfrak{g}'_{\mathbb{R}}) = \dim(\mathcal{C}(\mathfrak{g}'_{\mathbb{R}})) = 8$.

Chapter 7

Lie algebras related to finite p–groups

As explained in Section 3.1, one can obtain from a (real or complex) Lie group a Lie algebra (over $\mathbb{R}$ respectively $\mathbb{C}$). In this chapter, we will show that it is also possible to relate to a finite p–group G a Lie algebra $L_{\mathcal{F}}(G)$ which is defined over $\mathbb{F}_p$. In Section 7.1, strongly central series $\mathcal{F}$ of groups are introduced to provide a way to construct $L_{\mathcal{F}}(G)$. Further, we already stated some well-known results for class preserving automorphisms of a finite p–group G in Section 3.2. The goal of Section 7.2 is to compare some of these properties with the almost inner derivations of the associated Lie algebra $L_{\mathcal{F}}(G)$ over $\mathbb{F}_p$. We look at a class of two-step nilpotent groups where there is a nice correspondence between $\mathrm{Aut}_c(G)$ and $\mathrm{AID}(L_{\mathcal{F}}(G))$. Further, we also show that this does not work in general, so the construction is useful to some extent, but not always.

7.1 Strongly central series of groups

In this section, we present a way to associate a Lie algebra $L(G)$ over $\mathbb{F}_p$ to a finite p–group G. The same method has proven its value in the study of the restricted Burnside problem. There, properties were first stated and proven for modular Lie algebras (which are Lie algebras over a field of positive characteristic) and then translated to finite p–groups, so the approach is different. The notions and results in this section are standard and can be found in classical books about finite groups, such as [49], [53] or [73].

Definition 7.1.1 (Strongly central series)**.** Let G be a group. A series $\mathcal{F}$ of subgroups

$$G = G_1 \geq G_2 \geq \cdots \geq G_n = 1$$

of G is called strongly central if $[G_i, G_j] \leq G_{i+j}$ for all $1 \leq i, j \leq n$.

Take an arbitrary $1 \leq i \leq n-1$, so $[G, G_i] \leq G_{i+1}$. For all $g \in G$ and all $g_i \in G_i$, we have that $g^{-1}g_i^{-1}gg_i \in G_{i+1}$ and thus $gg_iG_{i+1} = g_igG_{i+1}$. Hence, $G_i/G_{i+1} \leq Z(G/G_{i+1})$ and $\mathcal{F}$ is a central series. In particular, G_i/G_{i+1} is abelian. Moreover, $[G, G_j] \leq G_j$ implies that G_j is normal in G for all $1 \leq j \leq n$, so $\mathcal{F}$ is a normal series as well.

Let $\mathcal{F}$ be a strongly central series. Define

$$L_{\mathcal{F}}(G) := \bigoplus_{i=1}^{n-1} \frac{G_i}{G_{i+1}} \tag{7.1}$$

as the direct sum of the abelian groups. Let $1 \leq i, j \leq n-1$ and consider homogeneous elements $xG_{i+1} \in G_i/G_{i+1}$ and $yG_{j+1} \in G_j/G_{j+1}$, with $x \in G_i$ and $y \in G_j$. Introduce a product on $L_{\mathcal{F}}(G)$ by the bilinear extension of

$$[xG_{i+1}, yG_{j+1}] = [x, y]G_{i+j+1}. \tag{7.2}$$

The condition that $\mathcal{F}$ is strongly central implies that $[x, y] \in G_{i+j}$ and that $[xG_{i+1}, yG_{j+1}]$ is independent of the chosen representatives.

Theorem 7.1.2. *Let G be a group with strongly central series $\mathcal{F}$.*

- *Then $L_{\mathcal{F}}(G)$ is a Lie ring.*
- *If G is nilpotent of class c, then $L_{\mathcal{F}}(G)$ is nilpotent of class $\leq c$.*
- *If G is finite, then the order of G equals the order of the associated Lie ring $L_{\mathcal{F}}(G)$.*

We will use finite p–groups. When all quotients G_i/G_{i+1} (with $1 \leq i \leq n-1$) have exponent p, then $L(G)$ is a Lie algebra over $\mathbb{F}_p$, the field with p elements. In the following subsections, we will introduce two different strongly central series. The first one, the lower central series $\mathcal{F}_0$, is easier to work with and has nice properties, but does not produce a Lie algebra in general. However, this is the case for the lower exponent–p central series $\mathcal{F}_p$. When it is clear or not important which strongly central series $\mathcal{F}$ we use, we will simply denote $L(G)$ instead of $L_{\mathcal{F}}(G)$.

7.1.1 Lower central series

The easiest example of a strongly central series is the lower central series, which we denote with $\mathcal{F}_0$. Define $\gamma_1(G) := G$ and $\gamma_{i+1}(G) := [G, \gamma_i(G)]$ for all $i \in \mathbb{N}_0$.

Definition 7.1.3 (Lower central series)**.** Let G be a group. The series $\mathcal{F}_0$, which is defined as $G_i := \gamma_i(G)$ for all $i \in \mathbb{N}_0$ is called the lower central series.

When G is nilpotent, there exists $c \in \mathbb{N}$ such that $\gamma_c(G) = \{1\}$. This value c is called the nilpotency class of G. Note that $\gamma_i(G) = \{1\}$ for all $i \geq c$. This construction is very similar to the lower central series for Lie algebras. The following result is well-known and states that $\mathcal{F}_0$ is strongly central for nilpotent groups.

Lemma 7.1.4. *Let G be a nilpotent group of class c, then it holds for all $1 \leq i, j \leq c$ that $[\gamma_i(G), \gamma_j(G)] \leq \gamma_{i+j}(G)$.*

The construction of equation (7.1) for the lower central series is called the 'associated Lie ring' or the 'Magnus-Sanov Lie ring'. When G is nilpotent, we have a stronger statement than in Theorem 7.1.2 concerning the nilpotency class.

Lemma 7.1.5. *Let G be a nilpotent group of class c, then $L_{\mathcal{F}_0}(G)$ is a nilpotent Lie ring with nilpotency class c.*

Since all finite p–groups are nilpotent, we can use the above results. However, the Magnus-Sanov Lie ring $L_{\mathcal{F}_0}(G)$ is not always a Lie algebra over $\mathbb{F}_p$, since there exist p–groups G for which $G/[G,G]$ is not elementary abelian.

Example 7.1.6. Consider the group G with presentation

$$G := \langle a, b \mid a^8 = b^2 = 1, bab = a^5 \rangle.$$

We have that G is a 2–group with $2^4 = 16$ elements and $[G,G] = \langle 1, a^4 \rangle$. We find that $G/[G,G] \cong C_2 \oplus C_4$ is not elementary abelian.

This example shows that the associated Lie ring $L_{\mathcal{F}}(G)$ is not always a Lie algebra over $\mathbb{F}_p$. We will introduce a strongly central series $\mathcal{F}_p$ such that $L_{\mathcal{F}_p}(G)$ is a Lie algebra for all p–groups G.

7.1.2 Lower exponent–p central series

Another example of a strongly central series is the lower exponent–p central series, which is only defined for finite p–groups. We first define another notion.

Let G be a group and denote $G^p := \langle g^p \mid g \in G \rangle$ for the group generated by the p–th powers of the elements of G.

Definition 7.1.7 (Lower exponent–p central series)**.** Let G be a finite p–group. The series $\mathcal{F}_p$, which is defined inductively as $G_1 := G$ and $G_{i+1} := [G, G_i](G_i)^p$ is called the lower exponent–p central series.

It turns out that the lower exponent–p central series is strongly central.

Lemma 7.1.8. *Let G be a nilpotent group and denote G_i for the elements of the lower exponent p–central series. Then we have that $[G_i, G_j] \leq G_{i+j}$ for all $i, j \in \mathbb{N}_0$.*

There is a correspondence between the lower exponent–p central series and the Frattini subgroup (Definition 3.2.9). We collect some facts about the Frattini subgroup.

Proposition 7.1.9. *Let G be a group.*

- *If G is finite, then the Frattini subgroup $\Phi(G)$ is nilpotent.*
- *If G is a p–group, then $G/\Phi(G)$ is an elementary abelian p–group. Moreover, if H is a normal subgroup of G such that G/H is elementary abelian, then $\Phi(G) \leq H$.*

It can be shown that for a finite p–group G, we have that $G_2 = [G, G]G^p$ is the Frattini subgroup $\Phi(G)$. Since the Frattini subgroup is the smallest normal subgroup of G such that the quotient group is elementary abelian, the exponent–p central series is the most quickly descending series. It follows from the previous observations that when G is a finite p–group, then $L_{\mathcal{F}_p}(G)$ is a Lie algebra over $\mathbb{F}_p$. The lower exponent–p central series can be computed directly from the lower central series with the help of the following result.

Proposition 7.1.10 ([49], page 242)**.** *Let G be a finite group. Then the lower exponent p–central series can be obtained as*

$$G_i = \prod_{j=1}^{i} (\gamma_j(G))^{p^{i-j}}.$$

Let G be a finite p–group and consider the lower exponent–p central series. Theorem 7.1.2 implies that $L_{\mathcal{F}_p}(G)$ is nilpotent as well. However, the Lie algebra can have a smaller nilpotency class than the nilpotency class of G.

Example 7.1.11. Consider the group G from Example 7.1.6, which has presentation $G = \langle a, b \mid a^8 = b^2 = 1, bab = a^5 \rangle$. Then G is two-step nilpotent and has Frattini subgroup

$$\Phi(G) = [G, G]G^2 = \{1, a^2, a^4, a^6\} = G_2.$$

We further have $G_3 = \{1, a^4\}$ and $G_4 = \{1\}$. This means that

$$L_{\mathcal{F}_2}(G) = \frac{G_1}{G_2} \oplus \frac{G_2}{G_3} \oplus \frac{G_3}{G_4}$$

is a 4–dimensional Lie algebra over $\mathbb{F}_2$. We can take a basis $\{e_1, e_2, e_3, e_4\}$ for $L_{\mathcal{F}_2}(G)$, where

$$e_1 := aG_2, \qquad e_2 := bG_2, \qquad e_3 := a^2G_3 \quad \text{and} \quad e_4 := a^4G_4.$$

By using (7.2), we find that $L_{\mathcal{F}_2}(G)$ is abelian. It follows for instance that

$$[e_1, e_2] = [aG_2, bG_2] = [a, b]G_3 = a^4G_3 = 0,$$

where the last equality holds because $a^4 \in G_3$.

Since we are only interested in Lie rings which are Lie algebras, we will use the exponent–p central series in the following sections. Note that there are also exist other strongly central series, such as the upper exponent–p central series and the Jennings series, which have a more difficult definition. Let G be a finite p–group. In the following section, we will study to what extent $\mathrm{Aut}_c(G)$ determines $\mathrm{AID}(\mathcal{F}_p(G))$.

7.2 Relating a Lie algebra over $\mathbb{F}_p$ to a finite p–group

In this section, we start from a finite p–group G for which we know $\mathrm{Aut}_c(G)$. We wonder if this set gives information about the almost inner derivations of the associated Lie algebra $L_{\mathcal{F}_p}(G)$. It turns out that, for a certain class of 2–step nilpotent finite p–groups, there is a nice correspondence. However, in general, a lot of information is lost.

7.2.1 Finite Frattini-in-center groups

Let p be a prime number.

Definition 7.2.1 (Frattini-in-center group)**.** A group G is Frattini-in-center if and only if $[G,G] \leq \Phi(G) \leq Z(G)$.

This definition is equivalent with the condition that the inner automorphism group $\mathrm{Inn}(G) \cong G/Z(G)$ is abelian and has a trivial Frattini subgroup. Note that a Frattini-in-center group is 2–step nilpotent. It follows from Proposition 7.1.9 that a finite p–group is Frattini-in-center if and only if $G/Z(G)$ is elementary abelian.

It turns out that there is a nice relationship between some two-step nilpotent p–groups and the corresponding Lie algebras over $\mathbb{F}_p$. Let G be a two-step nilpotent p–group and define the set

$$L := \{f \in \mathrm{Hom}(G/Z(G), [G,G]) \mid f(gZ(G)) \in [g,G] \text{ for all } g \in G\}. \tag{7.3}$$

Then $(L,+)$ is an abelian group, since $[G,G]$ is abelian. The following result was used in [87].

Theorem 7.2.2. *Let G be a finite 2–step nilpotent group. Consider the map $f : \mathrm{Aut}_c(G) \to L$, where $D \in \mathrm{Aut}_c(G)$ is mapped to*

$$f_D : G/Z(G) \to [G,G] : g/Z(G) \mapsto f_D(gZ(G)) = g^{-1}D(g).$$

Then f is an isomorphism.

Proof. Let G be a finite 2–step nilpotent group and take $D \in \mathrm{Aut}_c(G)$. Then there exists a map $\varphi_D : G \to G$ such that $D(g) = g^{\varphi_D(g)}$ for all $g \in G$. Consider the map $\tilde{D} : G \to [G,G] : g \mapsto g^{-1}D(g)$ and take $g,h \in G$. We then have

$$\begin{aligned} \tilde{D}(gh) &= (gh)^{-1}D(gh) \\ &= h^{-1}g^{-1}D(g)D(h) \\ &= g^{-1}D(g)h^{-1}D(h) \\ &= \tilde{D}(g)\tilde{D}(h), \end{aligned}$$

where we use that $g^{-1}D(g) \in [G,G]$ belongs to the center of G, since G is two-step nilpotent. This implies that $\tilde{D}$ is a homomorphism. Moreover, all elements of $Z(G)$ are mapped to the neutral element. Hence, for arbitrary $D \in \mathrm{Aut}_c(G)$, we have that

$$f_D : G/Z(G) \to [G,G] : gZ(G) \mapsto g^{-1}D(g) \in L,$$

which implies that $f : \mathrm{Aut}_c(G) \to L : D \mapsto f_D$ is well-defined. We will show that f is an isomorphism. It follows from the previous observations that the map $D \mapsto f_D$ is a monomorphism of the group $\mathrm{Aut}_c(G)$ into L. Moreover, suppose that $f \in L$ and consider the map $D : G \to G : g \mapsto gf(gZ(G))$. Then $D(g) \in g^G$ for all $g \in G$ and we find that $D \in \mathrm{Aut}_c(G)$ with $f_D = f$. $\square$

Suppose that G is a finite p–group which is Frattini-in-center. Denote $\mathcal{F}_p$ for the lower exponent–p central series, then

$$L_{\mathcal{F}_p}(G) = \frac{G_1}{G_2} \oplus \frac{G_2}{G_3}.$$

Define $\mathfrak{g}_i := \frac{G_i}{G_{i+1}}$ for $1 \leq i \leq 2$. We can view $\ell \in L$ as a map $\ell : \mathfrak{g}_1 \to \mathfrak{g}_2$ with $\ell(x) \in [x, \mathfrak{g}]$ for all $x \in \mathfrak{g}_1$. Define $\tilde{\ell} : \mathfrak{g}_1 \oplus \mathfrak{g}_2 \to \mathfrak{g}_1 \oplus \mathfrak{g}_2 : (x_1, x_2) \mapsto (0, \ell(x_1))$, then $\tilde{\ell} \in \mathrm{AID}(\mathfrak{g})$. In this way, we can associate to each $D \in \mathrm{Aut}_c(G)$ an almost inner derivation of $L_{\mathcal{F}_p}(G)$ and vice versa. This gives rise to the following result.

Theorem 7.2.3. *Let G be a finite Frattini-in-center p–group with corresponding Lie algebra $\mathfrak{g} := L_{\mathcal{F}_p}(G)$ over $\mathbb{F}_p$. Then there is a one-to-one correspondence between* $\mathrm{Aut}_c(G)$ *and* $\mathrm{AID}(\mathfrak{g})$.

Let G be a special group, then $L_{\mathcal{F}_0}(G)$ and $L_{\mathcal{F}_p}(G)$ coincide. Since every special group is Frattini-in-center, we can use the previous property.

Proposition 7.2.4. *Let G be an extra special p–group. For the corresponding Lie algebra $\mathfrak{g} := L_{\mathcal{F}_2}(G)$, we have* $\dim([\mathfrak{g}, \mathfrak{g}]) = 1$.

For instance, $\mathcal{D}_4$ and $\mathcal{Q}_8$ both correspond to the Heisenberg Lie algebra over $\mathbb{F}_2$. In Chapter 9, we will show that for a Lie algebra $\mathfrak{g}$ with $\dim([\mathfrak{g}, \mathfrak{g}]) = 1$, then $\mathrm{AID}(\mathfrak{g}) = \mathrm{Inn}(\mathfrak{g})$ holds as well. As is illustrated in Theorem 3.2.18, a special Camina group corresponds to a nonsingular Lie algebra.

Proposition 7.2.5. *Let G be a special Camina p–group with corresponding Lie algebra $\mathfrak{g} := L_{\mathcal{F}_2}(G)$. Then $\mathfrak{g}$ is nonsingular over $\mathbb{F}_p$.*

Example 7.2.6. Let G be the group from Example 3.2.19. Then G is a special Camina group if and only if $p \equiv \pm 3 \mod 8$. The associated Lie algebra $\mathfrak{g} := L_{\mathcal{F}_2}(G)$ has a basis $\{e_1, \ldots, e_6\}$ and is given by

$$\begin{aligned} [e_1, e_3] &= e_5, \quad [e_1, e_4] = e_6, \\ [e_2, e_3] &= e_6, \quad [e_2, e_4] = 2e_5. \end{aligned}$$

The matrix pencil $P := \mu_5 A_5 + \mu_6 A_6$ for $\mathfrak{g}$ has determinant

$$\det \begin{pmatrix} 0 & 0 & \mu_5 & \mu_6 \\ 0 & 0 & \mu_6 & 2\mu_5 \\ -\mu_5 & -\mu_6 & 0 & 0 \\ -\mu_6 & -2\mu_5 & 0 & 0 \end{pmatrix} = (2\mu_5^2 - \mu_6^2)^2.$$

Define $\mu := \frac{\mu_6}{\mu_5}$, then $\det(P) = 0$ if and only if $\mu^2 - 2 = 0$. Note that 2 is a quadratic non-residue in $\mathbb{F}_p$ if and only if $p \equiv \pm 3 \mod 8$.

In Section 3.2, we gave an example of a special Camina group for which all automorphisms are class preserving. For the associated (nonsingular) Lie algebra, we have that $\mathcal{C}(\mathfrak{g}) = \mathrm{AID}(\mathfrak{g})$. However, there is at least one non-nilpotent derivation for 2–step nilpotent Lie algebras, which means that $\mathrm{Der}(\mathfrak{g}) \neq \mathrm{AID}(\mathfrak{g})$.

Example 7.2.7. Consider the group G from Example 3.2.23. Since G is a special 3–group, the constructions $L_{\mathcal{F}_0}(G)$ and $L_{\mathcal{F}_3}(G)$ coincide and define a nonsingular Lie algebra over $\mathbb{F}_3$. Indeed, we find that $\mathfrak{g}$ has basis $\mathcal{B} = \{e_1, \dots, e_8\}$ and is given over $\mathbb{F}_3$ by

$$\begin{array}{lllll}
[e_1, e_2] = e_7, & [e_1, e_3] = e_8, & [e_1, e_4] = e_8, & [e_1, e_5] = e_8, & [e_1, e_6] = e_8, \\
[e_2, e_3] = e_7, & [e_2, e_4] = e_8, & [e_2, e_5] = e_7, & [e_2, e_6] = e_8, & [e_3, e_4] = e_8, \\
[e_3, e_5] = e_8, & [e_3, e_6] = e_7, & [e_4, e_5] = e_7, & [e_4, e_6] = e_7, & [e_5, e_6] = e_8.
\end{array}$$

The determinant of the associated matrix pencil $\mu_7 A_7 + \mu_8 A_8$ is

$$\det \begin{pmatrix}
0 & \mu_7 & \mu_8 & \mu_8 & \mu_8 & \mu_8 \\
-\mu_7 & 0 & \mu_7 & \mu_8 & \mu_7 & \mu_8 \\
-\mu_8 & -\mu_7 & 0 & \mu_8 & \mu_8 & \mu_7 \\
-\mu_8 & -\mu_8 & -\mu_8 & 0 & \mu_7 & \mu_7 \\
-\mu_8 & -\mu_7 & -\mu_8 & -\mu_7 & 0 & \mu_8 \\
-\mu_8 & -\mu_8 & -\mu_7 & -\mu_7 & -\mu_8 & 0
\end{pmatrix} = (\mu_7^3 + 2\mu_7^2\mu_8 + \mu_8^3)^2.$$

Since $(0,0)$ is the only tuple $(x, y) \in \mathbb{F}_3 \times \mathbb{F}_3$ for which $x^3 + 2x^2y + y^3 = 0$ holds, we see that $\mathfrak{g}$ is nonsingular over $\mathbb{F}_3$. Consider the linear map

$$D : \mathfrak{g} \to \mathfrak{g} : x \mapsto \sum_{i=1}^{6} x_i e_i + 2x_7 e_7 + 2x_8 e_8$$

and take $1 \leq i < j \leq 6$, then

$$[D(e_i), e_j] + [e_i, D(e_j)] = [e_i, e_j] + [e_i, e_j] = 2[e_i, e_j].$$

Since $[e_i, e_j] \in Z(\mathfrak{g})$, we also have $D([e_i, e_j]) = 2[e_i, e_j]$, so D is a derivation. It is clear that D is not almost inner. Hence, for the Lie algebra $\mathfrak{g}$, we have that $\mathrm{AID}(\mathfrak{g}) \neq \mathrm{Der}(\mathfrak{g})$.

7.2.2 Other groups

By going from a p–group to the corresponding Lie algebra, one loses important information. Indeed, each of the abelian p–groups of order p^n corresponds to the unique abelian Lie algebra over $\mathbb{F}_p$ of dimension n. Moreover, there are also non-abelian p–groups for which the associated Lie algebra is abelian, as we already saw in Example 7.1.11. Further, we also find groups G with non-inner class preserving automorphisms such that $\mathrm{AID}(\mathfrak{g}) = \mathrm{Inn}(\mathfrak{g})$ holds for the corresponding Lie algebra $\mathfrak{g}$.

Example 7.2.8. Consider the group $G := H_1$ from Theorem 3.2.14. This is isomorphic to Wall's group and has presentation

$$G = \langle a, b, c \mid a^8 = b^2 = c^2 = [b, c] = 1, [a, b] = a^2, [a, c] = a^4\rangle.$$

It holds that $G_2 = \Phi(G) = \langle a^2\rangle$ and also $G_3 = \langle a^4\rangle$ and $G_4 = \{1\}$. This means that $\mathfrak{g} := L_{\mathcal{F}_2}(G)$ is a 5–dimensional Lie algebra over $\mathbb{F}_2$. We can take a basis $\{e_1, \ldots, e_5\}$, where

$$e_1 := bG_2, \quad e_2 := aG_2, \quad e_3 := a^2G_3, \quad e_4 := a^4G_4, \quad e_5 := cG_2.$$

Then $\mathfrak{g}$ is given by $[e_1, e_2] = e_3$ and $[e_1, e_3] = e_4$. We find for instance that

$$[e_1, e_2] = [bG_2, aG_2] = [b, a]G_3 = a^6G_3 = e_3.$$

This means that $L_{\mathcal{F}_2}(G) = \mathfrak{g}_{5,3}$ has no non-inner almost inner derivations, whereas G has non-inner class preserving automorphisms.

A similar computation shows that H_2 from Theorem 3.2.14 also corresponds to $\mathfrak{g}_{5,3}$ over $\mathbb{F}_2$. More extreme is the following example. We have a group G with $\mathrm{Aut}_c(G) = \mathrm{Aut}(G)$ for which the only almost inner derivations for the corresponding Lie algebra are the inner ones.

Example 7.2.9. Let G be the group with $2^7 = 128$ elements from Example 3.2.21, then $\mathrm{Aut}(G) = \mathrm{Aut}_c(G)$ holds. The corresponding Lie algebra $\mathfrak{g} := L_{\mathcal{F}_2}(G)$ over $\mathbb{F}_2$ has basis $\{e_1, \ldots, e_7\}$ and is given by

$$\begin{aligned}
&[e_1, e_2] = e_4, \quad [e_1, e_3] = e_5, \quad [e_1, e_6] = e_7,\\
&[e_2, e_5] = e_6, \quad [e_3, e_4] = e_6, \quad [e_4, e_5] = e_7.
\end{aligned}$$

A direct computation shows that an arbitrary derivation looks like

$$D = a_1\mathrm{ad}(e_1) + \cdots + a_6\mathrm{ad}(e_6) + d_1D_1 + \cdots + d_7D_7$$

and has matrix form

$$D = \begin{pmatrix}
d_1 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & d_3 & d_6 & 0 & 0 & 0 & 0\\
0 & d_4 & d_7 & 0 & 0 & 0 & 0\\
a_2 & a_1 & 0 & d_1 + d_3 & d_6 & 0 & 0\\
a_3 & 0 & a_1 & d_4 & d_1 + d_7 & 0 & 0\\
d_2 & a_5 & a_4 & a_3 & a_2 & d_1 + d_3 + d_7 & 0\\
a_6 & d_5 & d_8 & a_5 & a_4 & a_1 & d_3 + d_7
\end{pmatrix}.$$

Since $d_1D_1 + \cdots + d_7D_7$ is not $\mathcal{B}$–almost inner, we immediately find that $\mathrm{AID}(\mathfrak{g}) = \mathrm{Inn}(\mathfrak{g})$.

The other situation also occurs. We start with a group for which all automorphisms are class preserving and find that $\mathrm{AID}(\mathfrak{g}) \neq \mathrm{Inn}(\mathfrak{g})$ holds for the corresponding Lie algebra $\mathfrak{g}$.

Example 7.2.10. Consider the group $G = C_2^3 \rtimes C_4$ with presentation

$$G = \langle a, b, c, d \mid a^2 = b^2 = c^2 = d^4 = [a, b] = [a, c] = [b, c] = [c, d] = 1,$$
$$da = abcd, db = bcd\rangle.$$

It follows from Theorem 3.2.14 that $\mathrm{Aut}_c(G) = \mathrm{Inn}(G)$. For the basis vectors

$$e_1 := aG_2, \quad e_2 := dG_2, \quad e_3 := bG_3, \quad e_4 := d^2G_3, \quad e_5 := cG_4,$$

we find that $\mathfrak{g} := L_{\mathcal{F}_2}(G)$ is given by

$$[e_1, e_2] = e_3, \qquad [e_1, e_4] = e_5 \quad \text{and} \quad [e_2, e_3] = e_5.$$

Hence, $\mathfrak{g}$ is isomorphic to $\mathfrak{g}_{5,5}$, which means that $\mathrm{AID}(\mathfrak{g}) \neq \mathrm{Inn}(\mathfrak{g})$.

As the above examples illustrate, there is in general no correspondence between $\mathrm{Aut}_c(G)$ for a finite p–group and $\mathrm{AID}(L_{\mathcal{F}_p}(G))$ of the corresponding Lie algebra $L_{\mathcal{F}_p}(G)$ over $\mathbb{F}_p$.

In the first part of this thesis, we developed the motivation and some techniques to compute the almost inner derivations for (a class of) Lie algebras. In this part, we focused on the fact that the dimension of $\mathrm{AID}(\mathfrak{g})$ depends on the field $\mathbb{F}$ over which $\mathfrak{g}$ is defined. In the last part, we will use these observations to compute the set of almost inner derivations for different classes of Lie algebras.

Part III

Classes of Lie algebras

Chapter 8

Low-dimensional Lie algebras

In this chapter, we compute the almost inner derivations of low-dimensional Lie algebras. Our approach is the following. We restrict to a given class of Lie algebras (defined over a certain field) for which there exists a complete list. It suffices to compute the almost inner derivations for all those Lie algebras, since every Lie algebra of the given class is isomorphic to (at least) one Lie algebra of the inventory.

In the first section, we work with Lie algebras over an arbitrary field $\mathbb{F}$. We show that all almost inner derivations are inner for Lie algebras of dimension at most three. Further, we calculate the almost inner derivations for all solvable Lie algebras of dimension 4, where we use the classification of [19]. It turns out that for $\mathrm{char}(\mathbb{F}) \neq 2$, all almost inner derivations are inner as well. However, there exist Lie algebras $\mathfrak{g}$ over an (infinite) field $\mathbb{F}$ with $\mathrm{char}(\mathbb{F}) = 2$ such that $\mathrm{Inn}(\mathfrak{g}) \neq \mathrm{AID}(\mathfrak{g}) = \mathrm{Der}(\mathfrak{g})$ holds. Next, we compute the almost inner derivations of all nilpotent Lie algebras of dimension at most six. Therefore, we use the classification of [15].

For the second section, we restrict ourselves to a field $\mathbb{F}$ of characteristic zero. We first show that $\mathrm{AID}(\mathfrak{g}) = \mathrm{Inn}(\mathfrak{g})$ also holds when $\mathfrak{g}$ is a 4–dimensional non-solvable Lie algebra. Further, we compute the almost inner derivations for the Lie algebras of dimension five over $\mathbb{C}$ and $\mathbb{R}$. The complex case already appeared in [7].

We will omit some proofs, since they mainly consist of doing computations. An overview of all results is listed in the appendix. For each classification, the first table contains all non-vanishing Lie brackets. We denote most Lie algebras with $\mathfrak{g}_{i,j}$, where i is the dimension of the Lie algebra and j is the number in the classification used. Some Lie algebras also have parameters, which are described between parentheses. The second table gives an overview of some properties. Let $\mathfrak{g}$ be a Lie algebra from the classification, then $c(\mathfrak{g})$ denotes the nilpotency class of $\mathfrak{g}$ and $d(\mathfrak{g})$ stands for the derived length (when these notions are well-defined). Further, we will write $I(\mathfrak{g})$ and $C(\mathfrak{g})$ instead of $\dim(\mathrm{Inn}(\mathfrak{g}))$ respectively $\dim(\mathrm{CAID}(\mathfrak{g}))$. Similarly, $\dim(\mathrm{AID}(\mathfrak{g}))$ and $\dim(\mathrm{Der}(\mathfrak{g}))$ are denoted with $A(\mathfrak{g})$ and $D(\mathfrak{g})$. If the entry in the column with 'D' is non-zero, it gives examples of almost inner derivations, which together with the inner derivations generate $\mathrm{AID}(\mathfrak{g})$.

8.1 Over an arbitrary field

8.1.1 Lie algebras of dimension at most 3

Let $\mathbb{F}$ be an arbitrary field. Up to isomorphism, there is only one Lie algebra of dimension 1 (the abelian one, denoted by $\mathfrak{g}_{1,1}$). There are, up to isomorphism, only two Lie algebras of dimension two, namely the abelian one $\mathfrak{g}_{2,1}$ (where all brackets are zero) and the solvable one $\mathfrak{g}_{2,2}$ with basis $\{e_1, e_2\}$ and given by $[e_1, e_2] = e_2$. Note that all derivations of $\mathfrak{g}_{2,2}$ are inner. Since abelian Lie algebras do not have inner derivations (and hence also no almost inner ones), this means that all almost inner derivations are inner for all Lie algebras of dimension at most two. In this subsection, we prove that the same holds for all 3–dimensional Lie algebras. Therefore, we consider two cases. For the solvable Lie algebras of dimension at most 3, we use the classification from [19]. The non-solvable Lie algebras turn out to be simple in dimension three.

Lemma 8.1.1. *Let $\mathfrak{g}$ be a Lie algebra of dimension 3 over an arbitrary field. If $\mathfrak{g}$ is not solvable, then $\mathfrak{g}$ is simple.*

Proof. Let $\mathfrak{h}$ be a proper ideal of $\mathfrak{g}$. This means that $\dim(\mathfrak{h}), \dim(\mathfrak{g}/\mathfrak{h}) \in \{1, 2\}$, so $\mathfrak{h}$ and $\mathfrak{g}/\mathfrak{h}$ both are solvable. Hence, $\mathfrak{g}$ has to be solvable as well. This gives a contradiction, meaning that $\mathfrak{g}$ does not have a proper ideal. □

We will classify the simple Lie algebras of dimension three. Therefore, we will need the following terminology. We adapt the definitions to make it useful for fields of any characteristic.

Definition 8.1.2 (Symmetric and skew matrices)**.** Let A be a matrix with entries over an arbitrary field $\mathbb{F}$. Then A is called symmetric if $A^\top = A$ and alternate (or skew) if $A^\top = -A$ and all diagonal elements are zero.

If $\mathrm{char}(\mathbb{F}) \neq 2$, the last condition is redundant and every non-zero symmetric matrix with entries over $\mathbb{F}$ is non-alternate. However, it ensures that there is a difference between the notions when $\mathrm{char}(\mathbb{F}) = 2$. If this is the case, a non-alternate symmetric matrix has at least one diagonal element which is non-zero.

Lemma 8.1.3. *Let $\mathbb{F}$ be a field of $\mathrm{char}(\mathbb{F}) = 2$. Let $A \in M_3(\mathbb{F})$ be symmetric. If A is alternate, then A is singular.*

Proof. Let $A \in M_3(\mathbb{F})$ be an alternate symmetric matrix. Hence, there exist $a, b, c \in \mathbb{F}$ such that

$$A := \begin{pmatrix} 0 & a & b \\ a & 0 & c \\ b & c & 0 \end{pmatrix}.$$

A direct computation shows that $\det(A) = 2abc = 0$, which means that A is singular. $\square$

These notions are used in a theorem of Albert.

Theorem 8.1.4 ([1])**.** *Let A be a matrix with entries over an arbitrary field $\mathbb{F}$. If A is non-alternate symmetric, then A is congruent to a diagonal matrix.*

When $\mathrm{char}(\mathbb{F}) \neq 2$, the condition that A is non-alternate is redundant and the result is well known. However, if $\mathrm{char}(\mathbb{F}) = 2$, the fact that A has to be non-alternate is a necessary condition. Indeed, the alternate symmetric matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

is not congruent to a diagonal matrix. Let

$$P := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{F})$$

be an arbitrary nonsingular matrix, so $\det(P) = ad + bc \neq 0$, then we have that

$$P^\top \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} P = \begin{pmatrix} 0 & \det(P) \\ \det(P) & 0 \end{pmatrix}.$$

The following result is due to Jacobson ([51]). He included a proof for fields of characteristic not two and stated that, by specifying some details, the proposition also holds in general. For completeness, a proof is added which is valid for fields of arbitrary characteristic. Therefore, we use the theorem of Albert from above.

Proposition 8.1.5. *Let $\mathfrak{g}$ be a simple Lie algebra of dimension 3 over an arbitrary field $\mathbb{F}$. Then there exist $\alpha, \beta \in \mathbb{F}^*$ and a basis $\{e_1, e_2, e_3\}$ for $\mathfrak{g}$ such that $\mathfrak{g}$ is given by*

$$[e_1, e_2] = e_3, \qquad [e_2, e_3] = \alpha e_1 \quad \textit{and} \quad [e_3, e_1] = \beta e_2. \tag{8.1}$$

Proof. Let $\mathfrak{g}$ be a simple Lie algebra of dimension 3 over an arbitrary field $\mathbb{F}$. Define a basis $x := \{x_1, x_2, x_3\}$ for $\mathfrak{g}$ and construct

$$y_1 := [x_2, x_3], \qquad y_2 := [x_3, x_1] \quad \text{and} \quad y_3 := [x_1, x_2].$$

Since $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$, we have that $y := \{y_1, y_2, y_3\}$ is a basis for $\mathfrak{g}$ as well. We call the matrix of change of basis from x to y the 'structure matrix of x' and denote it with $\mathrm{Id}_x^y := (a_{ij})_{ij}$. This means that we have that $y_i := \sum_{j=1}^3 a_{ji} x_j$ for all $1 \leq i \leq 3$. Note that Id_x^y is nonsingular. The Jacobi identity given by

$$\begin{aligned} 0 &= [x_1, y_1] + [x_2, y_2] + [x_3, y_3] \\ &= (a_{21} - a_{12})[x_1, x_2] + (a_{31} - a_{13})[x_1, x_3] + (a_{32} - a_{23})[x_2, x_3] \end{aligned}$$

implies that Id_x^y is symmetric. Every non-trivial condition for the Jacobi identity boils down to the one above, so every nonsingular symmetric matrix can be used to construct a 3-dimensional simple Lie algebra.

Consider another basis $\bar{x} := \{\bar{x}_1, \bar{x}_2, \bar{x}_3\}$ for $\mathfrak{g}$. Define $\mathrm{Id}_x^{\bar{x}} := (b_{ij})_{ij}$ for the matrix of change of basis from x to $\bar{x}$ and construct

$$\bar{y}_1 := [\bar{x}_2, \bar{x}_3], \qquad \bar{y}_2 := [\bar{x}_3, \bar{x}_1] \quad \text{and} \quad \bar{y}_3 := [\bar{x}_1, \bar{x}_2]$$

as before. Again, we have that $\bar{y} := \{\bar{y}_1, \bar{y}_2, \bar{y}_3\}$ is a basis. Note that

$$\begin{aligned} \bar{y}_1 &= [\bar{x}_2, \bar{x}_3] \\ &= [b_{12}x_1 + b_{22}x_2 + b_{32}x_3, b_{13}x_1 + b_{23}x_2 + b_{33}x_3] \\ &= (b_{22}b_{33} - b_{32}b_{23})y_1 + (b_{32}b_{13} - b_{12}b_{33})y_2 + (b_{12}b_{23} - b_{22}b_{13})y_3 \\ &= C_{11}y_1 + C_{21}y_2 + C_{31}y_3, \end{aligned}$$

where C_{ij} is the cofactor of the element b_{ij}. Similar computations for $\bar{y}_2$ and $\bar{y}_3$ show that $\mathrm{Id}_y^{\bar{y}}$ is given by $\mathrm{Id}_y^{\bar{y}} = (C_{ij})_{ij}$. The cofactor expansion of the determinant implies that

$$(\mathrm{Id}_x^{\bar{x}})^\top \cdot \mathrm{Id}_y^{\bar{y}} = \det(\mathrm{Id}_x^{\bar{x}}) I_3,$$

where I_3 is the identity matrix. Hence, we have that

$$\mathrm{Id}_{\bar{x}}^{\bar{y}} = \mathrm{Id}_y^{\bar{y}} \cdot \mathrm{Id}_x^y \cdot \mathrm{Id}_{\bar{x}}^x = \det(\mathrm{Id}_x^{\bar{x}})(\mathrm{Id}_x^{\bar{x}})^{-\top} \cdot \mathrm{Id}_x^y \cdot \mathrm{Id}_{\bar{x}}^x = \det(\mathrm{Id}_x^{\bar{x}})(\mathrm{Id}_{\bar{x}}^x)^\top \cdot \mathrm{Id}_x^y \cdot \mathrm{Id}_{\bar{x}}^x.$$

A change of basis with nonsingular $\mathrm{Id}_{\bar{x}}^{x}$ thus gives a structure matrix of $\bar{x}$ which is (up to a non-zero factor) congruent to the first one. Since Id_x^y is symmetric and nonsingular, it follows from Lemma 8.1.3 that Id_x^y is non-alternate as well. Hence, we can use Theorem 8.1.4, which states that the structure matrix Id_x^y is congruent to a diagonal matrix. By an appropriate scaling, we can choose one diagonal element to be equal to one. Hence, there exists a basis $\{e_1, e_2, e_3\}$ for $\mathfrak{g}$ and $\alpha, \beta \in \mathbb{F}^*$ such that

$$\begin{pmatrix} [e_2, e_3] \\ [e_3, e_1] \\ [e_1, e_2] \end{pmatrix} = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}.$$

This implies that the non-vanishing Lie brackets of $\mathfrak{g}$ are given by

$$[e_1, e_2] = e_3, \qquad [e_2, e_3] = \alpha e_1 \quad \text{and} \quad [e_3, e_1] = \beta e_2$$

for $\alpha, \beta \in \mathbb{F}^*$, which is what we had to prove. □

We denote the Lie algebra from equation (8.1) with $\mathfrak{g}(\alpha, \beta)$. Note that the result from the proposition does not determine the number of isomorphism classes. It can be made more precise when there is additional information about the field. For instance, when $\mathbb{F}$ is algebraically closed, a simple Lie algebra over $\mathbb{F}$ is isomorphic to $\mathfrak{g}(1, -1)$. For $\mathbb{F} = \mathbb{R}$, there are two isomorphism classes. The types $\mathfrak{g}(1, -1) \cong \mathfrak{sl}_2(\mathbb{R})$ and $\mathfrak{g}(1, 1) \cong \mathfrak{so}_3(\mathbb{R})$ correspond to Bianchi types VIII respectively IX in the Bianchi classification.

Collecting the previous observations from this chapter, we can obtain a complete list of all 3–dimensional Lie algebras over an arbitrary field.

Theorem 8.1.6. *Let $\mathfrak{g}$ be a Lie algebra of dimension at most 3 over an arbitrary field $\mathbb{F}$. Then $\mathfrak{g}$ is isomorphic to (at least) one of the Lie algebras from Table A.1.*

We used the results from [19] for the solvable Lie algebras. For the simple ones, the result follows from Proposition 8.1.5. Lie algebras with different indices are non-isomorphic. Further, $\mathfrak{g}_{3,3}(\varepsilon_1) \cong \mathfrak{g}_{3,3}(\varepsilon_2)$ if and only if $\varepsilon_1 = \varepsilon_2$. We also have that $\mathfrak{g}_{3,4}(\varepsilon_1) \cong \mathfrak{g}_{3,4}(\varepsilon_2)$ if and only if there exists $\alpha \in \mathbb{F}^*$ with $\varepsilon_1 = \alpha^2 \varepsilon_2$.

By computing the almost inner derivations for all Lie algebras from Table A.1, it is clear that all almost inner derivations are in fact inner ones. Hence, we showed the following result.

Theorem 8.1.7. *Let $\mathfrak{g}$ be a Lie algebra of dimension at most 3 over an arbitrary field $\mathbb{F}$. Then we have that $\mathrm{AID}(\mathfrak{g}) = \mathrm{Inn}(\mathfrak{g})$ holds.*

Table A.2 contains for each Lie algebra $\mathfrak{g}$ an overview of the dimensions of the different subalgebras of $\mathrm{Der}(\mathfrak{g})$.

Remark 8.1.8. We have a few comments about the results from the table.

- For the Lie algebras $\mathfrak{g}_{3,3}(\varepsilon)$ and $\mathfrak{g}_{3,4}(\varepsilon)$, we have to make a case distinction for $\varepsilon \neq 0$ and $\varepsilon = 0$.
- Up to isomorphism, there is only one nilpotent non-abelian Lie algebra of dimension 3, namely the Heisenberg algebra $\mathfrak{g}_{3,4}(0)$.
- For the Lie algebras $\mathfrak{g}_{3,4}(\varepsilon^*)$ and $\mathfrak{g}(\alpha, \beta)$ (with $\varepsilon^*, \alpha, \beta \in \mathbb{F}^*$), the dimension of the derivation algebra depends on the characteristic of the field $\mathbb{F}$. The largest number is the dimension when $\text{char}(\mathbb{F}) = 2$; the smaller number is for $\text{char}(\mathbb{F}) \neq 2$.

8.1.2 Solvable Lie algebras of dimension 4

For 4–dimensional Lie algebras over an arbitrary field, it is a lot more complicated to have a complete list, especially for the non-solvable Lie algebras. Therefore, we restrict ourselves to the solvable Lie algebras over an arbitrary field. We again use the classification of [19], but our notation slightly differs.

Theorem 8.1.9. *Let $\mathfrak{g}$ be a solvable Lie algebra of dimension 4 over an arbitrary field $\mathbb{F}$. Then $\mathfrak{g}$ is isomorphic to (at least) one Lie algebra of Table A.3.*

Due to the conditions on the parameters, Lie algebras with different indices are not isomorphic. However, Lie algebras with the same indices can be isomorphic for different values of the parameter(s). Moreover, some of the Lie algebras are only defined for specific values and/or over specific fields.

Remark 8.1.10. Here is an overview of all conditions on the field $\mathbb{F}$ and on the parameters, taken from [19].

- We have that $\mathfrak{g}_{4,3}(\varepsilon_1) \cong \mathfrak{g}_{4,3}(\varepsilon_2)$ if and only if $\varepsilon_1 = \varepsilon_2$.
- It follows that $\mathfrak{g}_{4,6}(\varepsilon_1, \delta_1) \cong \mathfrak{g}_{4,6}(\varepsilon_2, \delta_2)$ if and only if $\varepsilon_1 = \varepsilon_2$ and $\delta_1 = \delta_2$.
- Note that $\mathfrak{g}_{4,7}(\varepsilon_1, \delta_1) \cong \mathfrak{g}_{4,7}(\varepsilon_2, \delta_2)$ if and only if there is an $\alpha \in \mathbb{F}^*$ with $\varepsilon_1 = \alpha^3 \varepsilon_2$ and $\delta_1 = \alpha^2 \delta_2$.
- The Lie algebra $\mathfrak{g}_{4,9}(\varepsilon)$ is defined when $X^2 - X - \varepsilon$ has no roots in $\mathbb{F}$. Suppose that $\text{char}(\mathbb{F}) \neq 2$, then $\mathfrak{g}_{4,9}(\varepsilon_1) \cong \mathfrak{g}_{4,9}(\varepsilon_2)$ if and only if there is $\alpha \in \mathbb{F}^*$ with $\varepsilon_1 + \frac{1}{4} = \alpha^2(\varepsilon_2 + \frac{1}{4})$. When $\text{char}(\mathbb{F}) = 2$, then we have $\mathfrak{g}_{4,9}(\varepsilon_1) \cong \mathfrak{g}_{4,9}(\varepsilon_2)$ if and only if $X^2 + X + (\varepsilon_1 + \varepsilon_2)$ has roots in $\mathbb{F}$.

- The Lie algebra $\mathfrak{g}_{4,10}(\varepsilon)$ is only well-defined when $\operatorname{char}(\mathbb{F}) = 2$. We also require that $\varepsilon \notin \mathbb{F}^2$, since we have that $\mathfrak{g}_{4,10}(\varepsilon) \cong \mathfrak{g}_{4,13}(0)$ when $\varepsilon \in \mathbb{F}^2$. Note that we further have $\mathfrak{g}_{4,10}(\varepsilon_1) \cong \mathfrak{g}_{4,10}(\varepsilon_2)$ if and only if $Y^2+\varepsilon_2 X^2+\varepsilon_1$ has a solution $(X, Y) \in \mathbb{F} \times \mathbb{F}$ with $X \neq 0$.

- The Lie algebra $\mathfrak{g}_{4,11}(\varepsilon, \delta)$ is only defined when $\operatorname{char}(\mathbb{F}) = 2$. We also require that $\varepsilon \neq 0$ and $\delta \neq 1$, since $\mathfrak{g}_{4,11}(\varepsilon, 1) \cong \mathfrak{g}_{4,10}(\varepsilon)$ holds. Moreover, $\mathfrak{g}_{4,11}(0,0) \cong \mathfrak{g}_{4,12}$ and $\mathfrak{g}_{4,11}(0,\delta) \cong \mathfrak{g}_{4,13}(\frac{\delta+1}{\delta})$ when $\delta \notin \{0,1\}$. Note that $\mathfrak{g}_{4,11}(\varepsilon_1, \delta_1) \cong \mathfrak{g}_{4,11}(\varepsilon_2, \delta_2)$ if and only if both $\frac{\varepsilon_1}{\varepsilon_2}$ and $\frac{\gamma^2+(\delta_1+1)\gamma+\delta_1}{\varepsilon_1}$ are squares in $\mathbb{F}$, where $\gamma := \frac{\delta_1+1}{\delta_2+1}$.

- We have that $\mathfrak{g}_{4,13}(\varepsilon_2) \cong \mathfrak{g}_{4,13}(\varepsilon_2)$ if and only if $\varepsilon_1 = \varepsilon_2$.

- Since $\mathfrak{g}_{4,14}(0) \cong \mathfrak{g}_{4,7}(0,0)$ holds, we require that $\varepsilon \neq 0$. We further have $\mathfrak{g}_{4,14}(\varepsilon_1) \cong \mathfrak{g}_{4,14}(\varepsilon_2)$ if and only if there is an $\alpha \in \mathbb{F}^*$ with $\varepsilon_1 = \alpha^2\varepsilon_2$.

For each member of the list, we computed the almost inner derivations. The results are stated in Table A.4. We can summarise these computations as follows.

Theorem 8.1.11. *Let $\mathfrak{g}$ be a solvable Lie algebra of dimension 4 over an arbitrary field $\mathbb{F}$.*

- *For $\operatorname{char}(\mathbb{F}) \neq 2$, we have that $\operatorname{AID}(\mathfrak{g}) = \operatorname{Inn}(\mathfrak{g})$ holds.*

- *If $\operatorname{char}(\mathbb{F}) = 2$ and $\operatorname{AID}(\mathfrak{g}) \neq \operatorname{Inn}(\mathfrak{g})$, then $\mathfrak{g}$ is isomorphic to $\mathfrak{g}_{4,11}(\varepsilon, \delta)$, where $\varepsilon \neq 0$ and $\delta \neq 1$ are in such a way that $X^2 + \delta\varepsilon = 0$ has no solution over $\mathbb{F}$.*

Although we did all calculations ourselves, we will omit the proofs.

Remark 8.1.12. We mention some comments to explain the results.

- For $\mathfrak{g}_{4,3}(\varepsilon)$, we have to make a case distinction for $\varepsilon = 0$ and $\varepsilon \neq 0$.

- For $\mathfrak{g}_{4,6}(\varepsilon, \delta)$, there is a case distinction for $\varepsilon = 0$ and $\varepsilon \neq 0$. However, we don't have to do this for δ.

- For $\mathfrak{g}_{4,7}(\varepsilon, \delta)$, there are four cases, namely whether or not ε and/or δ equal zero. When $\varepsilon = 0$ and $\delta \neq 0$, we have that $\dim(\operatorname{Der}(\mathfrak{g}_{4,7}(0,\delta))) = 7$ when $\operatorname{char}(\mathbb{F}) = 2$ and $\dim(\operatorname{Der}(\mathfrak{g}_{4,7}(0,\delta))) = 6$ otherwise. Similarly, for $\varepsilon \neq 0$ and $\delta = 0$, we have that $\dim(\operatorname{Der}(\mathfrak{g}_{4,7}(\varepsilon, 0))) = 7$ when $\operatorname{char}(\mathbb{F}) = 3$ and $\dim(\operatorname{Der}(\mathfrak{g}_{4,7}(\varepsilon, 0))) = 6$ when this is not the case.

- For the Lie algebra $\mathfrak{g}_{4,9}(\varepsilon)$, where $\varepsilon \in \mathbb{F}$ is in such a way that $X^2-X-\varepsilon = 0$ does not have solutions over $\mathbb{F}$, we have $\dim(\mathrm{Der}(\mathfrak{g}_{4,9}(\varepsilon))) = 5$ when $4\varepsilon + 1 = 0$ and $\dim(\mathrm{Der}(\mathfrak{g}_{4,9}(\varepsilon))) = 4$ otherwise.

- For most of the Lie algebras, we see that $\mathrm{AID}(\mathfrak{g}) = \mathrm{Inn}(\mathfrak{g})$. However, for the Lie algebra $\mathfrak{g}_{4,11}(\varepsilon, \delta)$, this is not always the case. It depends on the parameters $\varepsilon, \delta \in \mathbb{F}$. We will elaborate on this example after the remarks.

- For the Lie algebra $\mathfrak{g}_{4,12}$, we have that $\mathrm{AID}(\mathfrak{g}) = \mathrm{Inn}(\mathfrak{g})$, although the dimension depends on the characteristic. We have that $\dim(\mathrm{Inn}(\mathfrak{g})) = 4$ when $\mathrm{char}(\mathbb{F}) \neq 2$. Otherwise, we have $\dim(\mathrm{Inn}(\mathfrak{g})) = 3$.

- For the Lie algebra $\mathfrak{g}_{4,13}(\varepsilon)$, we first consider the case that $\varepsilon = 0$. We have $\dim(\mathrm{Der}(\mathfrak{g}_{4,13}(0))) = 6$ when $\mathrm{char}(\mathbb{F}) = 2$ and $\dim(\mathrm{Der}(\mathfrak{g}_{4,13}(0))) = 5$ otherwise. When $\varepsilon \neq 0$, the dimension of the derivation algebra does not depend on the characteristic of the field.

- For $\mathfrak{g}_{4,14}(\varepsilon)$ with $\varepsilon \neq 0$, we have that $\dim(\mathrm{Der}(\mathfrak{g}_{4,14}(\varepsilon))) = 6$ when $\mathrm{char}(\mathbb{F}) = 2$ and $\dim(\mathrm{Der}(\mathfrak{g}_{4,14}(\varepsilon))) = 5$ otherwise.

Four-dimensional Lie algebra $\mathfrak{g}$ with $\mathrm{Inn}(\mathfrak{g}) \neq \mathrm{AID}(\mathfrak{g}) = \mathrm{Der}(\mathfrak{g})$

Lemma 8.1.13. *Consider the Lie algebra $\mathfrak{g} := \mathfrak{g}_{4,11}(\varepsilon, \delta)$ over a field $\mathbb{F}$ with $\mathrm{char}(\mathbb{F}) = 2$, where $\varepsilon \neq 0$ and $\delta \neq 1$. Take a basis $\{e_1, e_2, e_3, e_4\}$ for $\mathfrak{g}$ and non-zero Lie brackets*

$$\begin{aligned} &[e_1, e_2] = (1+\delta)e_2, \quad &&[e_1, e_3] = \delta e_3, \quad &&[e_1, e_4] = e_4 \\ &[e_2, e_3] = \varepsilon e_4, &&[e_2, e_4] = e_3. \end{aligned}$$

Then a basis for $\mathrm{Der}(\mathfrak{g})$ is given by $\{\mathrm{ad}(e_1), \mathrm{ad}(e_2), \mathrm{ad}(e_3), \mathrm{ad}(e_4), D\}$, where

$$D : \mathfrak{g} \to \mathfrak{g} : x = \sum_{i=1}^{4} x_i e_i \mapsto x_3 e_3 + x_4 e_4.$$

Proof. The statement is easily verified by a direct computation. $\square$

It turns out that the derivation D is almost inner for some, but not all fields of characteristic two. Next result makes this statement more precise.

Proposition 8.1.14. *Let $\mathbb{F}$ be a field of characteristic 2 and $\varepsilon, \delta \in \mathbb{F}$ with $\varepsilon \neq 0$ and $\delta \neq 1$. For the Lie algebra $\mathfrak{g} := \mathfrak{g}_{4,11}(\varepsilon, \delta)$ over $\mathbb{F}$, we have that*

$$\mathrm{Inn}(\mathfrak{g}) \neq \mathrm{AID}(\mathfrak{g}) = \mathrm{Der}(\mathfrak{g})$$

if and only if the equation $X^2 + \delta\varepsilon = 0$ has no solution over $\mathbb{F}$.

Proof. We prove the two implications. Note that $X^2 + \delta\varepsilon = 0$ has a solution over $\mathbb{F}$ if and only if $\varepsilon X^2 + \delta = 0$ has one.

- Assume that the conditions of the proposition are satisfied. It suffices by Lemma 8.1.13 to prove that $D : \mathfrak{g} \to \mathfrak{g} : x = \sum_{i=1}^4 x_i e_i \mapsto x_3 e_3 + x_4 e_4$ is almost inner. Consider the map $\varphi_D : \mathfrak{g} \to \mathfrak{g} : x = \sum_{i=1}^4 x_i e_i \mapsto \varphi_D(x)$, where $\varphi_D(x)$ is given by

$$\begin{cases} 0 & \text{if } x = 0, \\ \frac{1}{\varepsilon x_2^2 + \delta x_1^2}((x_2 x_4 + x_1 x_3)e_3 + (\varepsilon x_2 x_3 + \delta x_1 x_4)e_4) & \text{if } x_1 \neq 0 \text{ or } x_2 \neq 0, \\ \frac{1}{\delta\varepsilon x_3^2 + x_4^2}((x_4^2 + \varepsilon x_3^2)e_1 + (1+\delta)x_3 x_4 e_2) & \text{otherwise.} \end{cases}$$

  Note that the conditions on $\mathbb{F}$ and $\varepsilon, \delta \in \mathbb{F}$ ensure that the map φ_D is well-defined. We will show that $D(x) = [x, \varphi_D(x)]$ for all $x \in \mathfrak{g}$. Take an arbitrary $x = \sum_{i=1}^4 x_i e_i \in \mathfrak{g}$.

  - Assume that $x_1 \neq 0$ or $x_2 \neq 0$. We then have

$$\begin{aligned} [x, \varphi_D(x)] &= \frac{1}{\varepsilon x_2^2 + \delta x_1^2}[x, (x_2 x_4 + x_1 x_3)e_3 + (\varepsilon x_2 x_3 + \delta x_1 x_4)e_4] \\ &= \frac{1}{\varepsilon x_2^2 + \delta x_1^2}(x_1(x_2 x_4 + x_1 x_3)\delta e_3 + x_1(\varepsilon x_2 x_3 + \delta x_1 x_4)e_4 \\ &\quad + x_2(x_2 x_4 + x_1 x_3)\varepsilon e_4 + x_2(\varepsilon x_2 x_3 + \delta x_1 x_4)e_3) \\ &= \frac{1}{\varepsilon x_2^2 + \delta x_1^2}(x_3(\varepsilon x_2^2 + \delta x_1^2)e_3 + x_4(\varepsilon x_2^2 + \delta x_1^2)e_4) \\ &= x_3 e_3 + x_4 e_4. \end{aligned}$$

  - Suppose that $x \neq 0$, but $x_1 = x_2 = 0$. It follows that

$$\begin{aligned} [x, \varphi_D(x)] &= \frac{1}{\delta\varepsilon x_3^2 + x_4^2}[x, (x_4^2 + \varepsilon x_3^2)e_1 + (1+\delta)x_3 x_4 e_2] \\ &= \frac{1}{\delta\varepsilon x_3^2 + x_4^2}(\delta x_3(x_4^2 + \varepsilon x_3^2)e_3 + \varepsilon(1+\delta)x_3^2 x_4 e_4 \\ &\quad + (x_4^2 + \varepsilon x_3^2)x_4 e_4 + (1+\delta)x_3 x_4^2 e_3) \\ &= \frac{1}{\delta\varepsilon x_3^2 + x_4^2}(x_3(\delta\varepsilon x_3^2 + x_4^2)e_3 + x_4(\delta\varepsilon x_3^2 + x_4^2)e_4) \\ &= x_3 e_3 + x_4 e_4. \end{aligned}$$

- We show that $D : \mathfrak{g} \to \mathfrak{g} : x = \sum_{i=1}^{4} x_i e_i \mapsto x_3 e_3 + x_4 e_4$ is not almost inner when the condition is not fulfilled. Suppose that $D \in \mathrm{AID}(\mathfrak{g})$ is almost inner. Let α be a solution of $X^2 + \delta\varepsilon = 0$, so $\alpha^2 = \delta\varepsilon$. Take an arbitrary $x = \sum_{i=1}^{4} x_i e_i \in \mathfrak{g}$. Note that

$$[e_3 + \alpha e_4, x] = (\delta x_1 + \alpha x_2)e_3 + (\alpha x_1 + \varepsilon x_2)e_4.$$

Since we need that $D(e_3+\alpha e_4) = e_3+\alpha e_4$, we must have that $\delta x_1+\alpha x_2 = 1$ and $\alpha x_1 + \varepsilon x_2 = \alpha$. By multiplying this last equation by α and using the first equation, we find that

$$\delta\varepsilon = \delta\varepsilon x_1 + \alpha\varepsilon x_2 = \varepsilon(\delta x_1 + \alpha x_2) = \varepsilon.$$

Since $\varepsilon \neq 0$ and $\delta \neq 1$, we obtain a contradiction. □

Consider the (infinite) field $\mathbb{F} = \mathbb{F}_2(t)$ in the indeterminate t with $\varepsilon = 1$ and $\delta = t$. It follows from Proposition 8.1.14 that for the Lie algebra $\mathfrak{g} := \mathfrak{g}_{4,11}(\varepsilon, \delta)$, we have that

$$\mathrm{Inn}(\mathfrak{g}) \neq \mathrm{AID}(\mathfrak{g}) = \mathrm{Der}(\mathfrak{g}).$$

Note that, in a finite field of characteristic 2, every element is a square. This means that the conditions of the proposition cannot be satisfied for a finite field of characteristic 2 and hence, we have that all almost inner derivations are inner.

8.1.3 Nilpotent Lie algebras of dimension at most 6

For 5–dimensional and 6–dimensional Lie algebras, there doesn't exist (yet) a complete list which is valid over all fields. Therefore, we restrict ourselves to nilpotent Lie algebras. We will use the notation and the results from [15], where the authors provide a full classification of six-dimensional nilpotent Lie algebras over an arbitrary field. It is the first one which covers all ground fields (and hence also fields of characteristic two). This list also reveals the nilpotent Lie algebras of lower dimension. The ones with dimension at most 4 are already covered in the previous subsection, so we will focus on nilpotent Lie algebras of dimension 5 and 6. Let $\mathbb{F}$ be an arbitrary field. Take $\alpha, \beta \in \mathbb{F}$ and denote $\overset{*}{\sim}$ for the equivalence relation defined by

$$\alpha \overset{*}{\sim} \beta \qquad \Longleftrightarrow \qquad \exists\, \gamma \in \mathbb{F}^* : \alpha = \gamma^2 \beta.$$

The number of equivalence classes for this relation is given by $1 + s$, where s is the (possibly infinite) index of $(\mathbb{F}^*)^2$ in $\mathbb{F}^*$. For instance, for an algebraically closed field or a perfect field of characteristic two, there are only two equivalence classes ($\{0\}$ and $\mathbb{F}^*$), so $s = 1$. For $\mathbb{R}$, we have that $s = 2$, with equivalence classes $\{0\}, \mathbb{R}_0^+$ and $\mathbb{R}_0^-$.

When $\mathrm{char}(\mathbb{F}) = 2$, there is an additional equivalence relation $\overset{*+}{\sim}$. For $\alpha, \beta \in \mathbb{F}$, we have

$$\alpha \overset{*+}{\sim} \beta \qquad \iff \qquad \exists\, \gamma \in \mathbb{F}^* : \exists\, \delta \in \mathbb{F} : \alpha = \gamma^2\beta + \delta^2.$$

Note that when $\alpha \overset{*}{\sim} \beta$, then also $\alpha \overset{*+}{\sim} \beta$. For a set X and an equivalence relation $\sim$, denote $X/(\sim)$ for a transversal set that contains precisely one element from each of the equivalence classes of $\sim$. Hence, s stands for the number of elements of $\mathbb{F}^*/(\overset{*}{\sim})$.

Now consider the case that $\mathrm{char}(\mathbb{F}) = 2$ and view $\mathbb{F}$ as a vector space over $\mathbb{F}_2$. In [15], the map $\psi : \mathbb{F} \to \mathbb{F} : X \mapsto X^2 + X$ is studied, which is $\mathbb{F}_2$–linear with kernel $\mathbb{F}_2$. The authors claim that the image $\psi(\mathbb{F})$ is a subspace of codimension 1. While this is true for finite fields, this is not the case in general. Take for instance $\mathbb{F}(t)$ in the indeterminate t. Take an arbitrary element $q \in \mathbb{F}(t)$. Then there exist unique $a(t), b(t) \in \mathbb{F}_2[t]$ with $\gcd(a(t), b(t)) = 1$ such that $q = \frac{a(t)}{b(t)}$. We further have

$$\psi(q) = \psi\left(\frac{a(t)}{b(t)}\right) = \frac{a(t)}{b(t)} + \frac{a(t)^2}{b(t)^2} = \frac{b(t)(b(t) + a(t))}{b(t)^2}.$$

Since $\gcd(a(t), b(t)) = 1$, also $\gcd(b(t)^2, a(t)(b(t) + a(t))) = 1$ holds. Suppose that $t \in \psi(\mathbb{F}_2(t))$, then there exist $a(t), b(t) \in \mathbb{F}_2[t]$ with $t = \frac{t}{1} = \psi\left(\frac{a(t)}{b(t)}\right)$. This implies that $b(t)^2 = 1$ and hence $b(t) = 1$. It follows that $a(t)(a(t) + 1) = t$, but this is impossible. In a similar way, it can be shown that $1, 1 + t \notin \psi(\mathbb{F}_2(t))$. Hence, $1 + \mathrm{im}(\psi)$ and $t + \mathrm{im}(\psi)$ are different cosets of $\psi(\mathbb{F}_2(t))$ which are different from $\mathrm{im}(\psi)$. An analogous reasoning for polynomials of odd degree shows that $\mathbb{F}_2(t)/\psi(\mathbb{F}_2(t))$ is an infinite-dimensional vector space over $\mathbb{F}_2$. Take $\alpha, \beta \in \mathbb{F}$ and denote $\overset{\psi}{\sim}$ for the equivalence relation where $\alpha \overset{\psi}{\sim} \beta$ if and only if $\alpha + \psi(\mathbb{F}) = \beta + \psi(\mathbb{F})$. In other words, when $\alpha \overset{\psi}{\sim} \beta$, there exists $x \in \mathbb{F}$ such that $x^2 + x + \alpha + \beta = 0$. When $\mathbb{F}$ is finite, $\psi(\mathbb{F})$ is a subspace of codimension 1, so there are two equivalence classes for $\overset{\psi}{\sim}$. If $\mathbb{F}$ is algebraically closed, we have $|\mathbb{F}/(\overset{\psi}{\sim})| = 1$.

In [15], the authors claim that $\{0, \omega\}$ is a set of coset representatives for $\psi(\mathbb{F})$ in $\mathbb{F}$, where ω is a fixed element of $\mathbb{F} \setminus \{X^2 + X \mid X \in \mathbb{F}\}$. As we showed before, this is not true. However, this was corrected in [16] and we will use the adapted statements and use the notation $\mathbb{F}/(\overset{\psi}{\sim})$.

Theorem 8.1.15 ([15], see also [16]). *Let $\mathbb{F}$ be an arbitrary field.*

- *If* $\mathrm{char}(\mathbb{F}) \neq 2$*, then the isomorphism types of six-dimensional nilpotent Lie algebras are:*
  - $\mathfrak{g}_{6,k}$ *with* $k \in \{1, \ldots, 18, 20, 23, 25, \ldots, 28\}$,
  - $\mathfrak{g}_{6,k}(\varepsilon_1)$ *with* $k \in \{19, 21\}$ *and* $\varepsilon_1 \in \mathbb{F}^*/(\overset{*}{\sim})$,
  - $\mathfrak{g}_{6,k}(\varepsilon_2)$ *with* $k \in \{22, 24\}$ *and* $\varepsilon_2 \in \mathbb{F}/(\overset{*}{\sim})$.
- *If* $\mathrm{char}(\mathbb{F}) = 2$*, then the isomorphism types of six-dimensional nilpotent Lie algebras are:*
  - $\mathfrak{g}_{6,k}$ *with* $k \in \{1, \ldots, 18, 20, 23, 25, \ldots, 28\}$,
  - $\mathfrak{g}_{6,k}(\varepsilon_1)$ *with* $k \in \{19, 21\}$ *and* $\varepsilon_1 \in \mathbb{F}^*/(\overset{*}{\sim})$,
  - $\mathfrak{g}_{6,k}(\varepsilon_2)$ *with* $k \in \{22, 24\}$ *and* $\varepsilon_2 \in \mathbb{F}/(\overset{*+}{\sim})$,
  - $\mathfrak{g}_{6,k}^{(2)}$ *with* $k \in \{1, 2, 5, 6\}$,
  - $\mathfrak{g}_{6,k}^{(2)}(\varepsilon_3)$ *with* $k \in \{3, 4\}$ *and* $\varepsilon_3 \in \mathbb{F}^*/(\overset{*+}{\sim})$,
  - $\mathfrak{g}_{6,k}^{(2)}(\varepsilon_4)$ *with* $k \in \{7, 8\}$ *and* $\varepsilon_4 \in \mathbb{F}/(\overset{\psi}{\sim})$.

Table A.5 contains the non-vanishing Lie brackets (with respect to the basis $\{e_1, \ldots, e_6\}$) for Lie algebras over a field $\mathbb{F}$ of $\mathrm{char}(\mathbb{F}) \neq 2$. In the overview, the nine nilpotent Lie algebras of dimension 5 are added as well. They are denoted as $\mathfrak{g}_{5,i}$, where $1 \leq i \leq 9$, in such a way that $\mathfrak{g}_{6,i} = \mathfrak{g}_{5,i} \oplus \mathbb{F}$. When $\mathrm{char}(\mathbb{F}) = 2$, there are additional Lie algebras, for which the non-vanishing Lie brackets are stated in Table A.7. The Lie algebras from Theorem 8.1.15 form a complete set of representatives of the isomorphism classes of six-dimensional nilpotent Lie algebras. Two representatives from the equivalence class give rise to isomorphic Lie algebras, so the classification is irredundant. This means that a given six-dimensional nilpotent Lie algebra over some field is isomorphic to exactly one Lie algebra from the list.

As last result already suggests, the number of isomorphism types depends on the characteristic of the field, which explains why there is no uniform description for all fields. The next theorem is a summary (and a consequence) of the one before.

Theorem 8.1.16 ([15]).

- *Let* $\mathbb{F}$ *be a field with* $\mathrm{char}(\mathbb{F}) \neq 2$*. Denote* s *for the (possibly) infinite index of* $(\mathbb{F}^*)^2$ *in* $\mathbb{F}^*$*. Then the number of isomorphism types of nilpotent Lie algebras of dimension* 6 *over* $\mathbb{F}$ *is* $26 + 4s$*.*
- *Let* $\mathbb{F}$ *be a field with* $\mathrm{char}(\mathbb{F}) = 2$*. Denote* r *for the (possibly infinite) number of equivalence classes* $\overset{\psi}{\sim}$ *in* $\mathbb{F}$ *and* s *for the (possibly infinite)*

index of $(\mathbb{F}^*)^2$ *in* $\mathbb{F}^*$ *and* t *for the (possibly infinite) number of equivalence classes of* $\overset{*+}{\sim}$ *in* $\mathbb{F}$. *Then the number of isomorphism types of nilpotent Lie algebras of dimension* 6 *over* $\mathbb{F}$ *is* $26 + 2r + 2s + 4t$.

To determine the almost inner derivations of all six-dimensional nilpotent Lie algebras, it suffices to do the computations for all Lie algebras from the classification from Theorem 8.1.15. Tables A.6 and A.8 contain all results. Although we computed all results ourselves, we will omit the proofs.

Remark 8.1.17. We only mention some remarks about the results in Table A.6 and Table A.8.

- For the Lie algebras $\mathfrak{g}_{5,i}$ and $\mathfrak{g}_{6,i}$ (with $1 \leq i \leq 9$), all values are the same, except for $\dim(\mathrm{Der}(\mathfrak{g}))$. In the table, the first number is for $\mathfrak{g}_{5,i}$ and the second one for $\mathfrak{g}_{6,i}$.
- For some other Lie algebras, there are two numbers in the column $\dim(\mathrm{Der}(\mathfrak{g}))$ as well. That is because the dimension of the derivation algebra depends on the characteristic of the field $\mathbb{F}$. The largest number is the dimension when $\mathrm{char}(\mathbb{F}) = 2$; the smaller number is for $\mathrm{char}(\mathbb{F}) \neq 2$. Note that the dimension of the almost inner derivations does not depend on the characteristic. In other words, the linear maps which are derivations if and only if $\mathrm{char}(\mathbb{F}) = 2$ are not almost inner.
- For the Lie algebras $\mathfrak{g}_{6,22}(\varepsilon_2)$ and $\mathfrak{g}_{6,24}(\varepsilon_2)$, we distinguish the cases where $\varepsilon_2 = 0$ and $\varepsilon_2 \neq 0$. In the last case, we have $\varepsilon_2 \in \mathbb{F}^*/(\overset{*}{\sim})$ when $\mathrm{char}(\mathbb{F}) \neq 2$ and $\varepsilon_2 \in \mathbb{F}^*/(\overset{*+}{\sim})$ for $\mathrm{char}(\mathbb{F}) = 2$. Here we use the fact that $\mathbb{F}/(\overset{*}{\sim}) = \mathbb{F}^*/(\overset{*}{\sim}) \cup \{0\}$.
- The Lie algebras $\mathfrak{g}_{6,22}(\varepsilon_2)$ and $\mathfrak{g}_{6,24}(\varepsilon_2)$ appear twice in the list. In these cases, the dimension of the almost inner derivations depend on whether or not ε_2 is a square. It turns out that, for both Lie algebras $\mathfrak{g}$, we have that

  $$\dim(\mathrm{AID}(\mathfrak{g})) = \dim(\mathrm{Inn}(\mathfrak{g}))$$

  when $X^2 - \varepsilon_2 = 0$ has a solution in $\mathbb{F}$ and $\dim(\mathrm{AID}(\mathfrak{g})) = \dim(\mathrm{Inn}(\mathfrak{g})) + 4$ otherwise. For $\mathfrak{g}_{6,22}(\varepsilon)$, this is worked out in more detail in 8.1.3.
- Table A.6 is a generalisation of Remark 8.6 from [7]. There we made a similar list for six-dimensional nilpotent Lie algebras over $\mathbb{C}$, using the classification of [61]. The last column of the table indicates the isomorphic Lie algebra from Magnin's classification when we work over $\mathbb{C}$. In that case, we only have one case for $\mathfrak{g}_{6,22}(\varepsilon)$ and $\mathfrak{g}_{6,24}(\varepsilon)$, since the equation $X^2 - \varepsilon = 0$ always has solutions over $\mathbb{C}$.

- For the Lie algebras $\mathfrak{g}_{6,7}^{(2)}(\varepsilon_4)$ and $\mathfrak{g}_{6,8}^{(2)}(\varepsilon_4)$, there are non-inner almost inner derivations if and only if $X^2+X+\varepsilon_4=0$ has no solutions over $\mathbb{F}$.

We will elaborate on the computations for the Lie algebra $\mathfrak{g} := \mathfrak{g}_{6,22}(\varepsilon)$ (with $\varepsilon \in \mathbb{F}$). Note that the dimension of $\mathrm{Der}(\mathfrak{g})$ is different when $\mathrm{char}(\mathbb{F})=2$ and that the result depends on whether $\varepsilon=0$ or not. We give a basis of the derivation algebra and determine which derivations are almost inner. For the other Lie algebras, a similar computation can be made, but we omit the details. However, for the Lie algebras of the list with non-trivial almost inner derivations, we give examples of almost inner derivations, which together with the inner derivations form a basis of $\mathrm{AID}(\mathfrak{g})$.

Detailed computations for $\mathfrak{g} := \mathfrak{g}_{6,22}(\varepsilon)$

Consider the Lie algebra $\mathfrak{g} := \mathfrak{g}_{6,22}(\varepsilon)$, where $\varepsilon \in \mathbb{F}$. The non-zero Lie brackets are

$$\begin{aligned} [e_1,e_2] &= e_5, & [e_1,e_3] &= e_6, \\ [e_2,e_4] &= \varepsilon e_6, & [e_3,e_4] &= e_5. \end{aligned}$$

The associated matrix pencil, which we denote as $\mu A + \lambda B$ has determinant

$$\det(\mu A+\lambda B) = \det\begin{pmatrix} 0 & \mu & \lambda & 0 \\ -\mu & 0 & 0 & \varepsilon\lambda \\ -\lambda & 0 & 0 & \mu \\ 0 & -\varepsilon\lambda & -\mu & 0 \end{pmatrix} = (\varepsilon\lambda^2-\mu^2)^2.$$

- Consider the case that $\varepsilon=0$.

  When $\mathrm{char}(\mathbb{F})=2$, an arbitrary derivation D is given by

  $$D = a_1\mathrm{ad}(e_1)+\cdots+a_4\mathrm{ad}(e_4)+d_1D_1+\cdots+d_{10}D_{10}+e_{6,1}E_{6,1}+\cdots+e_{6,4}E_{6,4}$$

  (with coefficients in $\mathbb{F}$) and has matrix form

  $$D = \begin{pmatrix} d_1 & 0 & -d_6 & -d_{10} & 0 & 0 \\ d_2 & d_5 & d_7 & -d_3 & 0 & 0 \\ d_3 & d_{10} & d_8 & 0 & 0 & 0 \\ d_4 & d_6 & d_9 & d_1+d_5-d_8 & 0 & 0 \\ -a_2 & a_1 & -a_4 & a_3 & d_1+d_5 & -d_4+d_7 \\ e_{6,1}-a_3 & e_{6,2} & e_{6,3}+a_1 & e_{6,4} & d_{10} & d_1+d_8 \end{pmatrix},$$

  which means that $\dim(\mathrm{Der}(\mathfrak{g}))=18$.

For $\mathrm{char}(\mathbb{F}) \neq 2$, an arbitrary derivation D looks like

$$D = a_1\mathrm{ad}(e_1)+\cdots+a_4\mathrm{ad}(e_4)+d_1D_1+\cdots+d_9D_9+e_{6,1}E_{6,1}+\cdots+e_{6,4}E_{6,4}$$

(with coefficients in $\mathbb{F}$) and has matrix form

$$D = \begin{pmatrix} d_1 & 0 & -d_6 & 0 & 0 & 0 \\ d_2 & d_5 & d_7 & -d_3 & 0 & 0 \\ d_3 & 0 & d_8 & 0 & 0 & 0 \\ d_4 & d_6 & d_9 & d_1+d_5-d_8 & 0 & 0 \\ -a_2 & a_1 & -a_4 & a_3 & d_1+d_5 & -d_4+d_7 \\ e_{6,1}-a_3 & e_{6,2} & e_{6,3}+a_1 & e_{6,4} & 0 & d_1+d_8 \end{pmatrix},$$

so $\dim(\mathrm{Der}(\mathfrak{g})) = 17$. Note that $\mathfrak{g}$ is isomorphic to the Lie algebra from Example 4.3.7. This means that $\dim(\mathrm{AID}(\mathfrak{g})) = \dim(\mathrm{Inn}(\mathfrak{g})) + 2 = 6$. When $\varepsilon = 0$, a basis for $\mathrm{AID}(\mathfrak{g})$ is

$$\{\mathrm{ad}(e_1), \ldots, \mathrm{ad}(e_4), E_{6,1}, E_{6,3}\},$$

where the determination maps $\varphi_{E_{6,1}}, \varphi_{E_{6,3}} : \mathfrak{g} \to \mathfrak{g}$ are given by

$$\varphi_{E_{6,1}}(x) = \begin{cases} \frac{x_4}{x_1}e_2 + e_3 & \text{if } x_1 \neq 0, \\ 0 & \text{if } x_1 = 0, \end{cases}$$

$$\varphi_{E_{6,3}}(x) = \begin{cases} -e_1 - \frac{x_2}{x_3}e_4 & \text{if } x_3 \neq 0, \\ 0 & \text{if } x_3 = 0. \end{cases}$$

Note that $\mathfrak{g}$ is 2–step nilpotent, so only the central derivations can be almost inner. Take an arbitrary linear combination $D := aE_{6,2} + bE_{6,4}$, with $a, b \in \mathbb{F}$, then D is not $\mathcal{B}$–almost inner and hence, it is not almost inner.

- Suppose that $\varepsilon \neq 0$, then we have $\dim(\mathrm{Der}(\mathfrak{g})) = 18$ for $\mathrm{char}(\mathbb{F}) = 2$ and $\dim(\mathrm{Der}(\mathfrak{g})) = 16$ when $\mathrm{char}(\mathbb{F}) \neq 2$.

  When $\mathrm{char}(\mathbb{F}) = 2$, an arbitrary derivation D looks like

$$D = a_1\mathrm{ad}(e_1)+\cdots+a_4\mathrm{ad}(e_4)+d_1D_1+\cdots+d_{10}D_{10}+e_{6,1}E_{6,1}+\cdots+e_{6,4}E_{6,4}$$

  (with coefficients in $\mathbb{F}$) and has matrix form

$$\begin{pmatrix} d_1 & -\varepsilon d_8 & -d_6 & \varepsilon(d_4+d_7)-d_9 & 0 & 0 \\ d_2 & d_5 & d_7 & -d_3 & 0 & 0 \\ d_3 & d_9 & d_{10} & -\varepsilon d_2 & 0 & 0 \\ d_4 & d_6 & d_8 & d_1+d_5-d_{10} & 0 & 0 \\ -a_2 & a_1 & -a_4 & a_3 & d_1+d_5 & \varepsilon d_4+d_9 \\ e_{6,1}-a_3 & e_{6,2}-\varepsilon a_4 & e_{6,3}+a_1 & e_{6,4}+\varepsilon a_2 & d_4+d_7 & d_1+d_{10} \end{pmatrix}.$$

For $\mathrm{char}(\mathbb{F}) \neq 2$, a derivation D is given by

$$D = a_1\mathrm{ad}(e_1)+\cdots+a_4\mathrm{ad}(e_4)+d_1D_1+\cdots+d_8D_8+e_{6,1}E_{6,1}+\cdots+e_{6,4}E_{6,4}$$

and has matrix form

$$\begin{pmatrix} d_1 & -\varepsilon d_8 & -d_6 & \varepsilon d_4 & 0 & 0 \\ d_2 & d_5 & d_7 & -d_3 & 0 & 0 \\ d_3 & \varepsilon d_7 & d_5 & -\varepsilon d_2 & 0 & 0 \\ d_4 & d_6 & d_8 & d_1 & 0 & 0 \\ a_2 & a_1 & -a_4 & a_3 & d_1+d_5 & -d_4+d_7 \\ e_{6,1}-a_3 & e_{6,2}-\varepsilon a_4 & e_{6,3}+a_1 & e_{6,4}+\varepsilon a_2 & \varepsilon(d_7-d_4) & d_1+d_5 \end{pmatrix}.$$

We now consider two different cases (which do not depend on the characteristic, but on the value of ε). Note that we only have to consider the central derivations.

- Suppose that $X^2-\varepsilon=0$ does not have a solution over $\mathbb{F}$. Then $\mathfrak{g}$ is nonsingular over $\mathbb{F}$. It follows from Corollary 4.3.9 that $\mathrm{AID}(\mathfrak{g}) = \mathcal{C}(\mathfrak{g})$ and
$$\dim(\mathrm{AID}(\mathfrak{g})) = 2\dim(\mathrm{Inn}(\mathfrak{g})) = 8.$$
Take $i \in \{1,4\}$, then the derivation $E_{6,i} : \mathfrak{g} \to \mathfrak{g}$ is almost inner and a determination map $\varphi_{E_{6,i}} : \mathfrak{g} \to \mathfrak{g}$ is given by
$$\varphi_{E_{6,i}}(x) = \begin{cases} \frac{x_i}{x_1^2-\varepsilon x_4^2}(x_4e_2+x_1e_3) & \text{if } x_i \neq 0, \\ 0 & \text{if } x_i = 0. \end{cases}$$
For $j \in \{2,3\}$, the derivation $E_{6,j} : \mathfrak{g} \to \mathfrak{g}$ is almost inner with determination map
$$\varphi_{E_{6,j}}(x) = \begin{cases} \frac{x_j}{\varepsilon x_2^2-x_3^2}(x_3e_1+x_2e_4) & \text{if } x_j \neq 0, \\ 0 & \text{if } x_j = 0. \end{cases}$$
Hence, when $X^2-\varepsilon=0$ has no solution, a basis for $\mathrm{AID}(\mathfrak{g})$ is
$$\{\mathrm{ad}(e_1),\dots,\mathrm{ad}(e_4),E_{6,1},E_{6,2},E_{6,3},E_{6,4}\},$$
which means that every central derivation is almost inner.
- Assume that $X^2-\varepsilon=0$ has a solution $\alpha \in \mathbb{F}$, so $\alpha^2=\varepsilon$. Take an arbitrary $x=\sum_{i=1}^6 x_ie_i \in \mathfrak{g}$.
Consider $D := b_1E_{6,1}+b_2E_{6,2}+b_3E_{6,3}+b_4E_{6,4}$, where $b_1,\dots,b_4 \in \mathbb{F}$. Suppose that D is almost inner. Note that
$$[\alpha e_1+e_4,x] = (\alpha x_2-x_3)e_5+(-\varepsilon x_2+\alpha x_3)e_6.$$

We have that $D(\alpha e_1 + e_4) = (\alpha b_1 + b_4)e_6$, which means that $\alpha x_2 = x_3$. It follows that

$$-\varepsilon x_2 + \alpha x_3 = (\alpha^2 - \varepsilon)x_2 = 0,$$

so $b_1 = b_4 = 0$. Similarly, we have that

$$[e_2 + \alpha e_3, x] = (\alpha x_4 - x_1)e_5 + (\varepsilon x_4 - \alpha x_1)e_6.$$

Since $D(e_2 + \alpha e_3) = (b_2 + \alpha b_3)e_6$, last equation shows that $\alpha x_4 = x_1$. However, because

$$\varepsilon x_4 - \alpha x_1 = (\varepsilon - \alpha^2)x_4 = 0$$

holds, we obtain that $b_2 = b_3 = 0$.

This means that $D = b_1 E_{6,1} + b_2 E_{6,2} + b_3 E_{6,3} + b_4 E_{6,4}$ is almost inner if and only if $b_1 = b_2 = b_3 = b_4 = 0$. Since

$$\{\mathrm{ad}(e_1), \mathrm{ad}(e_2), \mathrm{ad}(e_3), \mathrm{ad}(e_4), E_{6,1}, E_{6,2}, E_{6,3}, E_{6,4}\}$$

forms a basis for the central derivations, we find as a conclusion that all almost inner derivations of $\mathfrak{g}$ are in fact inner.

The matrix pencil P for this Lie algebra has determinant $(\varepsilon\lambda^2 - \mu^2)^2$. Denote $X := \frac{\mu}{\lambda}$, then $\det(P) = 0$ if and only if $(X^2 - \varepsilon)^2 = 0$. Let l be the number of different linear factors of the determinant of P. As we observed at the end of Chapter 5, we have for this Lie algebra that

$$\dim(\mathrm{AID}(\mathfrak{g})/\mathrm{Inn}(\mathfrak{g})) = \deg(\det(P)) - 2l = 2 \cdot (2 - l).$$

Determination maps for the non-inner almost inner derivations

We can do the same computations as in the previous section to obtain the results from Tables A.6 and A.8. However, we will omit the calculations, since they are similar to those from the Lie algebra $\mathfrak{g}_{6,22}(\varepsilon)$. This subsection contains an overview of non-inner almost inner derivations, which form, together with the inner ones, a basis for $\mathrm{AID}(\mathfrak{g})$. We also present the determination maps.

- For $\mathfrak{g} \in \{\mathfrak{g}_{5,5}, \mathfrak{g}_{6,5}\}$, the derivation $E_{5,4} : \mathfrak{g} \to \mathfrak{g}$ is almost inner with determination map

$$\varphi_{E_{5,4}}(x) = \begin{cases} \frac{x_4}{x_1} e_3 & \text{if } x_1 \neq 0, \\ -e_2 & \text{if } x_1 = 0. \end{cases}$$

- For the Lie algebra $\mathfrak{g} \in \{\mathfrak{g}_{5,6}, \mathfrak{g}_{6,6}\}$, the derivation $E_{5,2} : \mathfrak{g} \to \mathfrak{g}$ is almost inner with determination map

$$\varphi_{E_{5,2}}(x) = \begin{cases} \frac{x_2}{x_1} e_4 & \text{if } x_1 \neq 0, \\ e_3 & \text{if } x_1 = 0. \end{cases}$$

- For the Lie algebra $\mathfrak{g} := \mathfrak{g}_{6,12}$, the derivation $E_{6,5} : \mathfrak{g} \to \mathfrak{g}$ is almost inner with determination map

$$\varphi_{E_{6,5}}(x) = \begin{cases} \frac{x_5}{x_1} e_4 & \text{if } x_1 \neq 0, \\ -e_2 & \text{if } x_1 = 0. \end{cases}$$

- For the Lie algebra $\mathfrak{g} := \mathfrak{g}_{6,13}$, the derivation $E_{6,4} : \mathfrak{g} \to \mathfrak{g}$ is almost inner with determination map

$$\varphi_{E_{6,4}}(x) = \begin{cases} \frac{x_4}{x_1} e_5 & \text{if } x_1 \neq 0, \\ -e_3 & \text{if } x_1 = 0. \end{cases}$$

- For the Lie algebra $\mathfrak{g} := \mathfrak{g}_{6,14}$, the derivation $E_{5,2} : \mathfrak{g} \to \mathfrak{g}$ is almost inner with determination map

$$\varphi_{E_{5,2}}(x) = \begin{cases} 0 & \text{if } x_2 = 0, \\ \frac{1}{x_2}(x_2 e_4 + x_3 e_5) & \text{if } x_2 \neq 0 \text{ and } x_1 \neq 0, \\ e_3 - \frac{x_4}{x_2} e_5 & \text{if } x_2 \neq 0 \text{ and } x_1 = 0. \end{cases}$$

- For the Lie algebra $\mathfrak{g} := \mathfrak{g}_{6,15}$, the derivation $E_{6,2} : \mathfrak{g} \to \mathfrak{g}$ is almost inner with determination map

$$\varphi_{E_{6,2}}(x) = \begin{cases} \frac{x_2}{x_1} e_5 & \text{if } x_1 \neq 0, \\ e_4 & \text{if } x_1 = 0. \end{cases}$$

- For the Lie algebra $\mathfrak{g} := \mathfrak{g}_{6,17}$, the derivation $E_{5,4} : \mathfrak{g} \to \mathfrak{g}$ is almost inner with determination map

$$\varphi_{E_{5,4}}(x) = \begin{cases} \frac{x_4}{x_1} e_3 & \text{if } x_1 \neq 0, \\ -e_2 & \text{if } x_1 = 0. \end{cases}$$

- We already elaborated on the detailed computations for the Lie algebra $\mathfrak{g}_{6,22}(\varepsilon)$ (with $\varepsilon \in \mathbb{F}$) in the previous section.

- For the Lie algebra $\mathfrak{g} := \mathfrak{g}_{6,23}$, the derivation $E_{6,1} : \mathfrak{g} \to \mathfrak{g}$ is almost inner with determination map

$$\varphi_{E_{6,1}}(x) = \begin{cases} \frac{-x_2}{x_1} e_3 + e_4 & \text{if } x_1 \neq 0, \\ 0 & \text{if } x_1 = 0. \end{cases}$$

  The derivation $E_{5,4} : \mathfrak{g} \to \mathfrak{g}$ is almost inner with determination map

$$\varphi_{E_{5,4}}(x) = \begin{cases} \frac{-x_2}{x_1} e_3 + e_4 & \text{if } x_1 \neq 0, \\ 0 & \text{if } x_1 = 0. \end{cases}$$

- For the Lie algebra $\mathfrak{g} := \mathfrak{g}_{6,24}(0)$, the derivation $E_{6,2} : \mathfrak{g} \to \mathfrak{g}$ is almost inner with determination map

$$\varphi_{E_{6,2}}(x) = \begin{cases} e_3 - \frac{x_1}{x_2} e_4 & \text{if } x_2 \neq 0, \\ 0 & \text{if } x_2 = 0. \end{cases}$$

  The derivation $E_{5,4} : \mathfrak{g} \to \mathfrak{g}$ is almost inner with determination map

$$\varphi_{E_{5,4}}(x) = \begin{cases} \frac{x_4}{x_2} e_4 & \text{if } x_2 \neq 0, \\ \frac{x_4}{x_1} e_3 & \text{if } x_2 = 0 \text{ and } x_1 \neq 0, \\ -\frac{x_4}{x_3} e_1 & \text{if } x_1 = x_2 = 0 \text{ and } x_3 \neq 0, \\ -e_2 & \text{if } x_1 = x_2 = x_3 = 0. \end{cases}$$

- For the Lie algebra $\mathfrak{g} := \mathfrak{g}_{6,24}(\varepsilon)$ with $\varepsilon \in \mathbb{F}$, we require that $X^2 - \varepsilon = 0$ has no solutions in $\mathbb{F}$. Take $i \in \{1, 2\}$, then the derivation $E_{6,i} : \mathfrak{g} \to \mathfrak{g}$ is almost inner and a determination map $\varphi_{E_{6,i}} : \mathfrak{g} \to \mathfrak{g}$ is given by

$$\varphi_{E_{6,i}}(x) = \begin{cases} \frac{x_i}{x_2^2 - \varepsilon x_1^2}(x_2 e_3 - x_1 e_4) & \text{if } x_i \neq 0, \\ 0 & \text{if } x_i = 0. \end{cases}$$

  For $j \in \{3, 4\}$, the derivation $E_{6,j} : \mathfrak{g} \to \mathfrak{g}$ is almost inner with determination map

$$\varphi_{E_{6,j}}(x) = \begin{cases} \frac{x_j}{x_2^2 - \varepsilon x_1^2}(x_2 e_3 - x_1 e_4) & \text{if } x_1 \neq 0, \\ \frac{x_j}{x_2} e_3 & \text{if } x_1 = 0 \text{ and } x_2 \neq 0, \\ \frac{x_j}{\varepsilon x_4^2 - x_3^2}(-x_4 e_1 + x_3 e_2) & \text{if } x_1 = x_2 = 0. \end{cases}$$

- For the Lie algebra $\mathfrak{g} := \mathfrak{g}_{6,1}^{(2)}$, the derivation $E_{6,4} : \mathfrak{g} \to \mathfrak{g}$ is almost inner with determination map

$$\varphi_{E_{6,4}}(x) = \begin{cases} \frac{x_4}{x_1} e_5 & \text{if } x_1 \neq 0, \\ e_3 & \text{if } x_1 = 0. \end{cases}$$

- For the Lie algebra $\mathfrak{g} := \mathfrak{g}_{6,2}^{(2)}$, the derivation $E_{6,4} : \mathfrak{g} \to \mathfrak{g}$ is almost inner with determination map

$$\varphi_{E_{6,4}}(x) = \begin{cases} \frac{x_4}{x_1} e_5 & \text{if } x_1 \neq 0, \\ e_3 & \text{if } x_1 = 0. \end{cases}$$

- For the Lie algebra $\mathfrak{g} := \mathfrak{g}_{6,3}^{(2)}(\varepsilon)$ with $\varepsilon \in \mathbb{F}^*$, the derivation $E_{5,2} : \mathfrak{g} \to \mathfrak{g}$ is almost inner with determination map

$$\varphi_{E_{5,2}}(x) = \begin{cases} \frac{1}{x_1}(x_2 e_4 + x_3 e_5) & \text{if } x_1 \neq 0, \\ e_4 + (\varepsilon_3 + \frac{x_4}{x_2}) & \text{if } x_1 = 0 \text{ and } x_2 \neq 0, \\ 0 & \text{if } x_1 = x_2 = 0. \end{cases}$$

- For the Lie algebra $\mathfrak{g} := \mathfrak{g}_{6,7}^{(2)}(\varepsilon)$ with $\varepsilon \in \mathbb{F}$, we require that $X^2 + X + \varepsilon = 0$ has no solutions over $\mathbb{F}$. Take $i \in \{1, 4\}$. The derivation $E_{6,i} : \mathfrak{g} \to \mathfrak{g}$ is almost inner and a determination map $\varphi_{E_{6,i}} : \mathfrak{g} \to \mathfrak{g}$ is given by

$$\varphi_{E_{6,i}}(x) = \begin{cases} \frac{x_i}{x_1^2 + x_1 x_4 + \varepsilon x_4^2}(x_4 e_2 + x_1 e_3) & \text{if } x_i \neq 0, \\ 0 & \text{if } x_i = 0. \end{cases}$$

  For $j \in \{2, 3\}$, the derivation $E_{6,j} : \mathfrak{g} \to \mathfrak{g}$ is almost inner with determination map

$$\varphi_{E_{6,j}}(x) = \begin{cases} \frac{x_j}{x_3^2 + x_2 x_3 + \varepsilon x_2^2}(x_3 e_1 + x_2 e_4) & \text{if } x_j \neq 0, \\ 0 & \text{if } x_j = 0. \end{cases}$$

- For the Lie algebra $\mathfrak{g} := \mathfrak{g}_{6,8}^{(2)}(\varepsilon)$ with $\varepsilon \in \mathbb{F}$, we require that $X^2 + X + \varepsilon = 0$ has no solutions over $\mathbb{F}$. Take $i \in \{1, 2\}$. The derivation $E_{6,i} : \mathfrak{g} \to \mathfrak{g}$ is almost inner and a determination map $\varphi_{E_{6,i}} : \mathfrak{g} \to \mathfrak{g}$ is given by

$$\varphi_{E_{6,i}}(x) = \begin{cases} \frac{x_i}{x_1^2 + x_1 x_2 + \varepsilon x_2^2}(x_2 e_3 + x_1 e_4) & \text{if } x_i \neq 0, \\ 0 & \text{if } x_i = 0. \end{cases}$$

  For $j \in \{3, 4\}$, the derivation $E_{6,j} : \mathfrak{g} \to \mathfrak{g}$ is almost inner with determination map

$$\varphi_{E_{6,j}}(x) = \begin{cases} \frac{x_j}{x_2^2 + x_1 x_2 + \varepsilon x_1^2}(x_2 e_3 + x_1 e_4) & \text{if } x_1 \neq 0, \\ \frac{x_j}{x_2} e_3 & \text{if } x_1 = 0 \text{ and } x_2 \neq 0, \\ \frac{x_j}{x_3^2 + x_3 x_4 + \varepsilon x_4^2}(x_4 e_1 + x_3 e_2) & \text{if } x_1 = x_2 = 0. \end{cases}$$

8.2 Over a field of characteristic zero

In the last section, we computed the almost inner derivations for all solvable Lie algebras (over fields of arbitrary characteristic) of dimension at most four. It turns out that if the characteristic is not two, then we only have the inner derivations. In this section, we will only look at Lie algebras over fields of characteristic zero. First, we will consider the non-solvable Lie algebras of dimension four. Then, we also compute the almost inner derivations for complex and real (non-decomposable) non-nilpotent Lie algebras of dimension five. The results over $\mathbb{C}$ already appeared in [7].

8.2.1 Lie algebras of dimension 4

There doesn't exist yet a complete list for all 4–dimensional Lie algebras over an arbitrary field. We already treated the solvable Lie algebras in the last section, so it suffices to consider the non-solvable ones. We will restrict ourselves Lie algebras over a field $\mathbb{F}$ of characteristic zero, since in that case, we can make use of the Levi-Mal'cev theorem.

Lemma 8.2.1. *Let $\mathfrak{g}$ be a non-solvable 4–dimensional Lie algebra over a field $\mathbb{F}$ of characteristic zero. If $\mathfrak{g}$ is non-solvable, then $\mathfrak{g}$ is isomorphic to $\mathfrak{g}(\alpha, \beta) \oplus \mathbb{F}$, where $\alpha, \beta \in \mathbb{F}^*$.*

Proof. Let $\mathfrak{g}$ be a 4–dimensional Lie algebra over a field $\mathbb{F}$ of characteristic zero. By the Levi decomposition, we can write $\mathfrak{g} = \mathfrak{r} \rtimes \mathfrak{s}$, where $\mathfrak{s}$ is semisimple and $\mathfrak{r}$ is the solvable radical. Over a field of characteristic zero, there are no semisimple Lie algebras of dimension 4. Since $\mathfrak{g}$ is non-solvable, we find from the previous section that $\mathfrak{s} = \mathfrak{g}(\alpha, \beta)$, for $\alpha, \beta \in \mathbb{F}^*$. Therefore, we must have that $\mathfrak{r} = \mathbb{F}$ and the semidirect product is in fact a direct sum. □

The number of isomorphism classes of non-solvable 4–dimensional Lie algebras over a field $\mathbb{F}$ of characteristic zero can be made more precise by specifying the field $\mathbb{F}$. For instance, over $\mathbb{C}$, there is only one, whereas there are two classes over $\mathbb{R}$, isomorphic to $\mathfrak{g}(1, -1) \oplus \mathbb{F}$ or $\mathfrak{g}(1, 1) \oplus \mathbb{F}$. This argument can not be used for a non-solvable Lie algebra over a field $\mathbb{F}$ of prime characteristic.

Example 8.2.2. Let $\mathbb{F}$ be a field with $\mathrm{char}(\mathbb{F}) = p$ and take $n \in \mathbb{N}_0$. Zassenhaus ([91]) defined a Lie algebra $W(1 : n)$ over $\mathbb{F}$ with basis $\{e_\alpha \mid \alpha \in \mathbb{F}_{p^n}\}$ and non-vanishing Lie brackets

$$[e_\alpha, e_\beta] = (\beta - \alpha) e_{\alpha+\beta}$$

for all $\alpha, \beta \in \mathbb{F}_{p^n}$. He proved that $W(1:n)$ is simple when $p > 2$ (and $n \in \mathbb{N}_0$). For $p = n = 2$, we find a non-solvable 4–dimensional Lie algebra with basis $\{e_1, \ldots, e_4\}$ and given by

$$\begin{aligned} &[e_1, e_2] = e_2, && [e_1, e_3] = \beta e_3, && [e_1, e_4] = (\beta+1)e_4, \\ &[e_2, e_3] = (\beta+1)e_4, && [e_2, e_4] = \beta e_3, && [e_3, e_4] = e_2, \end{aligned}$$

where $\beta^2 = \beta + 1$. A direct computation shows that an arbitrary derivation D is a linear combination

$$D = a_1 \mathrm{ad}(e_1) + \cdots + a_4 \mathrm{ad}(e_4) + d_1 D_1,$$

which has matrix form

$$D = \begin{pmatrix} 0 & 0 & 0 & 0 \\ a_2 & a_1 & a_4 & a_3 \\ \beta a_3 & \beta a_4 & \beta a_1 + d_1 & \beta a_2 \\ (\beta+1)a_4 & (\beta+1)a_3 & (\beta+1)a_2 & (\beta+1)a_1 + d_1 \end{pmatrix}.$$

Note that D_1 is not almost inner. Take an arbitrary $x = \sum_{i=1}^4 x_i e_i$, then

$$[e_1 + e_2 + e_3 + e_4, x] = (x_1 + x_2 + x_3 + x_4)(e_2 + \beta e_3 + (\beta+1)e_4).$$

However, we find that $D_1(e_1 + e_2 + e_3 + e_4) = e_3 + e_4$.

Proposition 4.1.8 and the observations from the last section imply the following result.

Proposition 8.2.3. *Let $\mathfrak{g}$ be a Lie algebra of dimension at most 4 over a field $\mathbb{F}$ of characteristic zero. Then $\mathrm{AID}(\mathfrak{g}) = \mathrm{Inn}(\mathfrak{g})$ holds.*

In 8.1.2, we studied an example of a 4–dimensional Lie algebra over a field $\mathbb{F}$ of $\mathrm{char}(\mathbb{F}) = 2$ with $\mathrm{AID}(\mathfrak{g}) \neq \mathrm{Inn}(\mathfrak{g})$.

8.2.2 Lie algebras of dimension 5

We showed before that Lie algebras of dimension $n \leq 4$ over a field of characteristic zero do not have non-inner almost inner derivations. As we already computed in Example 4.1.6, this is different in dimension five. In this subsection, we will compute the almost inner derivations for 5–dimensional Lie algebras over $\mathbb{C}$ and $\mathbb{R}$.

Complex Lie algebras

For complex Lie algebras, we will use the results from [29]. Here, the authors describe the moduli space for 5–dimensional complex Lie algebras, in terms of 24 families with up to 4 parameters. We will change the notation of that paper to be consistent with the rest of this chapter.

Theorem 8.2.4 ([29])**.** *Let $\mathfrak{g}$ be a Lie algebra of dimension 5 over $\mathbb{C}$. Then $\mathfrak{g}$ is isomorphic to (at least) one Lie algebra of Table A.9.*

Note that the Lie brackets of d_2 are not correct in Table 3 and in the definition on page 429 of [29]. There also exist other lists of 5–dimensional complex Lie algebras, see for instance [72]. However, the determination of almost inner derivations is much more efficient when there are less families of Lie algebras.

For each family, or type, we calculate the space $\mathrm{AID}(\mathfrak{g})$ for all possible parameters. The computation is easy for the types without parameters. Further, by choosing all parameters equal to zero, we obtain the nilpotent Lie algebras. We already know from the previous section that every complex nilpotent Lie algebra of dimension 5 having a non-inner almost inner derivation is isomorphic to $\mathfrak{g}_{5,5}$ or $\mathfrak{g}_{5,6}$. The hardest computations are for the families with several parameters. There, we often had to consider different cases. A long but straightforward computation shows that, for non-nilpotent algebras, the only family with non-inner almost inner derivations is $C_{5,12}(p,q,r)$, where we need that $p=0$. Therefore, we will consider this type in more detail.

Definition 8.2.5 (Lie algebra $A(q,r)$)**.** The family of 5–dimensional complex Lie algebras $A(q,r) := C_{5,12}(0,q,r)$ with $q,r \in \mathbb{C}$ has basis $\{e_1,\dots,e_5\}$ and is defined by the Lie brackets

$$\begin{aligned} &[e_1,e_5]=e_2, && [e_2,e_5]=(q+r)e_2, && [e_3,e_4]=e_2,\\ &[e_3,e_5]=e_1+qe_3, && [e_4,e_5]=e_3+re_4. \end{aligned}$$

It is straightforward to see that $\dim(\mathrm{Inn}(A(q,r)))=4$ for all $q,r\in\mathbb{C}$. We also have that $\dim(\mathrm{Der}(A(0,0)))=8$ and $\dim(\mathrm{Der}(A(q,r)))=7$ when $(q,r)\neq(0,0)$. A direct computation shows that $\mathrm{AID}(A(q,r))=\mathrm{Inn}(A(q,r))$ if and only if $q\cdot r\neq 0$ and $q+r\neq 0$. Otherwise, we have

$$\dim(\mathrm{AID}(A(q,r)))=\dim(\mathrm{Inn}(A(q,r))+1=5.$$

We can determine the Lie algebras $A(q,r)$ with non-inner almost inner derivations up to isomorphism.

Lemma 8.2.6. *Every Lie algebra $A(q,r)$ satisfying $q\cdot r=0$ or $q+r=0$ is either isomorphic to $A(1,0)$, to $A(1,-1)$ or to $A(0,0)\cong\mathfrak{g}_{5,6}$.*

Proof. It follows from [29] that $A(q,r) \cong A(r,q)$. Further, $A(0,0)$ is filiform nilpotent and isomorphic to $\mathfrak{g}_{5,6}$. We may assume that $(q,r) \neq (0,0)$. Suppose that $q \cdot r = 0$. Without loss of generality, we can take $q \neq 0$ and $r = 0$. Then there is a Lie algebra isomorphism $\varphi\colon A(q,0) \to A(1,0)$ given by

$$\begin{pmatrix} q^2 & 0 & 0 & 0 & 0 \\ 1-q^2 & q & 0 & 0 & 0 \\ 0 & 0 & q & 0 & 0 \\ 0 & 0 & 0 & 1 & \frac{1-q^2}{q} \\ 0 & 0 & 0 & 0 & q \end{pmatrix}.$$

Secondly, consider the case where $q + r = 0$ and $q \neq 0$. Then there is a Lie algebra isomorphism $\varphi\colon A(q,-q) \to A(1,-1)$ given by

$$\begin{pmatrix} 1 & 0 & \frac{q^2-1}{q} & \frac{q^2-1}{q^2} & 0 \\ 0 & q & \frac{q^2-1}{q} & 0 & 0 \\ 0 & 0 & q & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & q \end{pmatrix}.$$

Recall that a Lie algebra $\mathfrak{g}$ is 'unimodular' if and only if $\operatorname{tr}(\operatorname{ad}(x)) = 0$ for all $x \in \mathfrak{g}$. We find that $A(q,r)$ is unimodular if and only if $q+r = 0$. Hence $A(1,-1)$ is unimodular, but $A(1,0)$ is not, so they cannot be isomorphic. Both $A(1,-1)$ and $A(1,0)$ are solvable and non-nilpotent, whereas $A(0,0)$ is nilpotent. □

We can summarise the calculations for the 5–dimensional complex Lie algebras with the following result.

Proposition 8.2.7. *Let* $\mathfrak{g}$ *be a complex Lie algebra of dimension* 5*. If we have* $\operatorname{AID}(\mathfrak{g}) \neq \operatorname{Inn}(\mathfrak{g})$*, then* $\mathfrak{g}$ *is isomorphic to one of the Lie algebras*

$$\mathfrak{g}_{5,5}, \qquad \mathfrak{g}_{5,6}, \qquad A(1,0) \quad \text{or} \quad A(1,-1).$$

In each case, we have that $\dim(\operatorname{AID}(\mathfrak{g})) = \dim(\operatorname{Inn}(\mathfrak{g})) + 1 = 5$.

We present the determination maps for the non-nilpotent Lie algebras. Take an arbitrary $x = \sum_{i=1}^{5} x_i e_i \in \mathfrak{g}$.

- For the Lie algebra $\mathfrak{g} := A(1,0)$, the derivation $E_{1,5} : \mathfrak{g} \to \mathfrak{g}$ is almost inner with determination map

$$\varphi_{E_{1,5}}(x) = \begin{cases} \frac{1}{x_5}(x_4 - x_3)e_1 - e_3 - e_4 & \text{if } x_5 \neq 0, \\ 0 & \text{if } x_5 = 0. \end{cases}$$

- For $\mathfrak{g} := A(1,-1)$, the derivation $D : \mathfrak{g} \to \mathfrak{g} : x \mapsto x_1 e_2 + x_5(e_1 + e_4)$ is almost inner with determination map

$$\varphi_D(x) = \begin{cases} \frac{-1}{x_5}(x_1 - x_3 - x_4)e_1 - e_3 + e_4 & \text{if } x_5 \neq 0, \\ \frac{-x_1}{x_4} e_3 & \text{if } x_5 = 0 \text{ and } x_4 \neq 0, \\ \frac{x_1}{x_3} e_4 & \text{if } x_4 = x_5 = 0 \text{ and } x_3 \neq 0, \\ e_5 & \text{if } x_3 = x_4 = x_5 = 0. \end{cases}$$

Real Lie algebras

For 5–dimensional Lie algebras, the first list appeared in [66], albeit without the structure constants of the algebras. In [70], the authors give a classification of the non-decomposable Lie algebras. The same list is stated in [30].

Theorem 8.2.8 ([30])**.** *Let $\mathfrak{g}$ be a non-decomposable non-nilpotent Lie algebra of dimension 5 over $\mathbb{R}$. Then $\mathfrak{g}$ is isomorphic to at least one Lie algebra of Table A.10.*

Remark 8.2.9. In [30], the authors put extra conditions on the parameters $u, v, w \in \mathbb{R}$ to ensure that every non-decomposable non-nilpotent Lie algebra of dimension 5 is isomorphic to exactly one Lie algebra of the classification. However, we did not include these restrictions, since there are a lot of different cases and the list from Table A.10 is sufficient for doing the computations.

As for the 5–dimensional complex Lie algebras, we calculate the space $\mathrm{AID}(\mathfrak{g})$ for all possible parameters. The following result summarises the computations for the 5–dimensional real Lie algebras.

Proposition 8.2.10. *Let $\mathfrak{g}$ be a real Lie algebra of dimension 5, then*

$$\dim(\mathrm{AID}(\mathfrak{g})) = \dim(\mathrm{Inn}(\mathfrak{g})) + n,$$

where $n \in \{0, 1, 2\}$.

- *We have that $n = 1$ if and only if $\mathfrak{g}$ is isomorphic to one of the Lie algebras*

$$\mathfrak{g}_{5,5}, \qquad \mathfrak{g}_{5,6}, \qquad R_{5,20}(0), \qquad R_{5,26}(0,1) \qquad \textit{or} \qquad R_{5,29}.$$

- *Further, $n = 2$ holds if and only if $\mathfrak{g}$ is isomorphic to $R_{5,39}$.*

Note that all of the above Lie algebras are pairwise non-isomorphic over $\mathbb{R}$. The nilpotent Lie algebras $\mathfrak{g}_{5,5}$ and $\mathfrak{g}_{5,6}$ are discussed in 8.1.3. We present the determination maps for the non-nilpotent Lie algebras from the first case.

- For $\mathfrak{g} \in \{R_{5,20}(0), R_{5,26}(0,1)\}$, the derivation $E_{1,4} : \mathfrak{g} \to \mathfrak{g}$ is almost inner with determination map

$$\varphi_{E_{1,4}}(x) = \begin{cases} \frac{-x_4}{x_5} e_4 & \text{if } x_5 \neq 0, \\ \frac{x_4}{x_2} e_3 & \text{if } x_5 = 0 \text{ and } x_2 \neq 0, \\ \frac{-x_4}{x_3} e_2 & \text{if } x_2 = x_5 = 0 \text{ and } x_3 \neq 0, \\ e_5 & \text{if } x_2 = x_3 = x_5 = 0. \end{cases}$$

- For the Lie algebra $\mathfrak{g} := R_{5,29}$, the derivation $E_{3,5} : \mathfrak{g} \to \mathfrak{g}$ is almost inner with determination map

$$\varphi_{E_{3,5}}(x) = \begin{cases} \frac{-x_2}{x_5} e_1 - e_4 & \text{if } x_5 \neq 0, \\ 0 & \text{if } x_5 = 0. \end{cases}$$

For each of these three Lie algebras, we can take the complexification and compare with the results from Proposition 8.2.7. It turns out that the Lie algebras $\mathfrak{g}_1 := \mathbb{C} \otimes_{\mathbb{R}} R_{5,20}(0)$ and $\mathfrak{g}_2 := \mathbb{C} \otimes_{\mathbb{R}} R_{5,26}(0,1)$ are both isomorphic to $A(1,-1)$. For $1 \leq i \leq 2$, the isomorphism $\varphi_i : \mathfrak{g}_i \to A(1,-1)$ is given by

$$\begin{pmatrix} 1 & 2 & -3 & -1 & 0 \\ 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 2 & 0 & -2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & i & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & i & 0 & i \\ -1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & i \end{pmatrix}$$

for φ_1 respectively φ_2. Further, $\varphi_3 : \mathbb{C} \otimes_{\mathbb{R}} R_{5,29} \to A(1,0)$ is a Lie algebra isomorphism given by

$$\begin{pmatrix} 1 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

For the Lie algebra $R_{5,39}$, all derivations are almost inner.

Lemma 8.2.11. *Let* $\mathfrak{g} := R_{5,39}$ *be the Lie algebra over* $\mathbb{R}$ *with basis* $\{e_1, \ldots, e_5\}$ *and given by*

$$\begin{aligned} &[e_1, e_4] = e_1, \quad [e_1, e_5] = -e_2, \quad [e_2, e_4] = e_2, \\ &[e_2, e_5] = e_1, \quad [e_4, e_5] = e_3. \end{aligned}$$

We have that $\mathrm{Inn}(\mathfrak{g}) \neq \mathrm{AID}(\mathfrak{g}) = \mathrm{Der}(\mathfrak{g})$.

Proof. An arbitrary derivation D is given by

$$D = a_1\mathrm{ad}(e_1) + \cdots + a_4\mathrm{ad}(e_4) + e_{3,4}E_{3,4} + e_{3,5}E_{3,5}$$

and has matrix form

$$D = \begin{pmatrix} -a_4 & -a_5 & 0 & a_1 & a_2 \\ a_5 & -a_4 & 0 & a_2 & -a_1 \\ 0 & 0 & 0 & -a_5 + e_{3,4} & a_4 + e_{3,5} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

It turns out that $E_{3,4}$ is almost inner. A determination map $\varphi_{E_{3,4}} : \mathfrak{g} \to \mathfrak{g}$ is given by

$$\varphi_{E_{3,4}}(x) = \begin{cases} \frac{x_2x_4 + x_1x_5}{x_4^2 + x_5^2} e_1 + \frac{x_2x_5 - x_1x_4}{x_4^2 + x_5^2} e_2 + e_5 & \text{if } x_4 \neq 0, \\ 0 & \text{if } x_4 = 0. \end{cases}$$

Similarly, $E_{3,5}$ is almost inner with determination map

$$\varphi_{E_{3,5}}(x) = \begin{cases} \frac{x_2x_5 - x_1x_4}{x_4^2 + x_5^2} e_1 - \frac{x_1x_5 + x_2x_4}{x_4^2 + x_5^2} e_2 - e_4 & \text{if } x_5 \neq 0, \\ 0 & \text{if } x_5 = 0. \end{cases}$$

This shows that $\mathrm{Inn}(\mathfrak{g}) \neq \mathrm{AID}(\mathfrak{g}) = \mathrm{Der}(\mathfrak{g})$. $\square$

This result does not hold for the complexification of $R_{5,39}$. Indeed, for $\mathbb{C} \otimes_{\mathbb{R}} R_{5,39}$, the only almost inner derivations are the inner ones. Take $D := aE_{3,4} + bE_{3,5}$ (with $a, b \in \mathbb{C}$), then $D(e_1 + e_4 + ie_5) = (a + ib)e_3$. However, for arbitrary $x = \sum_{j=1}^{5} x_j e_j$, we find that

$$[e_1 + e_4 + ie_5, x] = (-x_1 - ix_2 + x_4)e_1 + (ix_1 - x_2 - x_5)e_2 + (-ix_4 + x_5)e_3.$$

When D is almost inner, we must have that $x_4 = x_1 + ix_2$ and $x_5 = ix_1 - x_2$. However, this implies that $-ix_4 + x_5 = 0$, so $a + ib = 0$. Similarly, it follows from $D(e_1 + e_4 - ie_5) = (a - ib)e_3$ and

$$[e_1 + e_4 - ie_5, x] = (-x_1 + ix_2 + x_4)e_1 + (-ix_1 - x_2 - x_5)e_2 + (ix_4 + x_5)e_3$$

that $a - ib = 0$. We conclude that $a = b = 0$.

Chapter 9

Two-step nilpotent Lie algebras

In this chapter, we study the almost inner derivations for two-step nilpotent Lie algebras. Let $\mathfrak{g}$ be a 2–step nilpotent Lie algebra. We will denote a basis of $\mathfrak{g}$ with $\{x_1, \ldots, x_n, y_1, \ldots, y_m\}$, where $\{y_1, \ldots, y_m\}$ is a basis for $[\mathfrak{g}, \mathfrak{g}]$. This means that $\mathfrak{g}$ is a nilpotent Lie algebra of type (n, m).

In the first section, we show that for Lie algebras determined by graphs, the only almost inner derivations are the inner ones. In Section 4.3, we associated matrix pencils to Lie algebras. This technique is in particular interesting for the computation of the almost inner derivations of 2–step nilpotent Lie algebras. In Section 9.2, we make use of this observation to give a complete answer for the 2–step nilpotent Lie algebras with 2–dimensional commutator algebra. We partially generalise the used methods for nonsingular Lie algebras. One of the results is that we can construct Lie algebras $\mathfrak{g}$ for which $\mathrm{AID}(\mathfrak{g})/\mathrm{Inn}(\mathfrak{g})$ is arbitrary large. In Section 9.4, we construct other examples of Lie algebras (where the commutator algebra has a larger dimension) with the same property. The first and last section appeared in [7] and the second in [9]. In each section, we will specify over which field $\mathbb{F}$ we work.

9.1 Lie algebras determined by graphs

Let $G(V, E)$ be a finite simple graph with vertices $V = \{x_1, x_2, \ldots, x_n\}$ and edges E. If there is an edge between vertex x_i and x_j with $1 \leq i < j \leq n$, we denote this edge by the symbol $y_{i,j}$. Let X be the vector space over an arbitrary field $\mathbb{F}$ with the elements of V as basis. Take Y for the vector space with basis the edges $y_{i,j}$. We define a two-step nilpotent Lie algebra $\mathfrak{g}$ over $\mathbb{F}$, where as a vector space $\mathfrak{g} = X \oplus Y$ and where the brackets are given by

$$[x_i, x_j] = \begin{cases} y_{i,j} & \text{if } y_{i,j} \in E, \\ 0 & \text{if there is no edge connecting } x_i \text{ with } x_j, \end{cases}$$

$$[x_i, y_{j,k}] = 0 \quad \text{for all } x_i \in V \text{ and all } y_{j,k} \in E,$$

$$[y_{i,j}, y_{k,l}] = 0 \quad \text{for all } y_{i,j}, y_{k,l} \in E.$$

For this class of Lie algebras, the only almost inner derivations are the inner ones.

Theorem 9.1.1. *Let $\mathfrak{g}$ be a 2–step nilpotent Lie algebra determined by a finite simple graph. Then* $\mathrm{AID}(\mathfrak{g}) = \mathrm{Inn}(\mathfrak{g})$.

Proof. Let $G(V, E)$ be a finite simple graph with vertices $V = \{x_1, \ldots, x_n\}$. Denote s for the number of edges and choose an order $p_1, p_2, \ldots, p_s$ for the edges. So any p_t corresponds to a unique edge $y_{i,j}$. This means that $\{x_1, x_2, \ldots, x_n, p_1, \ldots, p_s\}$ is a basis of $\mathfrak{g}$. Let $D \in \mathrm{AID}(\mathfrak{g})$ be determined by the map φ_D. We want to show that any basis vector is fixed for D. For $p_1, p_2, \ldots, p_s$, this is obvious, since these vectors belong to $Z(\mathfrak{g})$. Consider x_i with $1 \le i \le n$. If $x_i \in Z(\mathfrak{g})$, so when x_i is an isolated vertex, there is again nothing to show. Assume that $x_i \notin Z(\mathfrak{g})$, then there is at least one $x_j \notin C_{\mathfrak{g}}(x_i)$ (with $1 \le j \le n$). Hence $[x_j, x_i] = \pm p_l$ for some $1 \le l \le s$. Let $\alpha = t_i(\varphi_D(x_j))$. Consider any other basis vector $x_k \notin C_{\mathfrak{g}}(x_i)$. In order to show that x_i is fixed, we must show that also $t_i(\varphi_D(x_k)) = \alpha$. There exists an $1 \le m \le s$ with $[x_k, x_i] = \pm p_m$. As $\mathfrak{g}$ is determined by a graph, we have that $m \ne l$. We are in the following situation

$$\begin{aligned} &[x_j, x_i] \pm p_l = 0 \\ &[x_k, x_i] \pm p_m = 0 \\ &[x_j, \mathfrak{g}_i] \subseteq \mathfrak{g}_{l,m} \\ &[x_k, \mathfrak{g}_i] \subseteq \mathfrak{g}_{l,m}. \end{aligned}$$

We can apply Lemma 4.2.7 and we find that $t_i(\varphi_D(x_k)) = t_i(\varphi_D(x_j)) = \alpha$. Hence, x_i is indeed fixed for all $1 \le i \le n$. Corollary 4.2.6 finishes the proof. □

9.2 Lie algebras of genus 1 and 2

Two-step nilpotent Lie algebras have not been classified in general so far. For certain subclasses however, there is a complete description. If not said otherwise,

$\mathbb{F}$ denotes an arbitrary field. We will always assume that $\mathfrak{g}$ is a finite-dimensional Lie algebra over $\mathbb{F}$. In this section, we will study the almost inner derivations of Lie algebras of genus 1 and genus 2. The 'genus' of a Lie algebra $\mathfrak{g}$ is the number $\dim(\mathfrak{g}) - |S|$, where S is a minimal system of generators. If $\mathfrak{g}$ is nilpotent, then the genus is given by $\dim(\mathfrak{g}) - \dim(\mathfrak{g}/[\mathfrak{g},\mathfrak{g}]) = \dim([\mathfrak{g},\mathfrak{g}])$.

Let $\mathfrak{g}$ be a 2–step nilpotent Lie algebra over $\mathbb{F}$. Then $\gamma_3(\mathfrak{g}) = [\mathfrak{g},[\mathfrak{g},\mathfrak{g}]] = 0$ holds, so $[\mathfrak{g},\mathfrak{g}] \subseteq Z(\mathfrak{g})$. A nilpotent Lie algebra has genus 0 if and only if it is abelian. If $[\mathfrak{g},\mathfrak{g}] = \langle z \rangle$ for some $z \in \mathfrak{g}$, then $\mathfrak{g}$ has genus 1. An example is the Heisenberg Lie algebra of dimension $2s+1$, where $s \in \mathbb{N}_0$.

Example 9.2.1. Take $s \in \mathbb{N}_0$. We denote $\mathfrak{h}_{2s+1}$ for the 'Heisenberg algebra of dimension $2s+1$', which has basis $\{x_1, y_1, \ldots, x_s, y_s, z\}$ and non-vanishing Lie brackets

$$[x_1, y_1] = \cdots = [x_s, y_s] = z.$$

It is clear that $\mathfrak{h}_{2s+1}$ is 2–step nilpotent of genus 1 for all $s \in \mathbb{N}_0$.

It turns out that 2–step Lie algebras of genus 1 can be easily classified.

Proposition 9.2.2. *Let $\mathfrak{g}$ be a finite-dimensional 2–step nilpotent Lie algebra of genus 1. Then $\mathfrak{g}$ can be written as $\mathfrak{g} = \mathfrak{h}_{2s+1} \oplus \mathfrak{a}$, where $s \in \mathbb{N}_0$ and $\mathfrak{a}$ is abelian.*

Proof. Denote $n := \dim(\mathfrak{g})$. Suppose that $\mathfrak{g}$ has genus 1, then it is not abelian. For $x_1 \notin Z(\mathfrak{g})$, there exists $y_1 \in \mathfrak{g}$ with $[x_1, y_1] = z \neq 0$. Since $\mathrm{im}(\mathrm{ad}(x_1))$ is 1–dimensional, $C_{\mathfrak{g}}(x_1) = \ker(\mathrm{ad}(x_1))$ has dimension $n-1$. Similarly, we find that $\dim(C_{\mathfrak{g}}(y_1)) = n-1$. Consider $\mathfrak{g}_{n-2} := C_{\mathfrak{g}}(\langle x_1, y_1 \rangle) = C_{\mathfrak{g}}(x_1) \cap C_{\mathfrak{g}}(y_1)$. Note that $\dim(\mathfrak{g}_{n-2}) = n-2$. There are two possibilities.

- If $\mathfrak{g}_{n-2}$ is abelian, then $\mathfrak{g} = \langle x_1, y_1, z \rangle \oplus \mathfrak{a} = \mathfrak{h}_3 \oplus \mathfrak{a}$, where $\mathfrak{a}$ is an abelian subspace of $C_{\mathfrak{g}}(\langle x_1, y_1 \rangle)$.
- When $\mathfrak{g}_{n-2}$ is not abelian, there exist elements $x_2, y_2 \in \mathfrak{g}_{n-2}$ with $[x_2, y_2] \neq 0$. Since $\mathfrak{g}$ has genus 1, we can suppose (possibly after rescaling x_2 or y_2) that $[x_2, y_2] = z$. Consider $\mathfrak{g}_{n-4} := C_{\mathfrak{g}}(\langle x_1, y_1, x_2, y_2 \rangle)$, which has dimension $n-4$.

Inductively, we can repeat the previous reasoning, where we consider after s steps a subspace $\mathfrak{g}_{n-2s}$ of $\mathfrak{g}$ of dimension $n-2s$. After at most $\lfloor (n-1)/2 \rfloor$ steps, we obtain an abelian subspace and the process ends. $\square$

Lemma 4.2.3 and Corollary 4.2.6 imply that $\mathrm{AID}(\mathfrak{h}_{2s+1}) = \mathrm{Inn}(\mathfrak{h}_{2s+1})$ for all $s \in \mathbb{N}_0$. Let $\mathfrak{g}$ be a 2–step nilpotent Lie algebra of genus 1. It follows from the previous observations and Proposition 4.1.8 that the only almost inner derivations are the inner ones.

The next interesting case is genus 2, so Lie algebras $\mathfrak{g}$ satisfying $\dim([\mathfrak{g},\mathfrak{g}]) = 2$. Such algebras can be described in terms of matrix pencils. This has been studied for several purposes in the literature, see [31, 32, 33, 56, 57, 59, 80]. The results of this section also appeared in [9].

Let $\mathbb{F}[\lambda,\mu]$ be the polynomial ring in two variables.

Definition 9.2.3 (Matrix pencil)**.** Let $A, B \in M_n(\mathbb{F})$. A polynomial matrix $\mu A + \lambda B \in M_n(\mathbb{F}[\lambda,\mu])$ is called a matrix pencil or just a pencil. Two such pencils $\mu A + \lambda B$ and $\mu C + \lambda D$ are called 'strictly equivalent' if there are matrices $S, T \in GL_n(\mathbb{F})$ satisfying

$$S(\mu A + \lambda B)T = \mu C + \lambda D.$$

The pencil is called 'skew' if both A and B are skew-symmetric. Two skew-symmetric pencils $\mu A + \lambda B$ and $\mu C + \lambda D$ are called 'strictly congruent' if there is a matrix $S \in GL_n(\mathbb{F})$ such that $S^\top(\mu A + \lambda B)S = \mu C + \lambda D$. A pencil is called 'regular' or 'non-singular' if its determinant is not the zero polynomial in $\mathbb{F}[\lambda,\mu]$.

It is known that skew-symmetric pencils over an algebraically closed field of characteristic not two are strictly equivalent if and only if they are strictly congruent ([33]). The same is true over the field of real numbers ([56]).

In Section 4.3, we already introduced skew matrix pencils and the link with almost inner derivations. This section contains an overview of known properties of (skew) matrix pencils.

Let $\mathfrak{g}$ be a 2–step nilpotent Lie algebra over a field $\mathbb{F}$ with $\dim([\mathfrak{g},\mathfrak{g}]) = 2$. We fix a basis $\{x_1, x_2, \ldots, x_n, y_1, y_2\}$ of $\mathfrak{g}$, where $\{y_1, y_2\}$ is a basis of $[\mathfrak{g},\mathfrak{g}]$. Denote by $A = (a_{ij})$ and $B = (b_{ij})$ the skew-symmetric matrices of structure constants determined by

$$[x_i, x_j] = a_{ij}y_1 + b_{ij}y_2$$

for all $1 \le i, j \le n$. Let μ and λ be algebraically independent variables over $\mathbb{F}$. We will denote the 'pencil associated to $\mathfrak{g}$' by $\mu A + \lambda B\,(\in M_n(\mathbb{F}[\mu,\lambda]))$. The following proposition is a special case of [59, Proposition 4.1]

Proposition 9.2.4. *Let $\mathfrak{g}$ and $\mathfrak{h}$ be two-step nilpotent Lie algebras of genus 2 over an arbitrary field $\mathbb{F}$. If the pencils associated to $\mathfrak{g}$ and $\mathfrak{h}$ with respect to some bases of $\mathfrak{g}$ and $\mathfrak{h}$ are strictly congruent, then $\mathfrak{g}$ and $\mathfrak{h}$ are isomorphic.*

From this proposition, it follows that it is important for our study to be able to classify skew pencils up to strict congruence. We will give an overview in the next subsection.

9.2.1 Elementary divisors and minimal indices

For regular pencils, the classification of skew pencils was given by Weierstrass in terms of 'elementary divisors'. For a pencil $\mu A + \lambda B$ of rank r, let $G_m(\mu, \lambda)$ be the greatest common divisor of all its minor determinants of order m. Then $G_m(\mu, \lambda) \mid G_{m+1}(\mu, \lambda)$ for all $1 \leq m \leq r-1$. Let $i_1(\mu, \lambda) = G_1(\mu, \lambda)$ and

$$i_m(\mu, \lambda) = \frac{G_m(\mu, \lambda)}{G_{m-1}(\mu, \lambda)}$$

for $2 \leq m \leq r$.

Definition 9.2.5 (Invariant factors and elementary divisors)**.** The homogeneous polynomials $\{i_m(\mu, \lambda)\}_m$ are called the invariant factors of the pencil $\mu A + \lambda B$. Each polynomial $i_m(\mu, \lambda)$ can be written as a product of powers of prime polynomials because $\mathbb{F}[\lambda, \mu]$ is a unique factorisation domain. These prime power factors (which are only determined up to scalar multiple) are called the elementary divisors $e_a(\mu, \lambda)$ of the pencil $\mu A + \lambda B$ for $1 \leq a \leq t$. An elementary divisor is said to have 'multiplicity ν' if it appears exactly ν times in the factorisations of the invariant factors $i_m(\mu, \lambda)$ for $1 \leq m \leq r$.

Suppose that $\mathbb{F}$ is algebraically closed. In this case, the elementary divisors are all linear. Since the elementary divisors are only determined up to scalar multiple, each elementary divisor is either of type $(b\mu + \lambda)^e$ or of type μ^f. The first one is called of 'finite type'. The second one is called of 'infinite type', which means that the divisor belongs to $\mathbb{F}[\mu]$. Elementary divisors of infinite type exist if and only if $\det(B) = 0$. The elementary divisors $e_a(\mu, \lambda)$ of finite type correspond to the elementary divisors of the pencil $A + \lambda B \in \mathbb{F}[\lambda]$ as follows. Setting $\mu = 1$ in $e_a(\mu, \lambda)$, we clearly obtain the elementary divisors $e_a(\lambda)$ of $A + \lambda B$. These can be computed by the 'Smith normal form', because $\mathbb{F}[\lambda]$ is a PID. The diagonal elements of the Smith normal form are just the invariant polynomials. Conversely, from each elementary divisor $e_a(\lambda)$ of $A + \lambda B$ of degree e, we obtain the corresponding elementary divisor $e_a(\mu, \lambda)$ by $e_a(\mu, \lambda) = \mu^e e_a(\frac{\lambda}{\mu})$. In case $\mathbb{F} = \mathbb{R}$, apart from these elementary divisors of degree 1, there are also elementary divisors of degree 2 which are of the form $(\lambda - \mu(a + bi))(\lambda - \mu(a - bi)) = \lambda^2 - 2a\lambda\mu + (a^2 + b^2)\mu^2$, with $a \in \mathbb{R}$ and $b \in \mathbb{R}^*$.

Example 9.2.6. Let $\mathfrak{g}$ be the 6–dimensional real Lie algebra with basis $\{x_1, \ldots, x_4, y_1, y_2\}$ and Lie brackets defined by

$$\begin{aligned} [x_1, x_2] &= y_1, \quad [x_1, x_3] = y_2, \\ [x_2, x_4] &= y_2, \quad [x_3, x_4] = -y_1. \end{aligned}$$

Then the associated pencil is given by

$$\mu A + \lambda B = \begin{pmatrix} 0 & \mu & \lambda & 0 \\ -\mu & 0 & 0 & \lambda \\ -\lambda & 0 & 0 & -\mu \\ 0 & -\lambda & \mu & 0 \end{pmatrix}.$$

The pencil is regular because $\det(\mu A + \lambda B) = (\mu^2 + \lambda^2)^2$ is not the zero polynomial. Note that $\mathfrak{g}$ is nonsingular when considered over $\mathbb{R}$. The Smith normal form of $A + \lambda B$ is given by $\operatorname{diag}(1, 1, \lambda^2 + 1, \lambda^2 + 1)$. Hence there is one elementary divisor $e_1(\mu, \lambda) = \mu^2(1 + \frac{\lambda^2}{\mu^2}) = \mu^2 + \lambda^2$ of finite type of multiplicity 2 and there is no elementary divisor of infinite type. When we consider the complexification $\mathfrak{g} \otimes \mathbb{C}$ of this Lie algebra, then the corresponding pencil has two elementary divisors, $\lambda - i\mu$ and $\lambda + i\mu$, both of multiplicity 2.

For singular pencils, we still need another invariant. Let $\mu A + \lambda B$ be a singular pencil of size n. Then $(A + \lambda B)x = 0$ has a nonzero solution in $\mathbb{F}[\lambda]^n$. Let $x_1(\lambda)$ be such a nonzero solution of minimal degree ε_1. Of all solutions which are $\mathbb{F}[\lambda]$–independent of $x_1(\lambda)$, we select a solution $x_2(\lambda)$ of minimal degree ε_2. It is obvious that $\varepsilon_1 \leq \varepsilon_2$. By continuing this process we obtain a set $x_1(\lambda), \ldots, x_k(\lambda)$ of solutions, which is a maximal set of elements in $\mathbb{F}[\lambda]^n$ satisfying $(A + \lambda B)x_i(\lambda) = 0$ for $1 \leq i \leq k$ and being $\mathbb{F}[\lambda]$–independent. We have $k \leq n$. Note that this set is not uniquely determined, but that different sets have the same minimal degrees $\varepsilon_1 \leq \varepsilon_2 \leq \cdots \leq \varepsilon_k$. Hence the following notion is well-defined.

Definition 9.2.7 (Minimal indices)**.** Let $\mu A + \lambda B$ be a singular pencil. The associated numbers $\varepsilon_1, \ldots, \varepsilon_k$ are called the minimal indices of the pencil $\mu A + \lambda B$.

Let $x_i(\lambda)$ be a non-zero solution of degree ε_i, so $(A + \lambda B)x_i(\lambda) = 0$. Note that the constant term of $x_i(\lambda)$ is not the zero vector since otherwise, we can find a solution of smaller degree. It follows that

$$\left(\frac{1}{\lambda} A + B\right) \frac{x_i(\lambda)}{\lambda^{\varepsilon_i}} = 0$$

is a solution as well. By taking $\mu := 1/\lambda$, we obtain a non-zero solution $\tilde{x}_i(\mu)$ (of the same degree ε_i) of the equation $(\mu A + B)X = 0$. We can do the same for every non-zero solution x_i (with $1 \leq i \leq k$) and obtain a fundamental series $\tilde{x}_1(\mu), \ldots, \tilde{x}_k(\mu)$ of linearly independent solutions. By construction, the degrees for the new series will be the same as the minimal indices. This observation shows that we can look at non-zero solutions of $(\mu A + B)x = 0$ as well to determine the minimal indices of the matrix pencil $\mu A + \lambda B$.

Example 9.2.8. Let $\mathfrak{g}$ be the 7–dimensional real Lie algebra with basis $\{x_1, \ldots, x_5, y_1, y_2\}$ and Lie brackets defined by

$$\begin{aligned} [x_1, x_3] &= y_1, \quad [x_1, x_4] = y_2, \\ [x_2, x_4] &= y_1, \quad [x_2, x_5] = y_2. \end{aligned}$$

This is the same Lie algebra as in Example 4.3.5. The associated pencil is given by

$$\mu A + \lambda B = \begin{pmatrix} 0 & 0 & \mu & \lambda & 0 \\ 0 & 0 & 0 & \mu & \lambda \\ -\mu & 0 & 0 & 0 & 0 \\ -\lambda & -\mu & 0 & 0 & 0 \\ 0 & -\lambda & 0 & 0 & 0 \end{pmatrix}.$$

Since $\det(\mu A + \lambda B) = 0$, the pencil is singular. The equation $(A + \lambda B)x = 0$ has a non-zero solution $x_1(\lambda) = (0, 0, \lambda^2, -\lambda, 1)^\top$, and the set is maximal. Hence there is one minimal index $\varepsilon_1 = 2$. The Smith normal form of $A + \lambda B$ is given by $\operatorname{diag}(1, 1, 1, 1, 0)$, so there are no elementary divisors.

The following well-known result classifies skew pencils up to congruence, see Corollary 6.6 in [33] and Theorem 3.4 in [59].

Proposition 9.2.9. *Let $\mathbb{F}$ be an algebraically closed field of characteristic not 2 or the field of real numbers. Two skew-symmetric pencils of the same dimension are strictly congruent if and only if they have the same elementary divisors and the same minimal indices.*

For a skew pencil $\mu A + \lambda B$ over an algebraically closed field $\mathbb{F}$, the elementary divisors occur in pairs and we can arrange them as

$$\mu^{e_1}, \mu^{e_1}, \ldots, \mu^{e_s}, \mu^{e_s},$$

$$(\lambda - \mu\alpha_1)^{f_1}, (\lambda - \mu\alpha_1)^{f_1}, \ldots, (\lambda - \mu\alpha_t)^{f_t}, (\lambda - \mu\alpha_t)^{f_t},$$

where $\alpha_1,\ \alpha_2, \ldots, \alpha_t \in \mathbb{F}$.

When $\mathbb{F} = \mathbb{R}$, the elementary divisors still occur in pairs, and apart from the above set of elementary divisors (where of course $\alpha_1, \ldots, \alpha_t \in \mathbb{R}$), we can also have pairs of the form

$$\xi(a_1,b_1)^{m_1},\xi(a_1,b_1)^{m_1},\dots,\xi(a_p,b_p)^{m_p},\xi(a_p,b_p)^{m_p},$$

where $a_1,\dots,a_p\in\mathbb{R}$, $b_1,\dots,b_p\in\mathbb{R}^*$ and $\xi(a,b)=(\lambda-\mu(a+ib))(\lambda-\mu(a-bi))$ with $a,b\in\mathbb{R}$.

Since elementary divisors occur in pairs, we introduce a notation to indicate such pairs. Let $\mathbb{F}$ be an algebraically closed field or $\mathbb{F}=\mathbb{R}$. For given $\alpha\in\mathbb{F}\cup\{\infty\}$ or $\alpha\in\mathbb{C}\cup\{\infty\}$ in case $\mathbb{F}=\mathbb{R}$ and $e\in\mathbb{N}$, we denote by (α,e) the following pairs of elementary divisors

$$(\alpha,e):=\begin{cases}\mu^e,\mu^e & \text{if } \alpha=\infty,\\ (\lambda-\mu\alpha)^e,(\lambda-\mu\alpha)^e & \text{if } \alpha\in\mathbb{F},\\ \xi(a,b)^e,\xi(a,b)^e & \text{if } \mathbb{F}=\mathbb{R} \text{ and } \alpha=a+bi\in\mathbb{C}\setminus\mathbb{R}.\end{cases}$$

Hence, for a skew pencil $\mu A+\lambda B$ over an algebraically closed field $\mathbb{F}$, we can associate in a unique way a set of elementary divisors as follows:

$$(\infty,e_1),\dots,(\infty,e_s),\ (\alpha_1,f_1),\dots,(\alpha_t,f_t),$$

with $\alpha_1,\dots,\alpha_t\in\mathbb{F}$. For a skew pencil $\mu A+\lambda B$ over $\mathbb{R}$, we find a set of elementary divisors of the form

$$(\infty,e_1),\dots,(\infty,e_s),\ (\alpha_1,f_1),\dots,(\alpha_t,f_t),\ (\beta_1,m_1),\dots,(\beta_p,m_p),$$

where $\alpha_i\in\mathbb{R}$ and $\beta_i\in\mathbb{C}\setminus\mathbb{R}$.

For a given pair of elementary divisors (α,e) as above, there exists a canonical skew pencil having exactly that one pair of elementary divisors (α,e) (and no minimal indices or other elementary divisors). For a minimal index ε, there is a canonical skew pencil having no elementary divisors and exactly one minimal index ε.

These skew pencils are given by the following cases:
Case 1: For $(\alpha,e)=(\infty,e)$, the skew pencil is given by

$$F(\infty,e):=\begin{pmatrix}0 & \mu\Delta_e+\lambda\Lambda_e\\ -\mu\Delta_e-\lambda\Lambda_e & 0\end{pmatrix}\in M_{2e}(\mathbb{F}[\mu,\lambda]),$$

where

$$\Delta_e=\begin{pmatrix} & & & & 1\\ & & & 1 & \\ & & \iddots & & \\ & 1 & & & \\ 1 & & & & \end{pmatrix},\quad \Lambda_e=\begin{pmatrix} & & & & 0\\ & & & 0 & 1\\ & & \iddots & 1 & \\ & 0 & \iddots & & \\ 0 & 1 & & & \end{pmatrix}\in M_e(\mathbb{F}).$$

Case 2: For $(\alpha, f) = (\lambda - \mu\alpha)^f, (\lambda - \mu\alpha)^f$, the skew pencil is given by

$$F(\alpha, f) := \begin{pmatrix} 0 & (\lambda - \mu\alpha)\Delta_f + \mu\Lambda_f \\ -(\lambda - \mu\alpha)\Delta_f - \mu\Lambda_f & 0 \end{pmatrix} \in M_{2f}(\mathbb{F}[\mu, \lambda]).$$

Case 3: Only in case $\mathbb{F} = \mathbb{R}$ and for $(\alpha, m) = (a + bi, m) = \xi(a, b)^m, \xi(a, b)^m$ (with $a + bi \in \mathbb{C} \setminus \mathbb{R}$), the real skew pencil is given by

$$C(a, b, m) := \begin{pmatrix} 0 & T_m \\ -T_m & 0 \end{pmatrix} \in M_{4m}(\mathbb{R}[\mu, \lambda]),$$

where

$$T_m = \begin{pmatrix} & & & 0 & R \\ & & \iddots & R & \mu\Delta_2 \\ & & \iddots & & \\ 0 & R & \iddots & & \\ R & \mu\Delta_2 & & & \end{pmatrix} \in M_{2m}(\mathbb{R}[\mu, \lambda])$$

for $m \geq 2$ and

$$T_1 = R = \begin{pmatrix} -\mu b & \lambda - \mu a \\ \lambda - \mu a & \mu b \end{pmatrix} \in M_2(\mathbb{R}[\mu, \lambda]).$$

Case 4: For each minimal index $\varepsilon \geq 1$, the skew pencil is given by

$$M_\varepsilon = \begin{pmatrix} 0_{\varepsilon+1} & \mathcal{L}_\varepsilon(\mu, \lambda) \\ -\mathcal{L}_\varepsilon(\mu, \lambda)^t & 0_\varepsilon \end{pmatrix} \in M_{2\varepsilon+1}(\mathbb{F}[\mu, \lambda]),$$

with

$$\mathcal{L}_\varepsilon(\mu, \lambda) = \begin{pmatrix} \lambda & 0 & \cdots & 0 \\ \mu & \lambda & & 0 \\ & \ddots & \ddots & \\ 0 & & \mu & \lambda \\ 0 & \dots & 0 & \mu \end{pmatrix} \in M_{\varepsilon+1,\varepsilon}(\mathbb{F}[\mu, \lambda]).$$

For minimal index $\varepsilon = 0$, the skew pencil is just $M_0 = (0) \in M_1(\mathbb{F})$, the zero matrix.

Using the notations of above, we have the following result, see [33, 59].

Proposition 9.2.10. *Let $\mathbb{F}$ be an algebraically closed field of characteristic not 2. Any skew pencil over $\mathbb{F}$ with elementary divisors*

$$(\infty, e_1), \dots, (\infty, e_s),\ (\alpha_1, f_1), \dots, (\alpha_t, f_t)$$

and minimal indices $\varepsilon_1, \dots, \varepsilon_k$ is strictly congruent to the pencil consisting of a matrix with the blocks

$$F(\infty, e_1), \dots, F(\infty, e_s),\ F(\alpha_1, f_1), \dots, F(\alpha_t, f_t),\ M_{\varepsilon_1}, \dots, M_{\varepsilon_k}$$

on the diagonal. Any skew pencil over $\mathbb{R}$ having elementary divisors

$$(\infty, e_1), \dots,\ (\infty, e_s),\ (\alpha_1, f_1), \dots, (\alpha_t, f_t),\ (a_1 + b_1 i, m_1), \dots, (a_p + b_p i, m_p)$$

and minimal indices $\varepsilon_1, \dots, \varepsilon_k$ is strictly congruent to the pencil consisting of a matrix with the blocks

$$F(\infty, e_1), \dots, F(\infty, e_s),\ F(\alpha_1, f_1), \dots, F(\alpha_t, f_t),$$

$$C(a_1, b_1, m_1), \dots, C(a_p, b_p, m_p),\ M_{\varepsilon_1}, \dots, M_{\varepsilon_k}$$

on the diagonal.

Definition 9.2.11 (Canonical Lie algebra)**.** A 2–step nilpotent Lie algebra of genus 2 over an algebraically closed field $\mathbb{F}$ of characteristic not 2 or $\mathbb{F} = \mathbb{R}$ is called canonical if its associated skew pencil has a blocked diagonal form as in Proposition 9.2.10 above.

As an immediate consequence of Proposition 9.2.4, we obtain the following corollary.

Corollary 9.2.12. *Let $\mathbb{F}$ be an algebraically closed field of characteristic not 2 or $\mathbb{F} = \mathbb{R}$. Any 2–step nilpotent Lie algebra of genus 2 over $\mathbb{F}$ is isomorphic to a canonical one with the same elementary divisors and minimal indices.*

It follows that the computation of $\mathrm{AID}(\mathfrak{g})$ for 2–step nilpotent Lie algebras of genus 2 over $\mathbb{F}$ can be reduced to canonical Lie algebras.

9.2.2 Almost inner derivations of Lie algebras of genus 2

In this subsection, we determine the algebra $\mathrm{AID}(\mathfrak{g})$ for canonical Lie algebras in the sense of Definition 9.2.11 over $\mathbb{F}$, where $\mathbb{F}$ is either $\mathbb{R}$ or an algebraically closed field of characteristic not 2. We will start with the case that the canonical pencil only consists of one block. For the proofs, we will make use of the results from Section 4.3.

Lemma 9.2.13. *Let $\mathfrak{g}$ be a canonical Lie algebra over $\mathbb{F}$ with one pair of elementary divisors (∞, e) and no minimal indices. Then we have that* $\dim(\mathrm{Inn}(\mathfrak{g})) = 2e$ *and* $\dim(\mathrm{AID}(\mathfrak{g})) = 4e - 2$.

Proof. By assumption, the matrix pencil of $\mathfrak{g}$ is $F(\infty, e)$, so that the Lie brackets of $\mathfrak{g}$ in the usual basis $\mathcal{B}$ are given by

$$[x_i, x_{2e+1-i}] = y_1, \quad 1 \leq i \leq e,$$

$$[x_j, x_{2e+2-j}] = y_2, \quad 2 \leq j \leq e.$$

We have that $\dim(\mathrm{Inn}(\mathfrak{g})) = 2e$. We will compute $\mathrm{AID}(\mathfrak{g})$ with the aid of Proposition 4.3.4. A basis for $\mathcal{C}(\mathfrak{g})$ is given by the maps $D_{i,j} : \mathfrak{g} \to \mathfrak{g}$ for $1 \leq i \leq 2e$ and $1 \leq j \leq 2$, defined by

$$\sum_{k=1}^{2e} a_k x_k + (b_1 y_1 + b_2 y_2) \mapsto a_i y_j.$$

We have $\dim(\mathcal{C}(\mathfrak{g})) = 4e$. Note that the span of $D_{1,2}$ and $D_{e+1,2}$ has trivial intersection with $\mathrm{AID}(\mathfrak{g})$, since $\alpha D_{1,2} + \beta D_{e+1,2}$ is $\mathcal{B}$–almost inner if and only if $\alpha = \beta = 0$. It is easy to see that, except for $D_{1,2}$ and $D_{e+1,2}$, all $D_{i,j}$ with $1 \leq i \leq 2e$ and $1 \leq j \leq 2$ are $\mathcal{B}$–almost inner. This means that we have $\dim(\mathrm{AID}(\mathfrak{g})) = 4e - 2$ if we can show that all of the remaining $4e-2$ derivations are actually almost inner.

Let $D \in \mathcal{C}(\mathfrak{g})$ be $\mathcal{B}$–almost inner. We have $\det(\mu A + \lambda B) = \mu^{2e}$. For $\mu \neq 0$, condition (4.7) is satisfied, which means that we may assume that $\mu = 0$. Then the kernel of $\mu A + \lambda B = \lambda B = F(\infty, e)$ is equal to the set of all vectors of the form $a(x) = (k_1, 0, \ldots, 0, k_{e+1}, 0, \ldots, 0)^\top$, where $k_1, k_{e+1} \in \mathbb{F}$. For these vectors, we have $d_2(a(x)) = 0$, so that condition (4.7) is satisfied and the proof is finished. $\square$

Lemma 9.2.14. *Let $\mathfrak{g}$ be a canonical Lie algebra over $\mathbb{F}$ with one pair of elementary divisors (α, f) and no minimal indices. Then we have that* $\dim(\mathrm{Inn}(\mathfrak{g})) = 2f$ *and* $\dim(\mathrm{AID}(\mathfrak{g})) = 4f - 2$.

Proof. The Lie brackets of $\mathfrak{g}$ with respect to the usual basis $\{x_1, \ldots, x_{2f}, y_1, y_2\}$ and matrix pencil $F(\alpha, f)$ are given by

$$[x_i, x_{2f+1-i}] = y_2 - \alpha y_1, \quad 1 \leq i \leq f,$$

$$[x_j, x_{2f+2-j}] = y_1, \qquad 2 \leq j \leq e.$$

We may pass to the basis $\{x_1, \ldots, x_{2f}, y_2 - \alpha y_1, y_1\}$ so that $\mathfrak{g}$ coincides with the Lie algebra of Lemma 9.2.13. This finishes the proof. $\square$

The next lemma is only for the case $\mathbb{F} = \mathbb{R}$.

Lemma 9.2.15. *Let $\mathfrak{g}$ be a canonical Lie algebra over $\mathbb{R}$ with one pair of elementary divisors (β, m), where $\beta = a + bi$ and $b \neq 0$. Then we have that* $\dim(\mathrm{Inn}(\mathfrak{g})) = 4m$ *and* $\dim(\mathrm{AID}(\mathfrak{g})) = 8m$.

Proof. The Lie brackets of $\mathfrak{g}$ with respect to the usual basis $\{x_1, \ldots, x_{4m}, y_1, y_2'\}$ and matrix pencil $\mu A + \lambda B = C(a, b, m)$ are given by

$$\begin{aligned} [x_{2i-1}, x_{4m-2i+1}] &= -by_1, & 1 \leq i \leq m, \\ [x_{2i}, x_{4m-2i+2}] &= by_1, & 1 \leq i \leq m, \\ [x_j, x_{4m+1-j}] &= y_2' - ay_1, & 1 \leq j \leq 2m, \end{aligned}$$

and in addition

$$[x_k, x_{4m-k+3}] = y_1, \quad 3 \leq k \leq 2m.$$

for $m \geq 2$. We set $y_2 := y_2' - ay_1$ to obtain a basis $\mathcal{B} = \{x_1, \ldots, x_n, y_1, y_2\}$. Then a basis for $\mathcal{C}(\mathfrak{g})$ is given by the maps $D_{i,j} : \mathfrak{g} \to \mathfrak{g}$ for $1 \leq i \leq 4m$ and $1 \leq j \leq 2$, defined by

$$\sum_{k=1}^{4m} a_k x_k + (b_1 y_1 + b_2 y_2) \mapsto a_i y_j.$$

We have $\dim(\mathcal{C}(\mathfrak{g})) = 8m$ and $\det(\mu A + \lambda B) = (\lambda^2 + \mu^2 b^2)^{2m}$. It follows from Corollary 4.3.9 that $\mathrm{AID}(\mathfrak{g}) = \mathcal{C}(\mathfrak{g})$. □

We again consider Example 9.2.6.

Example 9.2.16. Let $\mathfrak{g}$ be the 6–dimensional Lie algebra over $\mathbb{R}$ with basis $\{x_1, \ldots, x_4, y_1, y_2\}$ and Lie brackets defined by

$$\begin{aligned} [x_1, x_2] &= y_1, & [x_1, x_3] &= y_2, \\ [x_2, x_4] &= y_2, & [x_3, x_4] &= -y_1. \end{aligned}$$

This is a canonical Lie algebra with one pair of elementary divisors $(\beta, m) = (i, 1)$. Hence, Lemma 9.2.15 says that $\dim(\mathrm{Inn}(\mathfrak{g})) = 4$ and $\dim(\mathrm{AID}(\mathfrak{g})) = 8$. We already saw in Example 9.2.6 that $\mathfrak{g}$ is nonsingular over $\mathbb{R}$, so the same result follows from Corollary 4.3.9.

Lemma 9.2.17. *Let $\mathfrak{g}$ be a canonical Lie algebra over $\mathbb{F}$ with minimal index $\varepsilon \geq 1$. Then it holds* $\dim(\mathrm{Inn}(\mathfrak{g})) = 2\varepsilon + 1$ *and* $\dim(\mathrm{AID}(\mathfrak{g})) = 3\varepsilon$.

Proof. Consider the usual basis $\mathcal{B} = \{x_1, \ldots, x_{2\varepsilon+1}, y_1, y_2\}$ and matrix pencil $\mu A + \lambda B = M_\varepsilon$. The Lie brackets of $\mathfrak{g}$ are given by

$$[x_i, x_{i+\varepsilon+1}] = y_2, \quad 1 \le i \le \varepsilon,$$

$$[x_{j+1}, x_{j+\varepsilon+1}] = y_1, \quad 1 \le j \le \varepsilon.$$

It is easy to see that $Z(\mathfrak{g}) = \langle y_1, y_2 \rangle$, so we have $\dim(\mathrm{Inn}(\mathfrak{g})) = 2\varepsilon + 1$. For $\varepsilon = 1$ we have $\mathrm{AID}(\mathfrak{g}) = \mathrm{Inn}(\mathfrak{g})$ by Theorem 9.1.1, since $\mathfrak{g}$ is determined by a graph. For $\varepsilon \ge 2$, a basis of $\mathcal{C}(\mathfrak{g})$ is given by the maps $D_{i,j} : \mathfrak{g} \to \mathfrak{g}$ for $1 \le i \le 2\varepsilon + 1$ and $1 \le j \le 2$, defined by

$$\sum_{k=1}^{2\varepsilon+1} a_k x_k + (b_1 y_1 + b_2 y_2) \mapsto a_i y_j.$$

Hence, we have $\dim(\mathcal{C}(\mathfrak{g})) = 4\varepsilon + 2$ and $\mathrm{AID}(\mathfrak{g}) \subseteq \mathcal{C}(\mathfrak{g})$. Suppose that

$$D = \sum_{i=1}^{2\varepsilon+1} \alpha_i D_{i,1} + \sum_{i=1}^{2\varepsilon+1} \beta_i D_{i,2}$$

is an element of $\mathrm{AID}(\mathfrak{g})$. Then for any $b \in \mathbb{F}$, we have

$$D\left(\sum_{i=1}^{\varepsilon+1} b^i x_i\right) = \sum_{i=1}^{\varepsilon+1} \alpha_i b^i y_1 + \sum_{i=1}^{\varepsilon+1} \beta_i b^i y_2. \tag{9.1}$$

Since $D \in \mathrm{AID}(\mathfrak{g})$, there exist $c_j(b) \in \mathbb{F}$ for all $\varepsilon + 2 \le j \le 2\varepsilon + 1$ such that

$$\begin{aligned} D\left(\sum_{i=1}^{\varepsilon+1} b^i x_i\right) &= \left[\sum_{i=1}^{\varepsilon+1} b^i x_i, \sum_{j=\varepsilon+2}^{2\varepsilon+1} c_j(b) x_j\right] \\ &= \sum_{i=2}^{\varepsilon+1} b^i c_{i+\varepsilon}(b) y_1 + \sum_{i=1}^{\varepsilon} b^i c_{i+\varepsilon+1}(b) y_2. \end{aligned} \tag{9.2}$$

We also have

$$b\left(\sum_{i=1}^{\varepsilon} b^i c_{i+\varepsilon+1}(b)\right) = \sum_{i=2}^{\varepsilon+1} b^i c_{i+\varepsilon}(b). \tag{9.3}$$

Comparing coefficients of y_1 and y_2 in (9.1) and (9.2) and using (9.3), we find that

$$b \sum_{i=1}^{\varepsilon+1} \beta_i b^i - \sum_{i=1}^{\varepsilon+1} \alpha_i b^i = 0,$$

so that
$$-\alpha_1 b + \sum_{i=2}^{\varepsilon+1} (\beta_{i-1} - \alpha_i) b^i + \beta_{\varepsilon+1} b^{\varepsilon+2} = 0.$$
Since this holds for all $b \in \mathbb{F}$, we obtain $\alpha_1 = \beta_{\varepsilon+1} = 0$ and $\alpha_i = \beta_{i-1}$ for all $2 \leq i \leq \varepsilon + 1$. This means that $\mathrm{AID}(\mathfrak{g})$ is contained in V, which is the subspace
$$\{D = \sum_{i=2}^{2\varepsilon+1} \alpha_i D_{i,1} + \sum_{i=1}^{2\varepsilon+1} \beta_i D_{i,2} \in \mathcal{C}(\mathfrak{g}) \mid \beta_{\varepsilon+1} = 0, \alpha_i = \beta_{i-1} \text{ for } 2 \leq i \leq \varepsilon + 1\}.$$
Note that $\dim(V) = \dim(\mathcal{C}(\mathfrak{g})) - (\varepsilon + 2) = 4\varepsilon + 2 - (\varepsilon + 2) = 3\varepsilon$. Hence, we have that $\dim(\mathrm{AID}(\mathfrak{g})) \leq 3\varepsilon$ and we claim that there holds equality. More precisely, we will show that each $D_{j,1}$ for $\varepsilon + 2 \leq j \leq 2\varepsilon$ is almost inner. Here we do not consider $D_{2\varepsilon+1,1}$ because it already coincides with the inner derivation $\mathrm{ad}(x_{\varepsilon+1})$, so it is almost inner. Let
$$x = \sum_{i=1}^{2\varepsilon+1} a_i x_i + (b_1 y_1 + b_2 y_2)$$
be an element in $\mathfrak{g}$. If $a_j = 0$, then $D_{j,1}(x) = [x, 0] = 0$. Otherwise, we have
$$D_{j,1}(x) = \frac{-a_j}{a_\ell}[x, x_{\ell-\varepsilon}] = a_j y_1$$
for $\ell := \max\{j \leq k \leq 2\varepsilon + 1 \mid a_k \neq 0\}$. This shows that $D_{j,1}$ is almost inner for all $\varepsilon + 2 \leq j \leq 2\varepsilon$. Consider the subspace W of $\mathrm{AID}(\mathfrak{g})$ generated by all $D_{j,1}$, where we again take $\varepsilon + 2 \leq j \leq 2\varepsilon$. We claim that
$$W \cap \mathrm{Inn}(\mathfrak{g}) = 0.$$
So assume that $D = \sum_{j=\varepsilon+2}^{2\varepsilon} \alpha_j D_{j,1} \in W \cap \mathrm{Inn}(\mathfrak{g})$ with $D = \mathrm{ad}(x)$ for some $x = \sum_{i=1}^{2\varepsilon+1} k_i x_i$. We will show that $\alpha_j = 0$ for all $\varepsilon + 2 \leq j \leq 2\varepsilon$ and $k_i = 0$ for all $1 \leq i \leq 2\varepsilon + 1$. Because of
$$0 = D(x_m) = \mathrm{ad}(x)(x_m) = \left[\sum_{i=1}^{2\varepsilon+1} k_i x_i, x_m\right] = -k_{\varepsilon+m+1} y_2 - k_{\varepsilon+m} y_1$$
for $2 \leq m \leq \varepsilon$, we have $k_{\varepsilon+2} = \cdots = k_{2\varepsilon+1} = 0$. It follows that
$$\begin{aligned} \alpha_{\varepsilon+m} y_1 &= D(x_{\varepsilon+m}) \\ &= \mathrm{ad}(x)(x_{\varepsilon+m}) \\ &= [k_1 x_1 + \cdots + k_{\varepsilon+1} x_{\varepsilon+1}, x_{\varepsilon+m}] \\ &= k_{m-1} y_2 + k_m y_1 \end{aligned}$$

for all $2 \leq m \leq \varepsilon + 1$, where we take $\alpha_{2\varepsilon+1} = 0$. Hence, $\alpha_j = 0$ for all $\varepsilon + 2 \leq j \leq 2\varepsilon$ and $k_i = 0$ for all $1 \leq i \leq 2\varepsilon + 1$ and we have shown that $W \cap \mathrm{Inn}(\mathfrak{g}) = 0$. We know that $\dim(\mathrm{Inn}(\mathfrak{g})) = 2\varepsilon + 1$ and $\dim(W) = \varepsilon - 1$, so that $\mathrm{Inn}(\mathfrak{g}) \oplus W$ is a 3ε–dimensional subspace of $\mathrm{AID}(\mathfrak{g})$. This implies $\dim(\mathrm{AID}(\mathfrak{g})) \geq 3\varepsilon$ and hence there holds equality. $\square$

Remark 9.2.18. For a minimal index $\varepsilon = 0$, the corresponding Lie algebra $\mathfrak{g}$ is just the abelian 3-dimensional Lie algebra (with basis $\{x_1, y_1, y_2\}$) over $\mathbb{F}$. In this case, $\mathrm{Inn}(\mathfrak{g}) = \mathrm{AID}(\mathfrak{g}) = 0$ holds.

Example 9.2.19. For $\varepsilon = 2$, the canonical Lie algebra $\mathfrak{g}$ of Lemma 9.2.17 is isomorphic to the Lie algebra of Example 4.3.5 and Example 9.2.8. In all cases, we have $\dim(\mathrm{Inn}(\mathfrak{g})) = 2\varepsilon + 1 = 5$ and $\dim(\mathrm{AID}(\mathfrak{g})) = 3\varepsilon = 6$, which coincides with the result of Example 4.3.5.

For the next lemma, let $\mathfrak{g}$ be a 2–step nilpotent Lie algebra over an arbitrary field $\mathbb{F}$ with basis $\{x_1, \ldots, x_n, y_1, \ldots, y_m, z_1, \ldots, z_p\}$, where $[\mathfrak{g}, \mathfrak{g}] = \langle z_1, \ldots, z_p \rangle$. Define Lie subalgebras by

$$\mathfrak{g}_x = \langle x_1, \ldots, x_n, z_1, \ldots, z_p \rangle,$$

$$\mathfrak{g}_y = \langle y_1, \ldots, y_m, z_1, \ldots, z_p \rangle.$$

Lemma 9.2.20. *Let $\mathfrak{g}$ be a 2–step nilpotent Lie algebra over a field $\mathbb{F}$ with the above basis such that $[x_i, y_j] = 0$ for all $1 \leq i \leq n$ and $1 \leq j \leq m$. Then we have*

$$\dim(\mathrm{AID}(\mathfrak{g})) = \dim(\mathrm{AID}(\mathfrak{g}_x)) + \dim(\mathrm{AID}(\mathfrak{g}_y)).$$

Proof. Let $D \in \mathrm{AID}(\mathfrak{g})$ and write $e = x + y + z$, where $x \in \langle x_1, \ldots, x_n \rangle$, $y \in \langle y_1, \ldots, y_m \rangle$ and $z \in [\mathfrak{g}, \mathfrak{g}]$. Then there are maps $\varphi_{D_x} : \mathfrak{g}_x \to \mathfrak{g}_x$ and $\varphi_{D_y} : \mathfrak{g}_y \to \mathfrak{g}_y$ such that

$$D(e) = D(x + y + z) = [x, \varphi_{D_x}(x)] + \left[y, \varphi_{D_y}(y)\right].$$

This means that $D_x : \mathfrak{g}_x \to \mathfrak{g}_x,\ x \mapsto D_{|_{\mathfrak{g}_x}}(x) \in \mathrm{AID}(\mathfrak{g}_x)$ with determination map φ_{D_x} and $D_y : \mathfrak{g}_y \to \mathfrak{g}_y,\ y \mapsto D_{|_{\mathfrak{g}_y}}(y) \in \mathrm{AID}(\mathfrak{g}_y)$ is determined by φ_{D_y}. Conversely, any almost inner derivation of $\mathfrak{g}_x$ or $\mathfrak{g}_y$ can be extended to an almost inner derivation of $\mathfrak{g}$. $\square$

Finally, we can state the main result of this section by combining the previous lemmas. For clarity, we formulate this result as two separate theorems depending on the type of field $\mathbb{F}$ we are considering. We only give a proof for the last theorem in case $\mathbb{F} = \mathbb{R}$. The proof for the other case is similar.

Theorem 9.2.21. *Let $\mathfrak{g}$ be a 2–step nilpotent Lie algebra of genus 2 over an algebraically closed field $\mathbb{F}$ of characteristic not 2 with minimal indices $\varepsilon_1, \ldots, \varepsilon_k$ and elementary divisors*

$$(\infty, e_1), \ldots, (\infty, e_s),\ (\alpha_1, f_1), \ldots, (\alpha_l, f_t)$$

with $\alpha_j \in \mathbb{F}$ for all $1 \leq j \leq t$. Then we have

$$\dim(\mathrm{AID}(\mathfrak{g})) = \dim(\mathrm{Inn}(\mathfrak{g})) + \sum_{\substack{j=1 \\ \varepsilon_j \neq 0}}^{k} (\varepsilon_j - 1) + 2\sum_{j=1}^{s}(e_j - 1) + 2\sum_{j=1}^{t}(f_t - 1).$$

Theorem 9.2.22. *Let $\mathfrak{g}$ be a 2–step nilpotent Lie algebra of genus 2 over $\mathbb{R}$ with minimal indices $\varepsilon_1, \ldots, \varepsilon_k$ and elementary divisors*

$$(\infty, e_1), \ldots, (\infty, e_s),\ (\alpha_1, f_1), \ldots, (\alpha_t, f_t),\ (\beta_1, m_1), \ldots, (\beta_p, m_p)$$

with $\alpha_j \in \mathbb{R}$ and $\beta_r = a_r + b_r i \in \mathbb{C} \setminus \mathbb{R}$ for all $1 \leq j \leq l$ and $1 \leq r \leq p$. Then we have

$$\dim(\mathrm{AID}(\mathfrak{g})) = \dim(\mathrm{Inn}(\mathfrak{g})) + \sum_{\substack{j=1 \\ \varepsilon_j \neq 0}}^{k} (\varepsilon_j - 1) + 2\sum_{j=1}^{s}(e_j - 1) + 2\sum_{j=1}^{t}(f_t - 1) + 4\sum_{j=1}^{p} m_j.$$

Proof. We may assume that $\mathfrak{g}$ is canonical with $N := k + s + t + p$ blocks. We will show the result by induction on N. For $N = 1$, the claim follows from the previous lemmas. If we have a canonical Lie algebra $\mathfrak{g}$ with $N + 1$ blocks, we take a basis $\{x_1, \ldots, x_n, y_1, \ldots, y_m, z_1, z_2\}$, where $\langle z_1, z_2 \rangle = [\mathfrak{g}, \mathfrak{g}]$ and where $\{x_1, \ldots, x_n\}$ corresponds to the first N blocks and $\{y_1, \ldots, y_m\}$ to the last block. Since we have $[x_i, y_j] = 0$ for all $1 \leq i \leq n$ and $1 \leq j \leq m$, we can apply Lemma 9.2.20 to show that the result holds for $N + 1$ blocks if it holds for N blocks. $\square$

We can apply these theorems to Example 9.2.6 with $n = 4$.

Example 9.2.23. Let $\mathfrak{g}$ be the Lie algebra over a field $\mathbb{F}$ with basis $\{x_1, \ldots, x_4, y_1, y_2\}$ and Lie brackets defined by

$$\begin{aligned} [x_1, x_2] &= y_1, \quad [x_1, x_3] = y_2, \\ [x_2, x_4] &= y_2, \quad [x_3, x_4] = -y_1. \end{aligned}$$

Then for $\mathbb{F} = \mathbb{R}$, we have that $\det(\mu A + \lambda B) = (\mu^2 + \lambda^2)^2 = 0$ if and only if $\mu = \lambda = 0$, so that $\mathrm{AID}(\mathfrak{g}) = \mathcal{C}(\mathfrak{g})$ and $\dim(\mathrm{AID}(\mathfrak{g})) = 2 \cdot \dim(\mathrm{Inn}(\mathfrak{g})) = 8$.

However, when $\mathbb{F}$ is an algebraically closed field of characteristic not 2, there are no minimal indices and two pair of elementary divisors, namely $(\alpha, 1)$ and $(-\alpha, 1)$, where $\alpha^2 = -1$. It follows from Theorem 9.2.21 that $\mathrm{AID}(\mathfrak{g}) = \mathrm{Inn}(\mathfrak{g})$. Denote l for the number of different linear factors of $\det(\mu A + \lambda B)$ over $\mathbb{F}$, then we have that

$$\dim(\mathrm{AID}(\mathfrak{g})/\mathrm{Inn}(\mathfrak{g})) = \deg(\det(\mu A + \lambda B)) - 2l.$$

This corresponds to the observations at the end of Chapter 5.

9.3 Nonsingular Lie algebras

In Section 4.3, we already defined nonsingular Lie algebras as 2–step nilpotent Lie algebras for which the determinant of the associated pencil only has the trivial solution. Let $\mathfrak{g}$ be a 2–step nilpotent Lie algebra over $\mathbb{F}$. We fix a basis $\{x_1, \ldots, x_n, y_1, \ldots, y_m\}$ of $\mathfrak{g}$, where $\{y_1, \ldots, y_m\}$ is a basis of $[\mathfrak{g}, \mathfrak{g}]$. When $\mathfrak{g}$ is nonsingular, we have that $\mathrm{AID}(\mathfrak{g}) = \mathcal{C}(\mathfrak{g})$ and

$$\dim(\mathrm{AID}(\mathfrak{g})) = m \dim(\mathrm{Inn}(\mathfrak{g})) = mn,$$

so the dimension of $\mathrm{AID}(\mathfrak{g})$ is 'as large as can be'. This makes nonsingular Lie algebras interesting for the study of almost inner derivations. When $\mathbb{F}$ is algebraically closed, then $\mathfrak{g}$ is nonsingular if and only if it is a generalised Heisenberg Lie algebra. This means that $\mathfrak{g}$ is not nonsingular if $m \geq 2$. When $\mathbb{F}$ is not algebraically closed, there also exist other nonsingular Lie algebras with $m \geq 2$. However, it is not easy to find examples for large $m \in \mathbb{N}$. More information and examples of real nonsingular Lie algebras can be found in [57].

Example 9.3.1. Consider the real Lie algebra with basis $\{x_1, \ldots, x_4, y_1, y_2, y_3\}$ and given by

$$\begin{aligned}[x_1, x_2] = [x_3, x_4] &= y_1,\\ [x_1, x_3] = [x_4, x_2] &= y_2,\\ [x_1, x_4] = [x_2, x_3] &= y_3.\end{aligned}$$

The determinant of the corresponding matrix pencil

$$\det \begin{pmatrix} 0 & \mu_1 & \mu_2 & \mu_3 \\ -\mu_1 & 0 & \mu_3 & -\mu_2 \\ -\mu_2 & -\mu_3 & 0 & \mu_1 \\ -\mu_3 & \mu_2 & -\mu_1 & 0 \end{pmatrix} = (\mu_1^2 + \mu_2^2 + \mu_3^2)^2$$

implies that $\mathfrak{g}$ is nonsingular over $\mathbb{R}$. It follows from Corollary 4.3.9 that $\dim(\mathrm{AID}(\mathfrak{g})) = 3 \dim(\mathrm{Inn}(\mathfrak{g})) = 12$.

In this section, we will give a way to construct a nonsingular Lie algebra of genus 2, where $\dim(\mathfrak{g}/[\mathfrak{g},\mathfrak{g}])$ is arbitrary large. Therefore, we will partially generalise the results from the previous section to 2–step nilpotent Lie algebras of genus 2 over an arbitrary field $\mathbb{F}$.

In the previous section, we assumed that $\mathbb{F} = \mathbb{R}$ or algebraically closed of characteristic not two. We showed that every 2–step nilpotent Lie algebra of genus 2 over $\mathbb{F}$ is isomorphic to a canonical Lie algebra (consisting of blocks on the diagonal) with the same elementary divisors and minimal indices. Underneath, the important steps for this result are discussed.

- Kronecker's theorem states that matrix pencils are classified up to strict equivalence by their elementary divisors and the minimal indices (see [32, Volume II, page 40] and [34, Proposition 6.5]). This result holds for every subfield of $\mathbb{C}$.
- If two matrix pencils are strictly congruent over $\mathbb{F}$, they are strictly equivalent over $\mathbb{F}$ by definition. The converse is true when $\mathbb{F}$ is algebraically closed of characteristic not two (see [34, Proposition 6.1]) and when $\mathbb{F} = \mathbb{R}$ (see [56, Lemma 13.1] and [80, pages 345–347]).
- When two matrix pencils over an arbitrary field $\mathbb{F}$ are strictly congruent, the corresponding Lie algebras are isomorphic (see Proposition 9.2.4).

The crucial step is to find for which fields strict equivalence and strict congruence are the same. The difficulty is that the proofs from [56] and [80] use specific properties of $\mathbb{R}$ or the fact that $\mathbb{F}$ is algebraically closed. Hence, the proofs cannot be easily adapted for other fields and another strategy has to be found.

In this subsection, we will give a generalisation of the elementary divisors. This will allow us to construct for an arbitrary field (which is not algebraically closed) a Lie algebra of genus 2 which is nonsingular over $\mathbb{F}$.

Take an arbitrary field $\mathbb{F}$ (of any characteristic) and consider a polynomial $p(x) = x^n + c_{n-1}x^{n-1} + \cdots + c_1 x + c_0$ which is irreducible over $\mathbb{F}$. Denote

$$C_p = \begin{pmatrix} 0 & \cdots & 0 & -c_0 \\ 1 & & & -c_1 \\ & \ddots & & \vdots \\ & & 1 & -c_{n-1} \end{pmatrix} \in M_n(\mathbb{F})$$

for the companion matrix of $p(x)$. We define

$$T_m := \begin{pmatrix} & & & 0 & C_p(\mu,\lambda) \\ & & \iddots & C_p(\mu,\lambda) & \mu\mathbb{I}_n \\ & & \iddots & & \\ 0 & C_p(\mu,\lambda) & \iddots & & \\ C_p(\mu,\lambda) & \mu\mathbb{I}_n & & & \end{pmatrix} \in M_{mn}(\mathbb{F}[\mu,\lambda])$$

for $m \geq 2$ and

$$T_1 = C_p(\mu,\lambda) = \lambda\mathbb{I}_n - \mu C_p.$$

Proposition 9.3.2. *Take $m \in \mathbb{N}$ and let $p(x) = x^n + c_{n-1}x^{n-1} + \cdots + c_1 x + c_0$ be a monic irreducible polynomial over a field $\mathbb{F}$. The elementary divisors of the matrix pencil*

$$\mu A + \lambda B := \begin{pmatrix} 0 & T_m \\ -T_m & 0 \end{pmatrix} \in M_{2mn}(\mathbb{F}[\mu,\lambda])$$

are $(\mu^n p(\lambda/\mu))^m, (\mu^n p(\lambda/\mu))^m$.

Proof. It suffices to compute the Smith normal form of the matrix pencil $A+\lambda B$, so we can take $\mu = 1$. By permuting the rows and columns of T_m, we obtain the matrix

$$\tilde{T}_m := \begin{pmatrix} \mathbb{I}_n & 0 & & & C_p(1,\lambda) \\ C_p(1,\lambda) & \mathbb{I}_n & \ddots & & \\ & & \ddots & & \\ & & \ddots & \mathbb{I}_n & 0 \\ & & & C_p(1,\lambda) & 0 \end{pmatrix}.$$

Note that the Smith normal forms of T_m and $\tilde{T}_m$ are the same. Since $\tilde{T}_m$ is Gaussian equivalent to the matrix

$$\begin{pmatrix} \mathbb{I}_n & 0 & & & C_p(1,\lambda) \\ & \mathbb{I}_n & \ddots & & -(C_p(1,\lambda))^2 \\ & & \ddots & & \vdots \\ & & & \mathbb{I}_n & \\ & & & & (-1)^{m+1}(C_p(1,\lambda))^m \end{pmatrix},$$

it suffices to compute the Smith normal form of $C_p(1,\lambda)^m$. It is a straightforward proof by induction on $m \in \mathbb{N}_0$ to show that this equals $\operatorname{diag}(1,\ldots,1,p(\lambda)^m)$. $\square$

We will denote the pair of elementary divisors from the previous proposition as (p, m).

Remark 9.3.3. This result generalises the elementary divisors from the previous section.

- For $p(x) = x - \alpha$ and $m = f$, we have $(\lambda - \alpha)^f, (\lambda - \alpha)^f$ and obtain the matrix $F(\alpha, f)$.
- For $p(x) = x^2 + 1$ over $\mathbb{R}$, the matrix $C(0, 1, m)$ can be obtained from the matrix of the proposition by permuting the rows.

Note that the results from 9.2.2 were only stated over $\mathbb{R}$ and over an algebraically closed field of characteristic not two. However, the proofs also work over arbitrary fields. In particular, consider a Lie algebra $\mathfrak{g}$ over a field $\mathbb{F}$ such that there are no minimal indices and such that the determinant of the corresponding matrix pencil splits in distinct linear factors. It follows from Lemma 9.2.14 and Lemma 9.2.20 that $\mathrm{AID}(\mathfrak{g}) = \mathrm{Inn}(\mathfrak{g})$. These observations imply the following result.

Corollary 9.3.4. *Let $p(x)$ be a monic polynomial of degree n which is irreducible over a field $\mathbb{F}_1$ and denote $\mathbb{F}_2$ for the splitting field of $\mathbb{F}_1$. Let $\mathfrak{g}_{\mathbb{F}_1}$ be a canonical Lie algebra over $\mathbb{F}_1$ with one pair of elementary divisors (p, m). Denote $\mathfrak{g}_{\mathbb{F}_2}$ for the Lie algebra over $\mathbb{F}_2$ with the same Lie brackets.*

- *We have* $\mathrm{AID}(\mathfrak{g}_{\mathbb{F}_1}) = \mathcal{C}(\mathfrak{g}_{\mathbb{F}_1})$, *so* $\dim(\mathrm{AID}(\mathfrak{g}_{\mathbb{F}_1})) = \dim(\mathrm{Inn}(\mathfrak{g}_{\mathbb{F}_1})) + 2mn$.
- *If* $\mathbb{F}_2 : \mathbb{F}_1$ *is separable, then* $\mathrm{AID}(\mathfrak{g}_{\mathbb{F}_2}) = \mathrm{Inn}(\mathfrak{g}_{\mathbb{F}_2})$.

Using the results from the previous section, the statement for $\mathrm{AID}(\mathfrak{g}_{\mathbb{F}_2})$ can be made more precise when we know the factorisation of $p(x)$ over $\mathbb{F}_2$. The corollary illustrates two interesting features of almost inner derivations. On the one hand, when a Lie algebra is considered over different fields (say $\mathbb{F}_1 \subseteq \mathbb{F}_2$), the difference between $\mathrm{AID}(\mathfrak{g}_{\mathbb{F}_1})$ and $\mathrm{AID}(\mathfrak{g}_{\mathbb{F}_2})$ can be arbitrary large. Over $\mathbb{F}_1$, we have a nonsingular Lie algebra, for which $\dim(\mathrm{AID}(\mathfrak{g}_{\mathbb{F}_1}))$ is in a sense 'as large as possible'. When $\mathbb{F}_2 : \mathbb{F}_1$ is separable, $\dim(\mathrm{AID}(\mathfrak{g}_{\mathbb{F}_2})$ is 'as small as possible', since the only almost inner derivations are the inner ones. On the other hand, the dimension of $\mathrm{AID}(\mathfrak{g}_{\mathbb{F}_1})/\mathrm{Inn}(\mathfrak{g}_{\mathbb{F}_1})$ can be an arbitrary large (even) number. In the next section, we will give a family of Lie algebras $\{\mathfrak{g}_n\}_n$ for which $\dim(\mathrm{AID}(\mathfrak{g}_n)/\mathrm{Inn}(\mathfrak{g}_n)) = n$.

9.4 Lie algebras with arbitrary large $\mathrm{AID}(\mathfrak{g})/\mathrm{Inn}(\mathfrak{g})$

In the next chapters, we will have many negative results concerning the existence of non-inner almost inner derivations. However, it is also possible to construct infinite families of Lie algebras $\{\mathfrak{g}_n\}_n$ such that $\mathrm{AID}(\mathfrak{g}_n)/\mathrm{Inn}(\mathfrak{g}_n)$ has an arbitrarily large dimension n, for any given $n \in \mathbb{N}$. In the previous section, we already showed examples of this phenomenon for Lie algebras of genus 2. In this section, we do the same for another type of 2–step nilpotent Lie algebras.

Consider the following family of 2–step nilpotent Lie algebras $\mathfrak{g}_n$ of dimension $4n+2$ over a general field $\mathbb{F}$, with basis $\{t_1, t_2, x_{1,i}, x_{2,i}, y_{1,i}, y_{2,i} \mid 1 \leq i \leq n\}$ and non-zero Lie brackets

$$[t_1, x_{1,i}] = y_{1,i}, \quad [t_1, x_{2,i}] = y_{2,i}, \quad [t_2, x_{2,i}] = y_{1,i}$$

for all $1 \leq i \leq n$. We have $\mathfrak{g}_n = \mathbb{F}^{4n} \rtimes \mathbb{F}^2$, where $\mathbb{F}^2$ is spanned by t_1 and t_2 and $\mathbb{F}^{4n}$ is the subspace spanned by $\{x_{p,i}, y_{p,i} \mid 1 \leq p \leq 2 \text{ and } 1 \leq i \leq n\}$.

Proposition 9.4.1. *For every $n \geq 2$, we have*

$$\dim(\mathrm{AID}(\mathfrak{g}_n)/\mathrm{Inn}(\mathfrak{g}_n)) = n.$$

Proof. Any element x of $\mathfrak{g}_n$ can be uniquely written in the form

$$x = \alpha_1 t_1 + \alpha_2 t_2 + v,$$

where $\alpha_1, \alpha_2 \in \mathbb{F}$ and $v \in \mathbb{F}^{4n} = \langle x_{p,i},\ y_{p,i} \mid 1 \leq p \leq 2 \text{ and } 1 \leq i \leq n\rangle$. Using this notation, we define for any $1 \leq i \leq n$ a map

$$\varphi_{D_i} : \mathfrak{g}_n \to \mathfrak{g}_n : x = \alpha_1 t_1 + \alpha_2 t_2 + v \mapsto \begin{cases} 0 & \text{if } \alpha_1 = 0, \\ -\frac{\alpha_2}{\alpha_1} x_{1,i} + x_{2,i} & \text{if } \alpha_1 \neq 0. \end{cases}$$

Define

$$D_i : \mathfrak{g}_n \to \mathfrak{g}_n : x \mapsto D_i(x) := [x, \varphi_{D_i}(x)].$$

If $\alpha_1 = 0$, then $D_i(\alpha_1 t_1 + \alpha_2 t_2 + v) = 0 = \alpha_1 y_{2,i}$. When $\alpha_1 \neq 0$, we have that

$$\begin{aligned} D_i(\alpha_1 t_1 + \alpha_2 t_2 + v) &= [\alpha_1 t_1 + \alpha_2 t_2 + v, -\frac{\alpha_2}{\alpha_1} x_{1,i} + x_{2,i}] \\ &= -\alpha_2 y_{1,i} + \alpha_1 y_{2,i} + \alpha_2 y_{1,i} \\ &= \alpha_1 y_{2,i}. \end{aligned}$$

Hence, $D_i : \mathfrak{g}_n \to \mathfrak{g}_n$ is a linear map having its image in the center of $\mathfrak{g}_n$ and so D_i is a derivation. By construction, $D_i \in \mathrm{AID}(\mathfrak{g}_n)$.

We claim that $\{D_i + \mathrm{Inn}(\mathfrak{g}_n) \mid 1 \le i \le n\}$ forms a basis of $\mathrm{AID}(\mathfrak{g}_n)/\mathrm{Inn}(\mathfrak{g}_n)$. First, we will show that this is a linearly independent set. Assume that $\sum_{i=1}^n \beta_i D_i \in \mathrm{Inn}(\mathfrak{g}_n)$, then

$$\sum_{i=1}^n \beta_i D_i = \mathrm{ad}(\alpha_1 t_1 + \alpha_2 t_2 + v)$$

for some $\alpha_1, \alpha_2 \in \mathbb{F}$ and $v \in \mathbb{F}^{4n}$. As $\sum_{i=1}^n \beta_i D_i(x_{1,1}) = 0$, it follows that

$$0 = [\alpha_1 t_1 + \alpha_2 t_2 + v, x_{1,1}] = \alpha_1 y_{1,1},$$

so that $\alpha_1 = 0$. Analogously, the fact that $\sum_{i=1}^n \beta_i D_i(x_{2,1}) = 0$ leads to $\alpha_2 = 0$, which means that $\sum_{i=1}^n \beta_i D_i = \mathrm{ad}(v)$ with $v \in \mathbb{F}^{4n}$. This implies that

$$\sum_{i=1}^n \beta_i y_{2,i} = \sum_{i=1}^n \beta_i D_i(t_1) = [v, t_1],$$

$$0 = \sum_{i=1}^n \beta_i D_i(t_2) = [v, t_2].$$

The second equation above shows that v has no component for $x_{2,i}$ (with $1 \le i \le n$) and thus, $[v, t_1] = 0$ holds. Using this in the first equation above leads to $\beta_1 = \beta_2 = \cdots = \beta_n = 0$.

Next, we have to verify that the set is generating. Let $D \in \mathrm{AID}(\mathfrak{g}_n)$ be determined by a map φ_D. We have to show that

$$D = \sum_{i=1}^n \beta_i D_i + \mathrm{ad}(x)$$

for some $\beta_1, \beta_2, \ldots, \beta_n \in \mathbb{F}$ and $x \in \mathfrak{g}_n$. Many of the basis vectors turn out to be fixed.

- Every basis vector which belongs to the center of $\mathfrak{g}_n$ is fixed.
- It follows from Lemma 4.2.3 that $x_{1,i}$ is fixed for all $1 \le i \le n$, since its centraliser is of codimension 1 in $\mathfrak{g}_n$.
- To see that t_2 is fixed, note that the basis vectors not belonging to $C_{\mathfrak{g}_n}(t_2)$ are the vectors $x_{2,i}$, with $1 \le i \le n$. Take $1 \le i \le n$, then we can apply Lemma 4.2.7 with $e_i = t_2$, $e_j = x_{2,i}$, $e_k = x_{2,j}$, $e_l = y_{1,i}$ and $e_m = y_{1,j}$. This shows that $t_{t_2}(\varphi_D(x_{2,i})) = t_{t_2}(\varphi_D(x_{2,j}))$, from which it follows that t_2 is fixed.

- We now will show that t_1 is fixed as well. Take arbitrary $1 \leq i, j \leq n$. We start with applying Lemma 4.2.7 with $e_i = t_1$, $e_j = x_{1,i}$, $e_k = x_{1,j}$, $e_l = y_{1,i}$ and $e_m = y_{1,j}$. This gives us that
$$t_{t_1}(\varphi_D(x_{1,i})) = t_{t_1}(\varphi_D(x_{1,j}))$$
for all $1 \leq i, j \leq n$. We apply the same Lemma 4.2.7 with $e_i = t_1$, $e_j = x_{1,i}$, $e_k = x_{2,j}$, $e_l = y_{1,i}$ and $e_m = y_{2,j}$, where we require that $i \neq j$. We find that
$$t_{t_1}(\varphi_D(x_{1,i})) = t_{t_1}(\varphi_D(x_{2,j}))$$
for all $1 \leq i, j \leq n$ with $i \neq j$. Together with the above (and knowing that $n \geq 2$), we can conclude that
$$t_{t_1}(\varphi_D(x_{1,i})) = t_{t_1}(\varphi_D(x_{1,j})) = t_{t_1}(\varphi_D(x_{2,k}))$$
for all $1 \leq i, j, k \leq n$, showing that t_1 is fixed.

This means that every basis vector is fixed, except for $x_{2,i}$, where $1 \leq i \leq n$. We apply Lemma 4.2.5 for every fixed basis vector and change D up to an inner derivation each time. Let x be a basis vector, then we may assume that $\varphi_D(x) = \sum_{i=1}^{n} \beta_i(x) x_{2,i}$, where $\beta_i(x) \in \mathbb{F}$ for all $1 \leq i \leq n$. We change D to $D + \mathrm{ad}(\varphi_D(t_2))$. In this way, we may suppose that $\varphi_D(t_2) = 0$ and also $D(t_2) = [t_2, \varphi_D(t_2)] = 0$. Take $x \in \{x_{p,i}, y_{p,i} \mid 1 \leq p \leq 2 \text{ and } 1 \leq i \leq n\}$, then we also have
$$D(x) = [x, \varphi_D(x)] = [x, \sum_{i=1}^{n} \beta_i(x) x_{2,i}] = 0.$$
Finally, it follows that
$$D(t_1) = [t_1, \varphi_D(t_1)] = [t_1, \sum_{i=1}^{n} \beta_i(t_1) x_{2,i}] = \sum_{i=1}^{n} \beta_i(t_1) y_{2,i}.$$
As a conclusion, we find that, after changing D up to an inner derivation, we obtain
$$D = \sum_{i=1}^{n} \beta_i(t_1) D_i.$$
This shows that the set $\{D_i + \mathrm{Inn}(\mathfrak{g}_n) \mid 1 \leq i \leq n\}$ forms a basis of $\mathrm{AID}(\mathfrak{g}_n)/\mathrm{Inn}(\mathfrak{g}_n)$. $\square$

Remark 9.4.2. For $n = 1$ the basis vector t_1 is not fixed. Then the algebra $\mathfrak{g}_1$ of the above family has 2–dimensional commutator Lie algebra, so we can compute the almost inner derivations with the techniques from Section 9.2. For this algebra, it turns out that $\dim(\mathrm{AID}(\mathfrak{g}_1)/\mathrm{Inn}(\mathfrak{g}_1)) = 2$.

Chapter 10

Filiform Lie algebras

Last chapter, we studied two-step nilpotent Lie algebras, which are (after the abelian ones) the 'most' nilpotent Lie algebras. This chapter is devoted to filiform Lie algebras. These are considered to be the 'least' nilpotent ones. Since we work with nilpotent Lie algebras, we can assume that the dimension is at least three. The following result is well-known.

Theorem 10.0.1. *Let $\mathfrak{g}$ be an n–dimensional filiform Lie algebras over a field $\mathbb{F}$, then there exists an adapted basis $\mathcal{B} = \{e_1, \ldots, e_n\}$ such that*

$$
\begin{aligned}
[e_1, e_i] &= e_{i+1}, & 2 \leq i \leq n-1,\\
[e_i, e_j] &\in \langle e_{i+j}, \ldots, e_n \rangle, & 4 \leq i+j \leq n,\\
[e_{i+1}, e_{n-i}] &= (-1)^i \alpha e_n & 1 \leq i \leq n-2.
\end{aligned}
$$

The undefined Lie brackets are zero. Moreover, $\alpha \in \mathbb{F}$ is zero when n is odd.

The above result also implies that $\dim(\mathrm{Inn}(\mathfrak{g})) = n-1$ holds. In this chapter, we will study the almost inner derivations for different types of filiform Lie algebras. The first section is about metabelian filiform Lie algebras, where we obtain a complete result. This also includes the classes L_n and R_n discussed in [42]. In the following section, we determine the almost inner derivations for the classes Q_n and W_n discussed in the same reference. Note that the notation is different from the one in [42], where the Lie algebras with subscript '$_n$' have dimension $n+1$ and where the adapted basis is denoted as $\{e_0, \ldots, e_n\}$. The last section of this chapter contains computations for a class of filiform Lie algebras where all derivations are almost inner (but not all inner). These Lie algebras were also studied in [10] for a different purpose. Most of the results from this chapter already appeared in [7] and [8]. However, the statements from [7] were formulated for complex Lie algebras, whereas they are valid for Lie algebras over a general field. We will specify at the start of each section over which field $\mathbb{F}$ we work. Therefore, we will not mention the fields in the definitions and results when it is not really necessary.

10.1 Metabelian filiform Lie algebras

In this section, we will study the almost inner derivations for the filiform Lie algebras which are also two-step solvable. It turns out that the adapted basis for metabelian filiform Lie algebras can be described in an easy way. We consider a general field $\mathbb{F}$ (over an arbitrary characteristic).

Definition 10.1.1 (Standard graded filiform Lie algebra)**.** Consider $\mathfrak{f}_n$, the n–dimensional Lie algebra with basis $\mathcal{B} = \{e_1, \ldots, e_n\}$ and Lie brackets defined by

$$[e_1, e_i] = e_{i+1}, \quad 2 \leq i \leq n-1.$$

The Lie algebra $\mathfrak{f}_n$ is called the 'standard graded filiform Lie algebra' of dimension n.

This is the standard and most easy example of a filiform Lie algebra. The smallest example is the Heisenberg Lie algebra $\mathfrak{f}_3 = \mathfrak{h}_3(\mathbb{F})$. In [42], the Lie algebra $\mathfrak{f}_n$ is denoted as L_{n-1}.

Proposition 10.1.2. *Let $n \geq 3$ and denote $\mathfrak{f}_n$ for the standard graded filiform nilpotent Lie algebra of dimension n, then* $\mathrm{AID}(\mathfrak{f}_n) = \mathrm{Inn}(\mathfrak{f}_n)$ *holds.*

Proof. The Lie algebra $\mathfrak{f}_n$ has a basis $\{e_1, \ldots, e_n\}$ for which the non-zero Lie brackets are given by $[e_1, e_i] = e_{i+1}$, where $2 \leq i \leq n-1$. Let $D \in \mathrm{AID}(\mathfrak{f}_n)$ be an arbitrary almost inner derivation determined by φ_D. Lemma 4.2.3 implies that the basis vector e_i is fixed for all $2 \leq i \leq n-1$. Take arbitrary $2 \leq j, k \leq n-1$ with $j \neq k$. We have

$$\begin{aligned}
&[e_j, e_1] + e_{j+1} \in \mathfrak{g}_{j+1,k+1} \\
&[e_k, e_1] + e_{k+1} \in \mathfrak{g}_{j+1,k+1} \\
&[e_j, \mathfrak{g}_1] \subseteq \mathfrak{g}_{j+1,k+1} \\
&[e_k, \mathfrak{g}_1] \subseteq \mathfrak{g}_{j+1,k+1}.
\end{aligned}$$

It follows from Lemma 4.2.7 that $t_1(\varphi_D(e_j)) = t_1(\varphi_D(e_k))$. Since j and k were arbitrary, we find that e_1 is fixed as well. Corollary 4.2.6 yields that D is inner, which concludes the proof. $\square$

For other metabelian filiform Lie algebras, this result does not hold. For instance, we studied in Example 4.1.6 a metabelian filiform nilpotent Lie algebra $\mathfrak{g}$ with $\dim(\mathrm{AID}(\mathfrak{g})) = \dim(\mathrm{Inn}(\mathfrak{g})) + 1$. It turns out that this example generalises to all metabelian filiform Lie algebras of dimension $n \geq 5$ which are not standard graded. We will look at the adapted basis for metabelian filiform Lie algebras in more detail.

Proposition 10.1.3 ([5])**.** *Let $\mathfrak{g}$ be a metabelian filiform Lie algebra $\mathfrak{g}$ over a field $\mathbb{F}$. Suppose that $\mathfrak{g}$ has dimension $n \geq 3$, then it has an 'adapted basis' $\mathcal{B} = \{e_1, \ldots, e_n\}$ such that*

$$\begin{aligned}
&[e_1, e_i] = e_{i+1}, && 1 \leq i \leq n-1,\\
&[e_2, e_j] = \alpha_{2,5}e_{2+j} + \ldots + \alpha_{2,n-j+3}e_n, && 3 \leq j \leq n-2,\\
&[e_k, e_l] = 0, && 3 \leq k, l \leq n,
\end{aligned}$$

with structure constants $\{\alpha_{2,j} \in \mathbb{F} \mid 5 \leq j \leq n\}$.

Clearly $\mathfrak{g} \cong \mathfrak{f}_n$ holds if and only if all structure constants $\alpha_{2,j}$ (with $5 \leq j \leq n$) are zero. The second set of equations only defines non-zero brackets if $n \geq 5$. Hence, if the dimension is at most 4, all metabelian filiform Lie algebras are standard graded.

Example 10.1.4. The Lie algebra R_{n-1} (for $n \geq 5$) from [42] has an adapted basis $\{e_1, \ldots, e_n\}$ and is defined by the Lie brackets

$$\begin{aligned}
&[e_1, e_i] = e_{i+1}, \quad 2 \leq i \leq n-1,\\
&[e_2, e_k] = e_{k+2}, \quad 3 \leq k \leq n-2.
\end{aligned}$$

This is a metabelian filiform Lie algebra with $\alpha_{2,5} = 1$ and where all other structure constants $\alpha_{2,j}$ (with $6 \leq j \leq n$) are zero.

With the aid of this adapted basis, it is possible to prove a general result concerning the almost inner derivations of metabelian filiform Lie algebras. First, we need the following lemma.

Lemma 10.1.5. *Let $\mathfrak{g}_n$ be a metabelian filiform Lie algebra of dimension $n \geq 3$ with adapted basis $\mathcal{B}$. Suppose that $D \in \mathrm{AID}(\mathfrak{g}_n)$, then there exist $x \in \mathfrak{g}$ and $\lambda \in \mathbb{F}$ such that*

$$D - \mathrm{ad}(x) = \lambda E_{n,2}.$$

Proof. We proceed by induction on the dimension n of $\mathfrak{g}$.

- Basis step: If $n < 5$, then $\mathfrak{g}_n$ is a standard filiform Lie algebra and all almost inner derivations are inner by Proposition 10.1.2. The result holds with $\lambda = 0$.

- Induction step: Assume that $n \geq 5$ and that the lemma is valid for metabelian filiform Lie algebras of smaller dimensions (induction hypothesis). Consider the adapted basis for $\mathfrak{g}_n$. Let $D \in \mathrm{AID}(\mathfrak{g}_n)$ be an arbitrary almost inner derivation. Then D induces an almost inner derivation $\bar{D}$ on $\mathfrak{g}_n/\langle e_n\rangle \cong \mathfrak{g}_{n-1}$. We may assume by the induction hypothesis that, after changing D up to an inner derivation, we have $\bar{D} = \mu E_{n-1,2}$ for some $\mu \in \mathbb{F}$. This implies that $D(e_1) = ae_n$ for some $a \in \mathbb{F}$. We replace D with $D' = D + \mathrm{ad}(ae_{n-1})$. Then we have $D'(e_1) = D(e_1) + [ae_{n-1}, e_1] = 0$ and

$$D'(e_i) = D(e_i) + [ae_{n-1}, e_i] = D(e_i)$$

for all $2 \leq i \leq n$. In particular, this means that

$$D'(e_2) = D(e_2) = \mu e_{n-1} + \lambda e_n$$

for some $\mu, \lambda \in \mathbb{F}$. It follows that

$$D'(e_3) = D'([e_1, e_2]) = [D'(e_1), e_2] + [e_1, D'(e_2)] = \mu e_n,$$

$$D'(e_4) = D'([e_1, e_3]) = [D'(e_1), e_3] + [e_1, D'(e_3)] = 0$$

and analogously $D'(e_i) = 0$ for $5 \leq i \leq n$. To finish the proof, we have to show that $\mu = 0$.

Suppose that $\mu \neq 0$. Since we have $D'(e_3) = \mu e_n$ and $D' \in \mathrm{AID}(\mathfrak{g})$, there must exist an element $\sum_{i=1}^n a_i e_i \in \mathfrak{g}$ with $[\sum_{i=1}^n a_i e_i, e_3] = \mu e_n$. This leads to the equation

$$a_1 e_4 + a_2 [e_2, e_3] = \mu e_n,$$

which expands to

$$a_1 e_4 + a_2(\alpha_{2,5} e_5 + \alpha_{2,6} e_6 + \cdots + \alpha_{2,n} e_n) = \mu e_n.$$

Since we assume that $\mu \neq 0$, this implies

$$\alpha_{2,5} = \alpha_{2,6} = \cdots = \alpha_{2,n-1} = 0.$$

As a conclusion thus far, we have found that when $\mu \neq 0$, then

$$[e_2, e_3] = \alpha_{2,n} e_n,$$

$$[e_2, e_i] = 0$$

for all $4 \leq i \leq n$. There also exists an element $\sum_{i=1}^{n} b_i e_i \in \mathfrak{g}$ with $D'(e_2) = [\sum_{i=1}^{n} b_i e_i, e_2]$. This leads to the equation

$$\mu e_{n-1} + \lambda e_n = b_1 e_3 - b_3 \alpha_{2,n} e_n.$$

This equation does not have a solution, because we suppose that $\mu \neq 0$. Hence, we obtain a contradiction and $\mu = 0$, which was to be shown.

- Conclusion: By the principle of induction, the statement follows from the basis and induction step. □

The lemma now easily implies the following result.

Proposition 10.1.6. *Let $\mathfrak{g}_n$ be a metabelian filiform Lie algebra of dimension $n \geq 5$ with adapted basis $\mathcal{B}$. If $\mathfrak{g}_n$ is different from $\mathfrak{f}_n$, then*

$$\mathrm{AID}(\mathfrak{g}) = \mathrm{CAID}(\mathfrak{g}) = \mathrm{Inn}(\mathfrak{g}) \oplus \langle E_{n,2} \rangle.$$

Proof. We only have to show that $D = E_{n,2}$ is an almost inner derivation. It is clear that it is a derivation, so we will only show that D satisfies the almost inner condition. Take an arbitrary $x = \sum_{i=1}^{n} x_i e_i \in \mathfrak{g}$. If $x_1 \neq 0$, then $D(x) = x_2 e_n = [x, \frac{x_2}{x_1} e_{n-1}]$. Otherwise, we have $x_1 = 0$. Since $\mathfrak{g}$ is not the standard graded algebra $\mathfrak{f}_n$, there exists a minimal index i with $5 \leq i \leq n$ such that $\alpha_{2,i} \neq 0$. Then, for $k = n - i + 3 \geq 3$ we have $D(x) = x_2 e_n = [x, \frac{1}{\alpha_{2,i}} e_k]$. Hence $D(x) \in [x, \mathfrak{g}]$ for all $x \in \mathfrak{g}$ and this concludes the proof. □

This result does not hold for filiform Lie algebras in general. We will show this in the next sections by computing the almost inner derivations for different classes of filiform Lie algebras.

10.2 Other filiform Lie algebras

In this section, we determine the almost inner derivations for the classes Q_n and W_n of filiform nilpotent Lie algebras discussed in [42, Chapter 4]. Note that we work with a different notation than in [42] for the adapted basis. Throughout this section, we consider a field $\mathbb{F}$ of characteristic zero.

Definition 10.2.1 (Lie algebra Q_n)**.** Let $n \geq 6$ be even. The Lie algebra Q_n is defined by the Lie brackets

$$[e_1, e_i] = e_{i+1}, \qquad 2 \leq i \leq n-1,$$

$$[e_j, e_{n-j+1}] = (-1)^{j+1} e_n, \quad 2 \leq j \leq \frac{n}{2}.$$

Let $x = \sum_{i=1}^n x_i e_i \in Q_n$. Define linear maps in $\mathrm{End}(Q_n)$ by

$$t_0(x) = x_2 e_n,$$

$$t_1(x) = x_1 e_1 + x_1 e_2 + \sum_{i=3}^{n-1} (i-2) x_i e_i + (n-3) x_n e_n,$$

$$t_2(x) = -x_1 e_2 + \sum_{i=2}^{n-1} x_i e_i + 2 x_n e_n,$$

$$h_s(x) = \sum_{i=2}^{n+1-2s} x_i e_{i-1+2s}, \quad 2 \leq s \leq \frac{n}{2} - 1.$$

A computation shows that these linear maps are derivations of Q_n. We have the following result, see [42].

Proposition 10.2.2. *Let $n \geq 6$ be even, then a basis of $\mathrm{Der}(Q_n)$ is given by $\{\mathrm{ad}(e_1), \ldots, \mathrm{ad}(e_{n-1}), t_0, t_1, t_2, h_2, h_3, \ldots, h_{\frac{n}{2}-1}\}$.*

This shows that $\dim \mathrm{Der}(Q_n) = \frac{3n}{2}$. Note that there is a mistake in the formulation and proof in [42], since the map t_0 is not taken into account, although it is a derivation. It corresponds with the map d_{n-2} of the proof, which is not zero as is claimed there. It turns out that every almost inner derivation of Q_n is inner.

Proposition 10.2.3. *Let $n \geq 6$ be even, then $\mathrm{AID}(Q_n) = \mathrm{Inn}(Q_n)$ holds.*

Proof. Take an arbitrary $D \in \langle t_0, t_1, t_2, h_s \mid 2 \leq s \leq \frac{n}{2} - 1 \rangle$, then there exist values $\alpha_0, \alpha_1, \alpha_2, \beta_s$ (with $2 \leq s \leq \frac{n}{2} - 1$) such that

$$D = \alpha_0 t_0 + \alpha_1 t_1 + \alpha_2 t_2 + \sum_{s=2}^{\frac{n}{2}-1} \beta_s h_s.$$

Suppose that $D \in \mathrm{AID}(Q_n)$. For $x = \sum_{i=1}^n x_i e_i \in Q_n$, we have

$$[e_1 + e_2, x] = (x_2 - x_1) e_3 + \sum_{i=4}^{n-1} x_{i-1} e_i.$$

Since $D(e_1 + e_2) = \alpha_0 e_n + \alpha_1 (e_1 + e_2) + \sum_{s=2}^{\frac{n}{2}-1} \beta_s e_{2s+1}$, we must have that $\alpha_0 = \alpha_1 = 0$.

Moreover, $D(e_2) = \alpha_2 e_2 + \sum_{s=2}^{\frac{n}{2}-1} \beta_s e_{2s+1}$, but $[e_2, Q_n] = \langle e_3, e_n \rangle$, which means that $\alpha_2 = \beta_s = 0$ (for all $2 \le s \le \frac{n}{2} - 1$). Hence, the only almost inner derivation in $\langle t_0, t_1, t_2, h_s \mid 2 \le s \le \frac{n}{2} - 1 \rangle$ is $D = 0$. $\square$

Definition 10.2.4 (Witt Lie algebra)**.** The Witt Lie algebra W_n for $n \ge 5$ is defined by the Lie brackets

$$\begin{aligned} [e_1, e_i] &= e_{i+1}, && 2 \le i \le n-1, \\ [e_i, e_j] &= \frac{6(j-i)}{j(j-1)\binom{j+i-2}{i-2}} e_{i+j}, && 2 \le i \le \frac{n-1}{2} \text{ and } i+1 \le j \le n-i. \end{aligned}$$

The Witt algebra W_n also has a basis $\{f_1, \ldots, f_n\}$ which is not adapted and where the non-zero Lie brackets are $[f_i, f_j] = (j - i) f_{i+j}$ for $1 \le i + j \le n$. However, in what follows, we will use the adapted one. Suppose that $n \ge 6$ and take $x = \sum_{i=1}^{n} x_i e_i \in W_n$. Define linear maps in $\mathrm{End}(W_n)$ by

$$\begin{aligned} t_1(x) &= x_2 e_n, \\ t_2(x) &= x_2 e_{n-1} + x_3 e_n, \\ t_3(x) &= x_2 e_{n-2} + x_3 e_{n-1} + x_4 e_n, \\ h(x) &= \sum_{i=1}^{n} i x_i e_i. \end{aligned}$$

These linear maps are derivations of W_n. Moreover, we have the following result, see [42].

Proposition 10.2.5. *Let $n \ge 6$. Then $\{\mathrm{ad}(e_1), \ldots, \mathrm{ad}(e_{n-1}), t_1, t_2, t_3, h\}$ is a basis of $\mathrm{Der}(W_n)$.*

This result does not hold for W_5, as we already computed in Example 4.1.6. From this proposition, we can compute the Lie algebra of almost inner derivations. It turns out that the dimension of $\mathrm{AID}(W_n)$ depends on the value of n.

Proposition 10.2.6. *Take $n \ge 5$ and consider the Witt Lie algebra W_n.*

- *For $5 \le n \le 6$, we have $\mathrm{AID}(W_n) = \mathrm{Inn}(W_n) \oplus \langle t_1 \rangle$.*
- *If $7 \le n \le 8$, then $\mathrm{AID}(W_n) = \mathrm{Inn}(W_n) \oplus \langle t_1, t_2 \rangle$.*
- *When $n \ge 9$, we have $\mathrm{AID}(W_n) = \mathrm{Inn}(W_n) \oplus \langle t_1, t_2, t_3 \rangle$.*

Proof. For all $2 \leq i < j \leq n-i$, we will write $[e_i, e_j] = c_{i,j}e_{i+j}$ for the coefficients appearing in the Lie brackets of W_n. For other values of i and j, the coefficient $c_{i,j}$ is not defined. The proof consists of different steps. In the first part, we will specify the determination maps φ_{t_i} for $1 \leq i \leq 3$. Let $x = \sum_{i=1}^n x_i e_i \in W_n$. It suffices to show that $t_i(x) = [x, \varphi_{t_i}(x)]$.

- Define a map $\varphi_{t_1} : W_n \to W_n$ by
$$\varphi_{t_1}(x) = \begin{cases} \frac{x_2}{x_1} e_{n-1} & \text{if } x_1 \neq 0, \\ \frac{1}{c_{2,n-2}} e_{n-2} & \text{if } x_1 = 0. \end{cases}$$
Here $n-2 \geq 3$ in $c_{2,n-2}$ since $n \geq 5$. When $x_1 \neq 0$, it follows that
$$\frac{x_2}{x_1}[x, e_{n-1}] = x_2[e_1, e_{n-1}] = x_2 e_n = t_1(x).$$
For $x_1 = 0$, we have $\frac{1}{c_{2,n-2}}[x, e_{n-2}] = \frac{x_2}{c_{2,n-2}}[e_2, e_{n-2}] = x_2 e_n = t_1(x)$.
- For $n \geq 7$, we define a map $\varphi_{t_2} : W_n \to W_n$ by
$$\varphi_{t_2}(x) = \begin{cases} \frac{x_2}{x_1} e_{n-2} + (\frac{x_3}{x_1} - \frac{c_{2,n-2}x_2^2}{x_1^2}) e_{n-1} & \text{if } x_1 \neq 0, \\ \frac{1}{c_{2,n-3}} e_{n-3} + \frac{(c_{2,n-3} - c_{3,n-3})x_3}{c_{2,n-2}c_{2,n-3}x_2} e_{n-2} & \text{if } x_1 = 0 \text{ and } x_2 \neq 0, \\ \frac{1}{c_{3,n-3}} e_{n-3} & \text{if } x_1 = x_2 = 0. \end{cases}$$
This is well-defined for the coefficients $c_{i,j}$, since $n \geq 7$. For $x_1 \neq 0$, we have
$$\begin{aligned} [x, \varphi_{t_2}(x)] &= x_2[e_1, e_{n-2}] + \left(x_3 - c_{2,n-2}\frac{x_2^2}{x_1} \right) [e_1, e_{n-1}] + \frac{x_2^2}{x_1}[e_2, e_{n-2}] \\ &= x_2 e_{n-1} + x_3 e_n - c_{2,n-2}\frac{x_2^2}{x_1} e_n + c_{2,n-2}\frac{x_2^2}{x_1} e_n \\ &= t_2(x). \end{aligned}$$
When $x_1 = 0$ and $x_2 \neq 0$, we have
$$\begin{aligned} [x, \varphi_{t_2}(x)] = \frac{x_2}{c_{2,n-3}}&[e_2, e_{n-3}] + \frac{c_{2,n-3} - c_{3,n-3}}{c_{2,n-2}c_{2,n-3}} x_3 [e_2, e_{n-2}] \\ &+ \frac{x_3}{c_{2,n-3}}[e_3, e_{n-3}] \\ &= x_2 e_{n-1} + x_3 \left(\frac{c_{2,n-3} - c_{3,n-3}}{c_{2,n-3}} + \frac{c_{3,n-3}}{c_{2,n-3}} \right) e_n \\ &= t_2(x). \end{aligned}$$

Finally, if $x_1 = x_2 = 0$, then

$$\frac{1}{c_{3,n-3}}[x, e_{n-3}] = \frac{x_3}{c_{3,n-3}}[e_3, e_{n-3}] = x_3 e_n = t_2(x).$$

- For $n \geq 9$, we define a map $\varphi_{t_3} : W_n \to W_n$ by

$$\varphi_{t_3}(x) = \begin{cases} \rho_1(x) & \text{if } x_1 \neq 0, \\ \rho_2(x) & \text{if } x_1 = 0 \text{ and } x_2 \neq 0, \\ \rho_3(x) & \text{if } x_1 = x_2 = 0 \text{ and } x_3 \neq 0, \\ \rho_4(x) & \text{if } x_1 = x_2 = x_3 = 0, \end{cases}$$

with

$$\begin{aligned} \rho_1(x) &= \frac{x_2}{x_1} e_{n-3} + \left(\frac{x_3}{x_1} - \frac{c_{2,n-3} x_2^2}{x_1^2} \right) e_{n-2} \\ &\quad + \left(\frac{x_4}{x_1} - \frac{(c_{2,n-2} + c_{3,n-3}) x_2 x_3}{x_1^2} + \frac{c_{2,n-2} c_{2,n-3} x_2^3}{x_1^3} \right) e_{n-1}, \\ \rho_2(x) &= \frac{1}{c_{2,n-4}} e_{n-4} + \frac{(c_{2,n-4} - c_{3,n-4}) x_3}{c_{2,n-3} c_{2,n-4} x_2} e_{n-3} \\ &\quad + \left(\frac{(c_{2,n-4} - c_{4,n-4}) x_4}{c_{2,n-2} c_{2,n-4} x_2} - \frac{(c_{2,n-4} - c_{3,n-4}) c_{3,n-3} x_3^2}{c_{2,n-2} c_{2,n-3} c_{2,n-4} x_2^2} \right) e_{n-2}, \\ \rho_3(x) &= \frac{1}{c_{3,n-4}} e_{n-4} + \frac{(c_{3,n-4} - c_{4,n-4}) x_4}{c_{3,n-3} c_{3,n-4} x_3} e_{n-3}, \\ \rho_4(x) &= \frac{1}{c_{4,n-4}} e_{n-4}. \end{aligned}$$

Since $n \geq 9$, all coefficients $c_{i,j} > 0$ are well-defined and hence also the determination map φ_{t_3}. We will check that $t_3(x) = [x, \varphi_{t_3}(x)]$ holds for all $x \in \mathfrak{g}$. For $x_1 \neq 0$, we have that

$$\begin{aligned} [x, \rho_1(x)] &= x_2[e_1, e_{n-3}] + x_3[e_1, e_{n-2}] - c_{2,n-3} \frac{x_2^2}{x_1} [e_1, e_{n-2}] \\ &\quad + x_4 [e_1, e_{n-1}] - (c_{2,n-2} + c_{3,n-3}) \frac{x_2 x_3}{x_1} [e_1, e_{n-1}] \\ &\quad + c_{2,n-2} \cdot c_{2,n-3} \frac{x_2^3}{x_1^2} [e_1, e_{n-1}] + \frac{x_2^2}{x_1} [e_2, e_{n-3}] \\ &\quad + \frac{x_2 x_3}{x_1} [e_2, e_{n-2}] - c_{2,n-3} \frac{x_2^3}{x_1^2} [e_2, e_{n-2}] + \frac{x_2 x_3}{x_1} [e_3, e_{n-3}]. \end{aligned}$$

The Lie brackets of $\mathfrak{g}$ imply that

$$\begin{aligned}[x,\rho_1(x)] &= x_2e_{n-2}+x_3e_{n-1}+(-c_{2,n-3}+c_{2,n-3})\frac{x_2^2}{x_1}e_{n-1}+x_4e_n\\ &\quad+\ \frac{x_2x_3}{x_1}(-c_{2,n-2}-c_{3,n-3}+c_{2,n-2}+c_{3,n-3})e_n\\ &\quad+\ \frac{x_2^3}{x_1^2}(c_{2,n-2}\cdot c_{2,n-3}-c_{2,n-2}\cdot c_{2,n-3})e_n\\ &= x_2e_{n-2}+x_3e_{n-1}+x_4e_n,\end{aligned}$$

so $[x,\rho_1(x)]=t_3(x)$ in this case. If $x_1=0$ and $x_2\neq 0$, then

$$\begin{aligned}[x,\rho_2(x)] &= \frac{1}{c_{2,n-4}}x_2[e_2,e_{n-4}]+\frac{c_{2,n-4}-c_{3,n-4}}{c_{2,n-3}\cdot c_{2,n-4}}x_3[e_2,e_{n-3}]\\ &\quad+\ \frac{c_{2,n-4}-c_{4,n-4}}{c_{2,n-2}\cdot c_{2,n-4}}x_4[e_2,e_{n-2}]\\ &\quad-\ \frac{(c_{2,n-4}-c_{3,n-4})\cdot c_{3,n-3}}{c_{2,n-2}\cdot c_{2,n-3}\cdot c_{2,n-4}}\frac{x_3^2}{x_2}[e_2,e_{n-2}]\\ &\quad+\ x_3\frac{1}{c_{2,n-4}}[e_3,e_{n-4}]+\frac{c_{2,n-4}-c_{3,n-4}}{c_{2,n-3}\cdot c_{2,n-4}}\frac{x_3^2}{x_2}[e_3,e_{n-3}]\\ &\quad+\ \frac{x_4}{c_{2,n-4}}[e_4,e_{n-4}]\\ &= x_2e_{n-2}+x_3\left(\frac{c_{2,n-4}-c_{3,n-4}}{c_{2,n-3}\cdot c_{2,n-4}}c_{2,n-3}+\frac{c_{3,n-4}}{c_{2,n-4}}\right)e_{n-1}\\ &\quad+\ x_4\left(\frac{c_{2,n-4}-c_{4,n-4}}{c_{2,n-2}\cdot c_{2,n-4}}c_{2,n-2}+\frac{c_{4,n-4}}{c_{2,n-4}}\right)e_n\\ &\quad+\ \frac{x_3^2}{x_2}\left(-\frac{(c_{2,n-4}-c_{3,n-4})\cdot c_{3,n-3}}{c_{2,n-2}\cdot c_{2,n-3}\cdot c_{2,n-4}}c_{2,n-2}\right.\\ &\quad+\ \left.\frac{c_{2,n-4}-c_{3,n-4}}{c_{2,n-3}\cdot c_{2,n-4}}c_{3,n-3}\right)e_n\\ &= x_2e_{n-2}+x_3e_{n-1}+x_4e_n\\ &= t_3(x).\end{aligned}$$

When $x_1 = x_2 = 0$ and $x_3 \neq 0$, then

$$\begin{aligned}
[x, \rho_3(x)] &= [x, \frac{1}{c_{3,n-4}} e_{n-4} + \frac{c_{3,n-4} - c_{4,n-4}}{c_{3,n-3} \cdot c_{3,n-4}} \frac{x_4}{x_3} e_{n-3}] \\
&= \frac{1}{c_{3,n-4}} x_3 [e_3, e_{n-4}] + \frac{c_{3,n-4} - c_{4,n-4}}{c_{3,n-3} \cdot c_{3,n-4}} x_4 [e_3, e_{n-3}] \\
&\quad + \frac{1}{c_{3,n-4}} x_4 [e_4, e_{n-4}] \\
&= \frac{c_{3,n-4}}{c_{3,n-4}} x_3 e_{n-1} + x_4 \left(\frac{c_{3,n-4} - c_{4,n-4}}{c_{3,n-4}} + \frac{c_{4,n-4}}{c_{3,n-4}} \right) e_n \\
&= x_3 e_{n-1} + x_4 e_n \\
&= t_3(x).
\end{aligned}$$

For $x_1 = x_2 = x_3 = 0$, we have

$$\frac{1}{c_{4,n-4}} [x, e_{n-4}] = \frac{x_4}{c_{4,n-4}} [e_4, e_{n-4}] = \frac{c_{4,n-4}}{c_{4,n-4}} x_4 e_n = x_4 e_n = t_3(x).$$

This finishes the first part of the proof. In the remainder, we will show that the above maps and the inner derivations indeed generate $\mathrm{AID}(W_n)$.

- Suppose that $5 \leq n \leq 6$, then $W_n = R_n$ is metabelian filiform. The result now follows from Proposition 10.1.6.
- Assume that $7 \leq n \leq 8$ and take a derivation $D = ah + bt_3$ for $a, b \in \mathbb{F}$. Suppose that D is almost inner. We find that $D(e_1) = ae_1$. Since we have that $[e_1, W_n] = \langle e_2, \ldots, e_n \rangle$, this shows that $a = 0$. Further, when $n = 7$, we see that $D(e_3) = be_6$. Since $[e_3, W_7] = \langle e_4, e_5, e_7 \rangle$, we must have that $b = 0$. Similarly, for $n = 8$, we have $D(e_4) = be_8$, but $e_8 \notin [e_4, W_8]$. In both cases, we find that D is almost inner if and only if $a = b = 0$.
- Take arbitrary $n \geq 9$. The derivation h does not satisfy the almost inner condition, since $h(e_1) = e_1 \notin [e_1, W_n]$.

The claim now follows from Proposition 10.2.5 and this finishes the proof. □

10.3 Lie algebras for which all derivations are almost inner

Throughout this thesis, one of the research questions is to see which of the inclusions in the chain $\mathrm{Inn}(\mathfrak{g}) \subseteq \mathrm{CAID}(\mathfrak{g}) \subseteq \mathrm{AID}(\mathfrak{g}) \subseteq \mathrm{Der}(\mathfrak{g})$ are actually equalities. We already considered various possibilities. For many Lie algebras, the only almost inner derivations are the inner ones. However, it is a lot more difficult to have examples of Lie algebras where the opposite happens, namely that all derivations are almost inner. In Chapter 8, we studied examples of solvable and non-nilpotent Lie algebras where this is the case, but this is only valid over some specific fields. In this section, we will give an example of a nilpotent Lie algebra where the same holds. We consider in this section a field $\mathbb{F}$ of characteristic zero.

Let $\mathfrak{g}$ be a nilpotent Lie algebra. Proposition 4.1.8 implies that $\mathrm{AID}(\mathfrak{g})$ is nilpotent and all $D \in \mathrm{AID}(\mathfrak{g})$ are nilpotent. Hence, a necessary condition is to have a Lie algebra for which all derivations are nilpotent. In [10], the authors introduced a family of filiform nilpotent Lie algebras $\mathfrak{f}_n$ for $n \geq 13$. They are closely related to the Witt algebras W_n. The Lie brackets are defined as follows:

$$
\begin{aligned}
&[e_1, e_i] = e_{i+1},\\
&[e_i, e_j] = \sum_{r=1}^{n} \left(\sum_{\ell=0}^{\lfloor \frac{j-i-1}{2} \rfloor} (-1)^{\ell} \binom{j-i-\ell-1}{\ell} \alpha_{i+\ell,\, r-j+i+2\ell+1} \right) e_r,
\end{aligned}
$$

where $2 \leq i \leq n-1$ and $i < j \leq n$. The parameters $\alpha_{k,s}$ for $1 \leq k, s \leq n$ are zero except for

$$
\begin{aligned}
\alpha_{\ell,2\ell+1} &= \frac{3}{\binom{\ell}{2}\binom{2\ell-1}{\ell-1}}, \quad 2 \leq \ell \leq \lfloor \tfrac{n-1}{2} \rfloor,\\
\alpha_{3,n-4} &= 1,\\
\alpha_{4,n-2} &= \frac{1}{7} + \frac{10}{21}\frac{(n-7)(n-8)}{(n-4)(n-5)},\\
\alpha_{4,n} &= \begin{cases} \frac{22105}{15246} & \text{if } n = 13,\\ 0 & \text{if } n \geq 14, \end{cases}
\end{aligned}
$$

and

$$\alpha_{5,n} = \frac{1}{42} - \frac{70(n-8)}{11(n-2)(n-3)(n-4)(n-5)} + \frac{25}{99}\frac{(n-6)(n-7)(n-8)}{(n-2)(n-3)(n-4)} + \frac{5}{66}\frac{(n-5)(n-6)}{(n-2)(n-3)} - \frac{65}{1386}\frac{(n-7)(n-8)}{(n-4)(n-5)}.$$

In [10], it was proven that these Lie brackets satisfy the Jacobi identity, so $\mathfrak{f}_n$ is a Lie algebra for all $n \geq 13$. Since many parameters vanish, the definition of the Lie brackets can be eased. For convenience, we consider the case $n = 13$ separately.

Lemma 10.3.1. *Let $\mathfrak{f}_n$ (with $n \geq 14$) be the Lie algebra over a field $\mathbb{F}$ with basis $\{e_1, \ldots, e_n\}$ as above. Then we have that*

$$[e_1, e_i] = e_{i+1}, \qquad 2 \leq i \leq n-1,$$
$$[e_i, e_j] = c_{i,j} e_{i+j} + d^n_{i,j} e_{i+j+n-11}, \quad 2 \leq i < j \ \text{ and } \ i + j \leq 11,$$

where the coefficients

$$c_{i,j} = \frac{6(j-i)}{j(j-1)\binom{j+i-2}{i-2}}$$

are the same as for the Witt Lie algebras W_n. Further, the values $d^n_{i,j}$ are defined as

$$d^n_{i,j} = \sum_{k=i}^{\lfloor \frac{i+j-1}{2} \rfloor} (-1)^{k-i} \binom{j-k-1}{k-i} \alpha_{k,2k+n-10}$$

when $2 \leq i < j$ and $7 \leq i + j \leq 11$ and zero otherwise.

Proof. Denote c^k_{ij} for the structure constants of $\mathfrak{f}_n$, so $1 \leq i, j, k \leq n$. By rewriting the Lie brackets from above, we find for all $2 \leq i < j \leq n$ that

$$[e_i, e_j] = \sum_{r=1}^{n} \left(\sum_{l=0}^{\lfloor \frac{j-i-1}{2} \rfloor} (-1)^l \binom{j-i-l-1}{l} \alpha_{i+l, r-(j-i-1)+2l} \right) e_r$$
$$= \sum_{r=1}^{n} \left(\sum_{k=i}^{\lfloor \frac{j+i-1}{2} \rfloor} (-1)^{k-i} \binom{j-k-1}{k-i} \alpha_{k, 2k+1+r-(i+j)} \right) e_r.$$

The definition of the parameters $\alpha_{k,s}$ (with $1 \leq k, s \leq n$) imply that $c_{ij}^r = 0$, unless for $r = i + j$ (if $i + j \leq n$) and $r = n + i + j - 11$ (if $7 \leq i + j \leq 11$). We obtain that

$$c_{i,j} = \sum_{k=i}^{\lfloor \frac{j+i-1}{2} \rfloor} (-1)^{k-i} \binom{j-k-1}{k-i} \frac{3}{\binom{k}{2}\binom{2k-1}{k-1}}$$

for all $2 \leq i < j$ with $i + j \leq n$. The result now follows from the Pfaff-Saalschütz sum formula, see [10]. This concludes the proof. $\square$

When $n = 13$, the Lie brackets are the same, but with one exception, since $\alpha_{4,13} \neq 0$. For $2 \leq i \leq 4$ and $i + j = 9$, we find that

$$[e_i, e_j] = c_{i,j} e_9 + d_{i,j}^{13} e_{11} + (-1)^i \alpha_{4,13} e_{13}.$$

Take $n \geq 13$ and consider the maps $t_i : \mathfrak{f}_n \to \mathfrak{f}_n$ (with $1 \leq i \leq 3$), where $x = \sum_{i=1}^n x_i e_i$ is mapped to

$$t_1(x) = x_2 e_n,$$

$$t_2(x) = x_2 e_{n-1} + x_3 e_n,$$

$$t_3(x) = x_2 e_{n-2} + x_3 e_{n-1} + x_4 e_n.$$

These maps generate, together with the inner derivations, the derivation algebra of $\mathfrak{f}_n$.

Proposition 10.3.2. *Consider* $n \geq 13$*, then a basis for* $\mathrm{Der}(\mathfrak{f}_n)$ *is given by* $\mathcal{B} = \{\mathrm{ad}(e_1), \ldots, \mathrm{ad}(e_{n-1}), t_1, t_2, t_3\}$.

Proof. With the Lie brackets from Lemma 10.3.1, it is easy to see that t_1, t_2 and t_3 belong to $\mathrm{Der}(\mathfrak{f}_n)$ for all $n \geq 13$. Moreover, a calculation (by hand or with a computer) shows that all elements of $\mathcal{B}$ are linearly independent.

Let $D \in \mathrm{Der}(\mathfrak{f}_n)$ be an arbitrary derivation. It suffices to show that there exist $x \in \mathfrak{f}_n$ and $\lambda, \mu, \sigma \in \mathbb{F}$ such that $D + \mathrm{ad}(x) = \lambda t_1 + \mu t_2 + \sigma t_3$. We will prove this by induction on the dimension n.

- Basis step: A direct computation shows that the result holds for $n = 13$.
- Induction step: Take $n \geq 14$ and suppose that the statement is true for $n - 1$ (induction hypothesis). Let $D \in \mathrm{Der}(\mathfrak{f}_n)$ be a derivation, then D induces a derivation $\bar{D}$ on $\mathfrak{f}_n / \langle e_n \rangle \cong \mathfrak{f}_{n-1}$. By the induction hypothesis, we can assume that, after changing D up to an inner derivation,

$$\bar{D} = \mu E_{n-1,2} + \sigma(E_{n-2,2} + E_{n-1,3}) + \tau(E_{n-3,2} + E_{n-2,3} + E_{n-1,4}),$$

with $\mu, \sigma, \tau \in \mathbb{F}$. Then we have that $D(e_1) = ae_n$, where $a \in \mathbb{F}$. Replace D by $D' := D + \mathrm{ad}(ae_{n-1})$, then we have $D'(e_1) = D(e_1) + [ae_{n-1}, e_1] = 0$ and

$$D'(e_i) = D(e_i) + [ae_{n-1}, e_i] = D(e_i)$$

for all $2 \leq i \leq n$. In particular, we find that

$$D'(e_2) = D(e_2) = \tau e_{n-3} + \sigma e_{n-2} + \mu e_{n-1} + \lambda e_n,$$

where also $\lambda \in \mathbb{F}$. Since $[e_1, e_i] = e_{i+1}$ for all $2 \leq i \leq n-1$, this implies

$$D'(e_3) = [D'(e_1), e_2] + [e_1, D'(e_2)] = \tau e_{n-2} + \sigma e_{n-1} + \mu e_n,$$

$$D'(e_4) = [D'(e_1), e_3] + [e_1, D'(e_3)] = \tau e_{n-1} + \sigma e_n,$$

$$D'(e_5) = [D'(e_1), e_4] + [e_1, D'(e_4)] = \tau e_n$$

and $D'(e_i) = 0$ for $6 \leq i \leq n$. To finish the proof, we have to show that $\tau = 0$. Assume that $\tau \neq 0$, then the map

$$\tilde{D} : \mathfrak{f}_n \to \mathfrak{f}_n : x = \sum_{i=1}^{n} x_i e_i \mapsto x_2 e_{n-3} + x_3 e_{n-2} + x_4 e_{n-1} + x_5 e_n$$

has to be a derivation. However, we have that $\tilde{D}([e_2, e_3]) = \tilde{D}(e_5) = e_n$, but

$$\begin{aligned}[\tilde{D}(e_2), e_3] + [e_2, \tilde{D}(e_3)] &= [e_{n-3}, e_3] + [e_2, e_{n-2}] \\ &= (-c_{3,n-3} + c_{2,n-2})e_n.\end{aligned}$$

This gives a contradiction, since

$$c_{2,n-2} - c_{3,n-3} = \frac{6(n^2 - 9n + 22)}{(n-2)(n-3)(n-4)} \neq 1$$

for all $n \geq 13$. Hence, we have $\tau = 0$, which means that

$$D' = D + \mathrm{ad}(x) = \lambda t_1 + \mu t_2 + \sigma t_3.$$

This concludes the proof.

- Conclusion: By the principle of mathematical induction, it follows from the basis and induction step that $\mathrm{Der}(\mathfrak{f}_n) = \langle \mathrm{ad}(e_1), \ldots, \mathrm{ad}(e_{n-1}), t_1, t_2, t_3 \rangle$ for all $n \geq 13$. $\square$

We will show that for the Lie algebras $\mathfrak{f}_n$ (with $n \geq 13$), all derivations are almost inner.

Proposition 10.3.3. *For all* $n \geq 13$, *we have* $\mathrm{Inn}(\mathfrak{f}_n) \neq \mathrm{AID}(\mathfrak{f}_n) = \mathrm{Der}(\mathfrak{f}_n)$.

Proof. It suffices to show that t_1, t_2 and t_3 are almost inner derivations for all $n \geq 13$. In Proposition 10.2.6, we showed that if $n \geq 9$, then t_1, t_2 and t_3 are almost inner for the Witt algebra $\mathcal{W}_n$ by giving the determination maps φ_{t_i}, with $1 \leq i \leq 3$. In the proof, we only used the Lie brackets

$$[e_1, e_k] = e_{k+1},$$

$$[e_i, e_j] = c_{i,j} e_{i+j}$$

for $2 \leq k \leq n-1$ and $2 \leq i \leq 4$ and $n-4 \leq j \leq n-i$. Hence, when $n \geq 14$, we can use the same determination maps $\varphi_{t_1}, \varphi_{t_2}$ and φ_{t_3} as in Proposition 10.2.6. When $n = 13$, the maps φ_{t_1} and φ_{t_2} are still valid. However, to show that t_3 is almost inner, we need another map since $c_{2,9}^{13} \neq 0$. Consider $\varphi'_{t_3} : \mathfrak{f}_{13} \to \mathfrak{f}_{13}$, where $x = \sum_{i=1}^{13} x_i e_i$ is mapped to

$$\begin{cases} \frac{12}{7}\left(e_9 + \frac{x_3}{x_2} e_{10}\right) + \left(\frac{22105}{2079} + \frac{380}{189}\frac{x_4}{x_2} - \frac{4}{27}\frac{x_3^2}{x_2^2}\right) e_{11} & \text{if } x_1 = 0 \text{ and } x_2 \neq 0, \\ \varphi_{t_3}(x) & \text{otherwise.} \end{cases}$$

Suppose that $x_1 = 0 \neq x_2$, then we have that

$$\begin{aligned}
[x, \varphi'_{t_3}(x)] = {} & \frac{12}{7} x_2 \left(\frac{7}{12} e_{11} - \frac{4421}{1452} e_{13}\right) + \frac{12}{7} \cdot \frac{8}{15} x_3 e_{12} \\
& + \frac{27}{55}\left(\frac{22105}{2079} x_2 + \frac{380}{189} x_4 - \frac{4}{27}\frac{x_3^2}{x_2}\right) e_{13} + x_3 \frac{12}{7} \cdot \frac{1}{20} e_{12} \\
& + \frac{12}{7} \cdot \frac{7}{165}\frac{x_3^2}{x_2} e_{13} + x_4 \frac{12}{7} \cdot \frac{1}{132} e_{13} \\
= {} & x_2 e_{11} + x_3 \left(\frac{12}{7} \cdot \frac{8}{15} + \frac{12}{7} \cdot \frac{1}{20}\right) e_{12} \\
& + \left(\frac{-12}{7} \cdot \frac{4421}{1452} + \frac{27}{55} \cdot \frac{22105}{2079}\right) x_2 e_{13} \\
& + \left(\frac{-27}{55} \cdot \frac{4}{27} + \frac{12}{7} \cdot \frac{7}{165}\right) \frac{x_3^2}{x_2} e_{13} + \left(\frac{27}{55} \cdot \frac{380}{189} + \frac{12}{7 \cdot 132}\right) x_4 e_{13} \\
= {} & x_2 e_{11} + x_3 e_{12} + x_4 e_{13}.
\end{aligned}$$

□

Chapter 11

Free nilpotent Lie algebras

In the last two chapters, we studied two different classes of nilpotent Lie algebras. First, we computed the almost inner derivations for the 'most' (non-abelian) nilpotent Lie algebras. Chapter 10 was about filiform Lie algebras, the so-called 'least' nilpotent Lie algebras. In this chapter, we will consider the free nilpotent Lie algebras, a type of nilpotent Lie algebras where all nilindices can occur. First, the notion of a free Lie algebra is explained and a suitable basis is worked out. Further, we compute the almost inner derivations for different free nilpotent Lie algebras. The results of Section 11.3 and Section 11.4 already appeared in [7], whereas the last section comes from [8]. In each section, we will specify over which field $\mathbb{F}$ we work.

11.1 Background

In this section, we will explain the concept of free (nilpotent) Lie algebras which are finitely generated. Therefore, we will make use of the Hall basis. We consider a general field $\mathbb{F}$.

Definition 11.1.1 (Free Lie algebra)**.** Let X be a finite set with r elements. Let $\mathfrak{f}$ be a Lie algebra and $i : X \to \mathfrak{f}$ a map of sets. The Lie algebra $\mathfrak{f}$ is free on X if for every Lie algebra $\mathfrak{g}$ with a map of sets $f : X \to \mathfrak{g}$, there is a unique Lie algebra morphism $\varphi : \mathfrak{f} \to \mathfrak{g}$ with $f = \varphi \circ i$.

For every finite set X, there is a unique free Lie algebra generated by X. This Lie algebra has $r := |X|$ generators and is denoted with $\mathfrak{f}_r$. A free Lie algebra on r generators is also called to be of rank r. More intuitively, one can think of $\mathfrak{f}_r$ as the Lie algebra generated by r elements $x_1, \ldots, x_r$ such that the only relations are the ones due to the skew-symmetry and the Jacobi-identity. When $r = 1$, we have the 1–dimensional (abelian) Lie algebra, so we assume from now on that $r > 1$.

A basis for $\mathfrak{f}_r$ as a vector space consists of more than r elements. Indeed, we have for instance that $[x_1, x_2] \in \mathfrak{f}_r$ is an element of the Lie algebra, but it is not spanned by $x_1, \ldots, x_r$. We will now introduce the Hall set, an ordered set which can be used as an explicit basis for the free Lie algebra.

Definition 11.1.2 (Hall set)**.** Let $\mathfrak{f}_r$ be a free Lie algebra on r generators. A totally ordered set $H := \bigcup_{n \in \mathbb{N}_0} H_n$ is called a Hall set for $\mathfrak{f}_r$ when it satisfies the following conditions.

- We have $H_1 := \{x_1, x_2, \ldots, x_r\}$, where the order is $x_1 < x_2 < \cdots < x_r$.
- Take $n \geq 2$ and suppose that H_k is defined for all $1 \leq k < n$ and that there is a total order imposed on $\bigcup_{k=1}^{n-1} H_k$.
- We define H_n as the set of all elements $[x, y]$, where $x \in H_k$ and $y \in H_l$ such that
  - $k + l = n$,
  - $x < y$ and
  - when there exist $y_1 \in H_{l_1}$ and $y_2 \in H_{l_2}$ (for $l_1, l_2 \in \mathbb{N}_0$) such that $y = [y_1, y_2]$, then $y_1 \leq x$.

  Choose a total order on H_n.
- We have a total order on $\bigcup_{k=1}^{n} H_k$ by requiring for all $i < j$ that $x < y$ when $x \in H_i$ and $y \in H_j$.

Marshall Hall introduced these Hall sets ([44]), based on work of Philip Hall on groups ([45]). Note that this Hall set depends on the choice of ordering we impose on the different sets H_k, with $k \geq 1$.

Example 11.1.3. The elements of H_2 are of the form

$$y_{i,j} := [x_i, x_j], \quad \text{where } 1 \leq i < j \leq r.$$

The elements of H_3 are of the form

$$z_{i,j,k} := [x_i, [x_j, x_k]], \quad \text{with } 1 \leq j < k \leq r \text{ and } 1 \leq j \leq i \leq r.$$

The elements of H_4 depend on the ordering on H_2.

When $i < j < k$, then

$$\begin{aligned}[x_i, y_{j,k}] &= [x_i, [x_j, x_k]] \\ &= [x_j, [x_i, x_k]] - [x_k, [x_i, x_j]] \\ &= z_{j,i,k} - z_{k,i,j}\end{aligned}$$

holds by the Jacobi identity. Hence, there is no need to add $[x_i, [x_j, x_k]]$ to H_3, since it can be expressed as a linear combination of other basis elements (namely $z_{j,i,k}$ and $z_{k,i,j}$).

It turns out that the Hall set can be used as a basis for a free Lie algebra.

Theorem 11.1.4. *Let $\mathfrak{f}_r$ be a free Lie algebra on r generators. A Hall set for $\mathfrak{f}_r$ defines a basis for $\mathfrak{f}_r$.*

Proof. A proof of this fact is given in for instance [18, Chapter 7]. □

Thus, the Hall set H for $\mathfrak{f}_r$ is a totally ordered basis for $\mathfrak{f}_r$, constructed inductively as a union $H := \bigcup_{n \in \mathbb{N}_0} H_n$, where H_n consists of n–fold Lie brackets. This basis has infinitely many elements. We define a free nilpotent Lie algebra as a quotient of a free Lie algebra.

Definition 11.1.5 (Free c–step nilpotent Lie algebra)**.** Let X be a finite set with r elements. Let $\mathfrak{f}_c$ be a Lie algebra and $i : X \to \mathfrak{f}_c$ a map of sets. The Lie algebra $\mathfrak{f}_c$ is free c–step nilpotent on X if for every c–step nilpotent Lie algebra $\mathfrak{g}$ with a map of sets $f : X \to \mathfrak{g}$, there is a unique Lie algebra morphism $\varphi : \mathfrak{f}_c \to \mathfrak{g}$ with $f = \varphi \circ i$.

For every finite set X and every $c \geq 2$, there is a unique free c–step nilpotent Lie algebra generated by X. This Lie algebra has $r := |X|$ generators and is denoted with $\mathfrak{f}_{r,c}$. Note that $\mathfrak{f}_{r,c}$ can be obtained as the quotient

$$\mathfrak{f}_{r,c} := \frac{\mathfrak{f}_r}{\gamma_{c+1}(\mathfrak{f}_r)}.$$

In this construction, we consider the free Lie algebra $\mathfrak{f}_r$, where all $(c+1)$–fold Lie brackets vanish. Hence, the result is indeed nilpotent and has nilpotency class c. Denote $H = \bigcup_{n \in \mathbb{N}_0} H_n$ for the Hall basis of $\mathfrak{f}_r$, then the natural projections of the elements of $H_1 \cup \cdots \cup H_c$ form a basis of $\mathfrak{f}_{r,c}$, which is also called a Hall basis.

The dimension of a free nilpotent Lie algebra $\mathfrak{f}_{r,c}$ on r generators and nilindex c can be computed explicitly due to a theorem of Witt. Therefore, some terminology has to be introduced first.

Definition 11.1.6 (Möbiusfunction)**.** The Möbiusfunction $\mu : \mathbb{N}_0 \to \{-1, 0, 1\}$ maps $d \in \mathbb{N}_0$ to

$$\mu(d) = \begin{cases} 1 & \text{if } d = 1, \\ (-1)^n & \text{if } d \text{ is the product of } n \text{ distinct primes}, \\ 0 & \text{if } d \text{ is not square-free.} \end{cases}$$

This function appears in Witt's theorem. The formula determines the number of basis vectors of a given length.

Theorem 11.1.7 ([85])**.** *Let $\mathfrak{f}_{r,c}$ be the free nilpotent Lie algebra on r generators and nilindex c. Let $H = \bigcup_{k=1}^{c} H_k$ be a Hall basis of $\mathfrak{f}_{r,c}$. For all $1 \leq k \leq c$, the dimension of $\gamma_k(\mathfrak{f}_{r,c})/\gamma_{k+1}(\mathfrak{f}_{r,c})$ is given by*

$$\#H_k = \frac{1}{k} \sum_{d|k} \mu(d)\, r^{k/d}.$$

From Witt's theorem, the dimension of $\mathfrak{f}_{r,c}$ immediately follows.

Corollary 11.1.8. *Let $\mathfrak{f}_{r,c}$ be the free nilpotent Lie algebra on r generators and nilindex c over a field $\mathbb{F}$. The dimension of $\mathfrak{f}_{r,c}$ is given by*

$$\dim(\mathfrak{f}_{r,c}) = \sum_{k=1}^{c} \#H_k.$$

A Hall basis makes it possible to describe the basis vectors and the corresponding Lie brackets. In the following sections, we will use this strategy for $c \leq 3$ to compute the almost inner derivations of $\mathfrak{f}_{r,c}$ over an arbitrary field $\mathbb{F}$. In Section 11.5, we will determine $\dim(\mathrm{AID}(\mathfrak{f}_{r,c}))$ for all $c \geq 2$ using another approach which is only valid for Lie algebras over a field of characteristic zero. In the next section, we also consider a duality theory for 2–step nilpotent Lie algebras.

11.2 Free 2–step nilpotent Lie algebras

Let $\mathbb{F}$ be an arbitrary field. Consider $\mathfrak{f}_{r,2}$, the free 2–step nilpotent Lie algebra over $\mathbb{F}$ with r generators $x_1, x_2, \ldots, x_r$. The Hall basis is given by $\{x_1, \ldots, x_r, y_{i,j} \mid 1 \leq i < j \leq r\}$ and the non-vanishing Lie brackets are

$$[x_i, x_j] = y_{i,j},$$

where $1 \leq i < j \leq r$. This means that $\mathfrak{f}_{r,2}$ is a 2–step nilpotent Lie algebra determined by the complete graph on r vertices. Hence, it follows from Theorem 9.1.1 that all almost inner derivations are inner.

Proposition 11.2.1. *Let $\mathfrak{f}_{r,2}$ be the free 2–step nilpotent Lie algebra on r generators over an arbitrary field $\mathbb{F}$, then*

$$\mathrm{AID}(\mathfrak{f}_{r,2}) = \mathrm{Inn}(\mathfrak{f}_{r,2}).$$

For the rest of this section, we study quotients of $\mathfrak{f}_{r,2}$. Therefore, we need the notion of a dual Lie algebra.

11.2.1 Dual Lie algebras

Scheuneman ([75]) introduced a duality theory for 2–step nilpotent Lie algebras. Later, Gauger ([33]) developed his own theory, but showed that they are in fact the same ([34]). We will use the approach from [33] and consider a field $\mathbb{F}$ of characteristic not two.

Let $x_1, \ldots, x_r$ be generators of $\mathfrak{f}_{r,2}$ as before and denote $V := \langle x_1, \ldots, x_r \rangle$. We can consider the vector space $V \oplus \bigwedge^2 V$ as a 2–step nilpotent Lie algebra by linearly extending the rules

$$[x_i, x_j] = x_i \wedge x_j,$$

$$[x_m, x_i \wedge x_j] = 0,$$

$$[x_i \wedge x_j, x_m] = 0,$$

$$[x_i \wedge x_j, x_k \wedge x_l] = 0$$

for all $1 \leq i, j, k, l, m \leq r$ with $i < j$ and $k < l$. A direct verification shows that $\mathfrak{f}_{r,2} \to V \oplus \bigwedge^2 V$ is an isomorphism when x_i is mapped to x_i for all $1 \leq i \leq r$. This means that $\mathfrak{f}_{r,2}$ can be identified with $V \oplus \bigwedge^2 V$.

Let $\mathfrak{g}$ be a 2–step nilpotent Lie algebra of type (r, m). Take a basis $\{x_1, \ldots, x_r, y_1, \ldots, y_m\}$ for $\mathfrak{g}$, where $\{y_1, \ldots, y_m\}$ is a basis for $[\mathfrak{g}, \mathfrak{g}]$. Note that $\mathfrak{g}$ can be considered as a quotient

$$\mathfrak{g} = \frac{V \oplus \bigwedge^2 V}{I},$$

where I is a subspace of $\bigwedge^2 V$. The space I describes the relations among the extra generators $x_1, \ldots, x_r$ for $\mathfrak{f}_{r,2}$. The dimension of I is therefore called the 'number of relations' defining $\mathfrak{g}$. Consider the pairing $(.,.) : \bigwedge^2 V \times \bigwedge^2(V^*) \to \mathbb{F}$, where

$$(x_i \wedge x_j, x_k^* \wedge x_l^*) = \det \begin{pmatrix} x_k^*(x_i) & x_l^*(x_i) \\ x_k^*(x_j) & x_l^*(x_j) \end{pmatrix} \tag{11.1}$$

for all $1 \leq i, j, k, l \leq r$. We need this pairing to define the dual Lie algebra.

Definition 11.2.2 (Dual Lie algebra)**.** Let $\mathfrak{g} = \mathfrak{f}_{r,2}/I = (V \oplus \bigwedge^2 V)/I$ be a 2–step nilpotent Lie algebra, where I is a subspace of $\bigwedge^2 V$. The 2–step nilpotent Lie algebra

$$\mathfrak{g}^* := \frac{V^* \oplus \bigwedge^2(V^*)}{I^\perp}$$

is called the dual Lie algebra of $\mathfrak{g}$, where

$$I^\perp = \{\alpha \wedge \beta \in \bigwedge^2(V^*) \mid (a \wedge b, \alpha \wedge \beta) = 0 \text{ for all } a \wedge b \in I\}$$

is the orthogonal complement of I with respect to (11.1).

With this definition, a Lie algebra and its dual satisfy several interesting properties.

Theorem 11.2.3. *Let $\mathfrak{g}$ be a 2–step nilpotent Lie algebra of type (n, m). Denote $\mathfrak{g}^*$ for the dual Lie algebra, then the following statements hold.*

- *The dual Lie algebra $\mathfrak{g}^*$ is 2–step nilpotent of type $(n, \binom{n}{2} - m)$.*
- *The Lie algebras $(\mathfrak{g}^*)^*$ and $\mathfrak{g}$ are isomorphic.*
- *Let $\mathfrak{h}$ be a 2–step nilpotent Lie algebra with dual $\mathfrak{h}^*$, then $\mathfrak{g} \cong \mathfrak{h}$ if and only if $\mathfrak{g}^* \cong \mathfrak{h}^*$.*

This means that the dual of a Lie algebra of genus k has k relations and vice versa.

Example 11.2.4. Let $\mathbb{F}$ be a field of characteristic not two. Consider the canonical Lie algebra $\mathfrak{g}$ over $\mathbb{F}$ with minimal index $\varepsilon = 2$. Then $\mathfrak{g}$ has a basis $\{x_1, \ldots, x_5, y_{1,4}, y_{2,4}\}$ and non-vanishing Lie brackets

$$\begin{aligned} [x_1, x_4] &= y_{1,4}, & [x_2, x_4] &= y_{2,4}, \\ [x_2, x_5] &= y_{1,4}, & [x_3, x_5] &= y_{2,4}. \end{aligned}$$

This means that $\mathfrak{g}$ can be considered as a quotient $\mathfrak{f}_{5,2}/I$, where

$$I = \langle y_{1,2},\ y_{1,3},\ y_{1,4} - y_{2,5},\ y_{1,5},\ y_{2,3},\ y_{2,4} - y_{3,5},\ y_{3,4},\ y_{4,5}\rangle.$$

The orthogonal complement of I is given by $I^\perp = \langle y^*_{1,4} + y^*_{2,5},\ y^*_{2,4} + y^*_{3,5}\rangle$. The dual Lie algebra $\mathfrak{g}^*$ has basis $\{x^*_1, \ldots, x^*_5, y^*_{1,2}, y^*_{1,3}, y^*_{1,4}, y^*_{1,5}, y^*_{2,3}, y^*_{2,4}, y^*_{3,4}, y^*_{4,5}\}$ and is given by

$$\begin{array}{llll} [x^*_1, x^*_2] = y^*_{1,2}, & [x^*_1, x^*_3] = y^*_{1,3}, & [x^*_1, x^*_4] = y^*_{1,4}, & [x^*_1, x^*_5] = y^*_{1,5}, \\ [x^*_2, x^*_3] = y^*_{2,3}, & [x^*_2, x^*_4] = y^*_{2,4}, & [x^*_2, x^*_5] = -y^*_{1,4}, & \\ [x^*_3, x^*_4] = y^*_{3,4}, & [x^*_3, x^*_5] = -y^*_{2,4}, & [x^*_4, x^*_5] = y^*_{4,5}. & \end{array}$$

We found that $\mathfrak{g}$ has type $(5, 2)$ and $\mathfrak{g}^*$ has type $(5, \binom{5}{2} - 2) = (5, 8)$.

Let $\mathfrak{g}$ be a Lie algebra with dual $\mathfrak{g}^*$. It is a natural question to ask what the correspondence is between $\mathrm{AID}(\mathfrak{g})$ and $\mathrm{AID}(\mathfrak{g}^*)$. This is the topic of the next subsections.

11.2.2 Lie algebras with one relation

Let $\mathfrak{g}$ be a 2–step nilpotent Lie algebra over a field $\mathbb{F}$ of characteristic not two. Suppose that $\mathfrak{g}$ has genus 1, then the only almost inner derivations are the inner ones, as we showed in Section 9.2. We will prove that $\mathrm{AID}(\mathfrak{g}^*) = \mathrm{Inn}(\mathfrak{g}^*)$ holds for the dual Lie algebra $\mathfrak{g}^*$ as well. Since $\mathfrak{g}^*$ has one relation, it can be considered as a Lie algebra $\frac{\mathfrak{f}_{r,2}}{\langle x\rangle}$, where $0 \neq x \in [\mathfrak{f}_{r,2}, \mathfrak{f}_{r,2}]$. We use the notions and results from [20].

Definition 11.2.5 (Length of a Lie bracket)**.** Let $\mathfrak{f}_{r,2}$ be the free 2–step nilpotent Lie algebra on r generators over a field $\mathbb{F}$. Take $0 \neq x \in [\mathfrak{f}_{r,2}, \mathfrak{f}_{r,2}]$. Then the length of x is defined as

$$\ell(x) = \min\{s \in \mathbb{N}_0 \mid x = [x_1, x_2] + \cdots + [x_{2s-1}, x_{2s}] \text{ for some } x_1, \ldots, x_{2s} \in \mathfrak{f}_{r,2}\}.$$

Proposition 11.2.6. *Let $\mathfrak{f}_{r,2}$ be the free 2–step nilpotent Lie algebra on r generators. Then for any $x \in [\mathfrak{f}_{r,2}, \mathfrak{f}_{r,2}]$ with $x \neq 0$, we have*

$$\ell(x) \leq \left\lfloor \frac{r}{2} \right\rfloor.$$

It turns out that the length can be used to classify the 2–step nilpotent Lie algebras with one relation.

Theorem 11.2.7. *Let $\mathfrak{f}_{r,2}$ be the free 2–step nilpotent Lie algebra on r generators over a field $\mathbb{F}$. Let $0 \neq x, y \in [\mathfrak{f}_{r,2}, \mathfrak{f}_{r,2}]$. Then*

$$\frac{\mathfrak{f}_{r,2}}{\langle x\rangle} \cong \frac{\mathfrak{f}_{r,2}}{\langle y\rangle} \qquad \text{if and only if} \qquad \ell(x) = \ell(y).$$

Let $\mathfrak{f}_{r,2}$ be the free 2–step nilpotent Lie algebra on r generators. Consider $\mathfrak{g} := \mathfrak{f}_{r,2}/\langle x\rangle$, with $0 \neq x \in [\mathfrak{f}_{r,2}, \mathfrak{f}_{r,2}]$. When $\ell(x) = 1$, then $\mathfrak{g}$ is isomorphic to a Lie algebra determined by a complete graph from which one edge is removed. It follows from Theorem 9.1.1 that $\mathrm{AID}(\mathfrak{g}) = \mathrm{Inn}(\mathfrak{g})$. For $\ell(x) \geq 2$, we define $s := \ell(x)$ and consider $x_i \in \mathfrak{f}_{r,2}$ for all $1 \leq i \leq 2s$ such that

$$x = [x_1, x_2] + \cdots + [x_{2s-1}, x_{2s}] \in \mathfrak{f}_{r,2}.$$

It follows that $\mathfrak{f}_{r,2}/\langle x\rangle$ is isomorphic to the Lie algebra $\mathfrak{g}_{r,s}$ with basis

$$\{x_1, \ldots, x_r, y_{i,j} \mid 1 \leq i < j \leq r \text{ and } (i,j) \neq (1,2)\},$$

where the non-zero Lie brackets are given by

$$\begin{aligned}
[x_1, x_2] &= -\sum_{i=2}^{s} y_{2i-1,2i},\\
[x_i, x_j] &= y_{i,j} \quad \text{for } \ 1 \leq i < j \leq r \ \text{ and } \ (i,j) \neq (1,2).
\end{aligned}$$

By the previous theorem, it suffices to compute the almost inner derivations of $\mathfrak{g}_{r,s}$ for all $r \geq 2$ and all $s \leq \lfloor r/2 \rfloor$. In what follows, we will use a variation of the notation from Section 4.2. Let $1 \leq i_1, j_1, i_2, j_2, \ldots, i_t, j_t \leq r$, then $\mathfrak{g}_{i_1,i_2,\ldots,i_t} = \langle x_i \mid i \notin \{i_1, i_2, \ldots, i_t\}\rangle$ and

$$\mathfrak{g}_{i_1j_1,i_2j_2,\ldots,i_tj_t} = \langle y_{i,j} \mid i \notin \{i_1, i_2, \ldots, i_t\} \text{ and } j \notin \{j_1, j_2, \ldots, j_t\}\rangle.$$

Proposition 11.2.8. *Let $\mathfrak{f}_{r,2}$ be the free 2–step nilpotent Lie algebra on r generators over a field $\mathbb{F}$ of characteristic not two. Let $0 \neq x \in [\mathfrak{f}_{r,2}, \mathfrak{f}_{r,2}]$. Then*

$$\mathrm{AID}\left(\frac{\mathfrak{f}_{r,2}}{\langle x\rangle}\right) = \mathrm{Inn}\left(\frac{\mathfrak{f}_{r,2}}{\langle x\rangle}\right).$$

Proof. Denote $s := \ell(x)$. Without loss of generality, we can assume that $r \geq 4$ and $s \geq 2$. Consider the Lie algebra $\mathfrak{g}_{r,s}$ from above. Let $D \in \mathrm{AID}(\mathfrak{g}_{r,s})$ be determined by a map φ_D. It is clear that all basis vectors in the center are fixed. We show that x_1 is fixed as well. Take arbitrary $1 \leq j < k \leq r$, then $x_j, x_k \notin C_{\mathfrak{g}_{r,s}}(x_1)$. There are two possibilities.

- If $j \neq 2$, we are in the situation that

$$\begin{aligned}
&[x_j, x_1] + y_{1,j} \in \mathfrak{g}_{1j,1k}\\
&[x_k, x_1] + y_{1,k} \in \mathfrak{g}_{1j,1k}\\
&[x_j, \mathfrak{g}_1] \subseteq \mathfrak{g}_{1j,1k}\\
&[x_k, \mathfrak{g}_1] \subseteq \mathfrak{g}_{1j,1k}.
\end{aligned}$$

  Lemma 4.2.8 implies that $t_1(\varphi_D(x_j)) = t_1(\varphi_D(x_k))$.

- Suppose that $j = 2$, then

$$\begin{aligned}[x_2, x_1] - y_{3,4} &\in \mathfrak{g}_{1k,34}\\ [x_k, x_1] + y_{1,k} &\in \mathfrak{g}_{1k,34}\\ [x_2, \mathfrak{g}_1] &\subseteq \mathfrak{g}_{1k,34}.\end{aligned}$$

  When $[x_k, \mathfrak{g}_1] \subseteq \mathfrak{g}_{1k,34}$ holds, we can apply Lemma 4.2.8 to obtain that $t_1(\varphi_D(x_2)) = t_1(\varphi_D(x_k))$. Otherwise, we have that $k \in \{3, 4\}$. We assume that $k = 3$; the other case is similar.

  Denote $a_i := t_i(\varphi_D(x_2))$ and $b_i := t_i(\varphi_D(x_3))$ for all $1 \leq i \leq r$. There exist values $v, v' \in \mathfrak{g}_{13,24,34}$ such that

$$D(x_2) = a_4 y_{2,4} + a_1 y_{3,4} + v, \tag{11.2}$$

$$D(x_3) = -b_1 y_{1,3} + b_4 y_{3,4} + v'. \tag{11.3}$$

  We further define $c_i := t_i(\varphi_D(x_2 + x_3))$ and $d_i := t_i(\varphi_D(x_2 - x_3))$ for all $1 \leq i \leq r$. It follows that

$$D(x_2 + x_3) = -c_1 y_{1,3} + c_4 y_{2,4} + (c_1 + c_4) y_{3,4} + w, \tag{11.4}$$

$$D(x_2 - x_3) = d_1 y_{1,3} + d_4 y_{2,4} + (d_1 - d_4) y_{3,4} + w', \tag{11.5}$$

  where $w, w' \in \mathfrak{g}_{13,24,34}$. By summing equations (11.2) and (11.3) and comparing the coefficients with these from (11.4), we obtain

$$(a_1 - b_1) + (b_4 - a_4) = 0.$$

  On the other hand, we can take the difference of (11.2) and (11.3). We compare the coordinates with (11.5) and find that

$$(a_1 - b_1) + (a_4 - b_4) = 0.$$

  These two last equations imply that $a_1 = b_1$ since $\mathrm{char}(\mathbb{F}) \neq 2$. Hence, we have that $t_1(\varphi_D(x_j)) = t_1(\varphi_D(x_k))$.

In both cases, we showed that $t_1(\varphi_D(x_j)) = t_1(\varphi_D(x_k))$. Since $1 \leq j, k \leq r$ were arbitrary, this implies that x_1 is fixed. To show that x_i is fixed for $2 \leq i \leq r$, a similar approach can be used. As before, we have to make a case distinction for $i + 1 \in \{j, k\}$ (if i is odd) or $i - 1 \in \{j, k\}$ (when i is even). □

11.2.3 Lie algebras with more relations

Motivated by the result from the previous subsection, one can think that there is correspondence between the almost inner derivations for a Lie algebra $\mathfrak{g}$ and the dual $\mathfrak{g}^*$. However, this is not the case.

Example 11.2.9. Consider the Lie algebras $\mathfrak{g}$ and $\mathfrak{g}^*$ from Example 11.2.4. Since $\mathfrak{g}$ is a canonical Lie algebra with minimal index $\varepsilon = 2$, it follows from Lemma 9.2.17 that $\dim(\mathrm{AID}(\mathfrak{g})) = \dim(\mathrm{Inn}(\mathfrak{g})) + 1 = 6$.

We will show that for the dual Lie algebra $\mathfrak{g}^*$, the only almost inner derivations are the inner ones. Let $D \in \mathrm{AID}(\mathfrak{g}^*)$ be an almost inner derivation determined by φ_D. We will show that all basis vectors are fixed. This is clear for all basis vectors in the center. We will illustrate that x_1^* is fixed. Take $j \in \{2, 4, 5\}$. Since we have

$$\begin{aligned}
&[x_j^*, x_1^*] + y_{1,j}^* \in \mathfrak{g}_{1j,13}^* \\
&[x_3^*, x_1^*] + y_{1,3}^* \in \mathfrak{g}_{1j,13}^* \\
&[x_j^*, \mathfrak{g}_1^*] \subseteq \mathfrak{g}_{1j,13}^* \\
&[x_3^*, \mathfrak{g}_1^*] \subseteq \mathfrak{g}_{1j,13}^*,
\end{aligned}$$

we can apply Lemma 4.2.7 to show that $t_1(\varphi_D(x_j^*)) = t_1(\varphi_D(x_3^*))$. This implies that x_1^* is fixed. With a similar approach, we can show that x_i^* is fixed as well, where $2 \leq i \leq 5$. Corollary 4.2.6 implies that $\mathrm{AID}(\mathfrak{g}^*) = \mathrm{Inn}(\mathfrak{g}^*)$.

Let $\mathbb{F}$ be $\mathbb{R}$ or an algebraically closed field of characteristic not two. Consider a 2–step nilpotent Lie algebra $\mathfrak{g}$ of genus 2 over $\mathbb{F}$. Denote $\mathfrak{g}^*$ for the dual Lie algebra, so $\mathfrak{g}^*$ has two relations. We conjecture that $\mathrm{AID}(\mathfrak{g}^*) = \mathrm{Inn}(\mathfrak{g}^*)$ will hold. It suffices to compute the almost inner derivations for the dual of the canonical Lie algebras from Section 9.2.

Theorem 11.2.10. *Let $\mathfrak{g}_1$ and $\mathfrak{g}_2$ be 2–step nilpotent 2–relation Lie algebras over $\mathbb{R}$ or an algebraically closed field $\mathbb{F}$ of characteristic not two. Then $\mathfrak{g}_1 \cong \mathfrak{g}_2$ if and only if $\mathfrak{g}_1^*$ and $\mathfrak{g}_2^*$ have the same minimal indices and elementary divisors.*

However, it is not straightforward to give a rigorous proof for this conjecture, since there is no uniform description of the Lie brackets which considers all possible cases at the same time.

Let $\mathfrak{g}$ be a 2–step nilpotent Lie algebra with 3 relations, then $\mathfrak{g}$ can have non-inner almost inner derivations.

Example 11.2.11. Consider the Lie algebra $\mathfrak{g}$ over an arbitrary field $\mathbb{F}$. Suppose that $\mathfrak{g}$ has basis $\mathcal{B} := \{x_1, \ldots, x_4, y_1, y_2, y_3\}$ and non-vanishing Lie brackets

$$\begin{aligned}
[x_1, x_2] &= y_1, \quad & [x_1, x_3] &= y_2, \\
[x_1, x_4] &= y_3, \quad & [x_3, x_4] &= y_1.
\end{aligned}$$

This is a 2–step nilpotent Lie algebra of type $(4, 3)$, so it has $\binom{4}{2} - 3 = 3$ relations. It is clear that $\dim(\mathrm{Inn}(\mathfrak{g})) = 4$. Take an arbitrary $D \in \mathcal{C}(\mathfrak{g})$, then the matrix of D is given by

$$D = \left(\begin{array}{cccc|c} & 0 & & & 0 \\ \hline r_1 & r_2 & r_3 & r_4 & \\ s_1 & s_2 & s_3 & s_4 & 0 \\ t_1 & t_2 & t_3 & t_4 & \end{array}\right).$$

Suppose that D is an almost inner derivation. Since D has to be $\mathcal{B}$–almost inner, we find that $s_2 = t_2 = t_3 = s_4 = 0$. Take $x = a_1x_1 + \cdots + a_4x_4$, then we have that

$$D(x_3 + x_4) = (r_3 + r_4)y_1 + s_3y_2 + t_4y_3.$$

Since $[x_3 + x_4, x] = (a_4 - a_3)y_1 - a_1(y_2 + y_3)$ holds, this implies that $s_3 = t_4$, so $\dim(\mathrm{AID}(\mathfrak{g})) \leq 7$. Take an arbitrary $x = a_1x_1 + \cdots + a_4x_4 + b_1y_1 + b_2y_2 + b_3y_3 \in \mathfrak{g}$. Consider the linear maps $D_1 : \mathfrak{g} \to \mathfrak{g} : x \mapsto D_1(x) = a_1y_2$ and

$$\varphi_{D_1} : \mathfrak{g} \to \mathfrak{g} : x \mapsto \begin{cases} \frac{a_4}{a_1}x_2 + x_3 & \text{if } a_1 \neq 0, \\ 0 & \text{if } a_1 = 0. \end{cases}$$

For $a_1 \neq 0$, we have

$$[x, \varphi_{D_1}(x)] = a_4y_1 + a_1y_2 - a_4y_1 = a_1y_2 = D_1(x).$$

When $a_1 = 0$, then $[x, 0] = 0 = D_1(x)$. This shows that $D_1 \in \mathrm{AID}(\mathfrak{g})$. An analogous computation shows that $D_2 : \mathfrak{g} \to \mathfrak{g} : x \mapsto D_2(x) = a_1y_3$ is an almost inner derivation, determined by

$$\varphi_{D_2} : \mathfrak{g} \to \mathfrak{g} : x \mapsto \begin{cases} \frac{-a_3}{a_1}x_2 + x_4 & \text{if } a_1 \neq 0, \\ 0 & \text{if } a_1 = 0. \end{cases}$$

Further, consider the derivation $D_3 : \mathfrak{g} \to \mathfrak{g} : x \mapsto D_3(x) = a_2y_1$. This is almost inner, determined by

$$\varphi_{D_3} : \mathfrak{g} \to \mathfrak{g} : x \mapsto \begin{cases} \frac{a_2}{a_1}x_2 & \text{if } a_1 \neq 0, \\ \frac{a_2}{a_3}x_4 & \text{if } a_1 = 0 \text{ and } a_3 \neq 0, \\ \frac{-a_2}{a_4}x_3 & \text{if } a_1 = a_3 = 0 \text{ and } a_4 \neq 0, \\ -x_1 & \text{if } a_1 = a_3 = a_4 = 0. \end{cases}$$

Since D_1, D_2 and D_3 are linearly independent and $\langle D_1, D_2, D_3 \rangle \cap \mathrm{Inn}(\mathfrak{g}) = \{0\}$, this implies that $\mathrm{AID}(\mathfrak{g}) = \langle \mathrm{ad}(x_1), \ldots, \mathrm{ad}(x_4), D_1, D_2, D_3 \rangle$.

A 2–step nilpotent Lie algebra with 3 relations can be nonsingular as well, as we already saw in Example 9.3.1 for a Lie algebra over $\mathbb{R}$.

11.3 Free 3–step nilpotent Lie algebras

Consider an arbitrary field $\mathbb{F}$. Let $\mathfrak{f}_{r,3}$ be the free 3–step nilpotent Lie algebra over $\mathbb{F}$ which has r generators $x_1, x_2, \dots, x_r$. For these generators, we can find a Hall basis of $\mathfrak{f}_{r,3}$, which is a basis of $\mathfrak{f}_{r,3}$ as a vector space and which is explicitly given by the following collection of vectors:

$$\begin{aligned} &x_i, && 1 \leq i \leq r, \\ y_{i,j} &= [x_i, x_j], && 1 \leq i < j \leq r, \\ z_{i,j,k} &= [x_i, y_{j,k}], && 1 \leq j < k \leq r \text{ and } 1 \leq j \leq i \leq r. \end{aligned}$$

Lemma 11.3.1. *Let $a, b \in \mathfrak{f}_{r,3}$. If $a - b \notin [\mathfrak{f}_{r,3}, \mathfrak{f}_{r,3}]$, then*

$$[a, [\mathfrak{f}_{r,3}, \mathfrak{f}_{r,3}]] \cap [b, [\mathfrak{f}_{r,3}, \mathfrak{f}_{r,3}]] = 0.$$

Proof. If either a or b belongs to $[\mathfrak{f}_{r,3}, \mathfrak{f}_{r,3}]$, there is nothing to show, since $\mathfrak{f}_{r,3}$ is 3–step nilpotent. In case both do not belong to $[\mathfrak{f}_{r,3}, \mathfrak{f}_{r,3}]$, the condition that $a - b \notin [\mathfrak{f}_{r,3}, \mathfrak{f}_{r,3}]$ means that we can choose a generating set

$$x_1, \dots, x_{r-2}, x_{r-1} = a, \ x_r = b$$

such that $\mathfrak{f}_{r,3}$ is the free 3–step nilpotent Lie algebra on that set of generators. Using the Hall basis introduced above, we see that

$$[a, [\mathfrak{f}_{r,3}, \mathfrak{f}_{r,3}]] = \langle z_{r-1,j,k} \mid 1 \leq j < k \leq r \rangle,$$

$$[b, [\mathfrak{f}_{r,3}, \mathfrak{f}_{r,3}]] = \langle z_{r,j,k} \mid 1 \leq j < k \leq r \rangle.$$

Note that all of the vectors $z_{r-1,j,k}$ and $z_{r,j,k}$ belong to the Hall set mentioned above and that the set of basis vectors $z_{r-1,j,k}$ is disjoint of the set of basis vectors $z_{r,j,k}$. We have that the subspaces spanned by those two sets only have the zero vector in common. □

Theorem 11.3.2. *Let $\mathfrak{f}_{r,3}$ be the free 3-step nilpotent Lie algebra on r generators. Then*

$$\mathrm{AID}(\mathfrak{f}_{r,3}) = \mathrm{Inn}(\mathfrak{f}_{r,3}).$$

Proof. Let $D \in \mathrm{AID}(\mathfrak{f}_{r,3})$ be an arbitrary almost inner derivation. Note that D induces an almost inner derivation $\bar{D}$ on $\mathfrak{f}_{r,3}/Z(\mathfrak{f}_{r,3}) \cong \mathfrak{f}_{r,2}$. Proposition 11.2.1 implies that $\bar{D}$ is an inner derivation. Hence, by adjusting D with an inner derivation, we may assume that $D(\mathfrak{f}_{r,3}) \subseteq Z(\mathfrak{f}_{r,3})$. Let $x_1, x_2, \dots, x_r$ be the generators of $\mathfrak{f}_{r,3}$. Since we must have that $D(x_i) \in Z(\mathfrak{f}_{r,3})$, there exist vectors $v_i \in [\mathfrak{f}_{r,3}, \mathfrak{f}_{r,3}]$ for all $1 \leq i \leq r$ such that

$$D(x_i) = [x_i, v_i].$$

Analogously, there are also vectors $w_j \in [\mathfrak{f}_{r,3}, \mathfrak{f}_{r,3}]$, for $2 \leq j \leq r$, with

$$D(x_1 + x_j) = [x_1 + x_j, w_j].$$

Take an arbitrary $2 \leq j \leq r$. By using the equation $D(x_1+x_j) = D(x_1)+D(x_j)$, we find that

$$[x_1, w_j] - [x_1, v_1] = [x_j, v_j] - [x_j, w_j].$$

The left hand side of the above expression belongs to $[x_1, [\mathfrak{f}_{r,3}, \mathfrak{f}_{r,3}]]$ and the right hand side to $[x_j, [\mathfrak{f}_{r,3}, \mathfrak{f}_{r,3}]]$. It follows from Lemma 11.3.1 that both expressions are zero. Hence, we have

$$[x_1, w_j - v_1] = [x_j, v_j - w_j] = 0.$$

Since the only elements of $[\mathfrak{f}_{r,3}, \mathfrak{f}_{r,3}]$ that commute with x_1, respectively with x_j, are those belonging to the center $Z(\mathfrak{f}_{r,3})$, we find that

$$w_j - v_1 \in Z(\mathfrak{f}_{r,3}) \quad \text{and} \quad v_j - w_j \in Z(\mathfrak{f}_{r,3}).$$

This means that $v_j - v_1 \in Z(\mathfrak{f}_{r,3})$. Therefore we can without any problem replace v_j by v_1. We find that $D(x_j) = [x_j, v_1]$ holds for all $2 \leq j \leq r$. For the derivation $D' := D + \mathrm{ad}(v_1)$, we see that $D'(x_i) = 0$ holds for all $1 \leq i \leq r$. This means that D' is a derivation which is zero on the generators, and hence D' is zero everywhere. It follows that $D = -\mathrm{ad}(v_1)$ is an inner derivation, which was to be shown. □

11.4 Free metabelian nilpotent Lie algebras on two generators

In this section, we consider free metabelian and c–step nilpotent Lie algebras on two generators. These Lie algebras are c–step nilpotent and metabelian, but do not have other relations. We will show that, when the Lie algebras are defined over an infinite field $\mathbb{F}$ (of any characteristic), all almost inner derivations are inner.

Let $\mathfrak{f}_2$ be the free Lie algebra on two generators, say a and b. The free metabelian c–step nilpotent Lie algebra $\mathfrak{m}_{2,c}$ is obtained as a quotient

$$\mathfrak{m}_{2,c} := \frac{\mathfrak{f}_2}{\mathfrak{f}_2^{(2)} + \gamma_{c+1}(\mathfrak{f}_2)}.$$

It is the largest quotient of the free Lie algebra $\mathfrak{f}_2$ which is both metabelian and c–step nilpotent. We will use x_1 and x_2 to denote the projection in $\mathfrak{m}_{2,c}$ of a respectively b. Take $m \geq 2$ and $1 \leq n \leq m-1$. We introduce the notation y_n^m for

$$y_n^m = [\underbrace{x_2, x_2, x_2, \ldots, x_2}_{n-1 \text{ times}}, \underbrace{x_1, x_1, x_1, \ldots, x_1}_{m-n \text{ times}}, x_2],$$

where for all $z_1, \ldots, z_{n-2}, z_{n-1}, z_n \in \mathfrak{g}$, the iterated bracket

$$[z_1, [\ldots, [z_{n-2}, [z_{n-1}, z_n]] \cdots]]$$

is denoted with $[z_1, \ldots, z_{n-2}, z_{n-1}, z_n]$. So y_n^m is an m–fold Lie bracket with $m-n$ appearances of x_1 and n appearances of x_2. It is well known that x_1, x_2 together with the elements y_n^m (where $1 \leq n < m \leq c$) form a basis of $\mathfrak{m}_{2,c}$, see for instance [3, Section 4.7]. In fact, for any $2 \leq i \leq c$, the Lie algebra $\gamma_i(\mathfrak{m}_{2,c})/\gamma_{i+1}(\mathfrak{m}_{2,c})$ is $(i-1)$–dimensional and has a basis consisting of the projections of the elements $y_1^i, y_2^i, \ldots, y_{i-1}^i$.

Lemma 11.4.1. *Take elements $z_1, z_2, \ldots, z_{n-2} \in \{x_1, x_2\}$ and define the number $k := \#\{1 \leq i \leq n-2 \mid z_i = x_2\} + 1$. Then we have*

$$[z_1, z_2, \ldots, z_{n-2}, x_1, x_2] = y_k^n.$$

Proof. Let $\mathfrak{g}$ be a metabelian Lie algebra $\mathfrak{g}$ and take $c \in \gamma_2(\mathfrak{g})$. For all $x, y \in \mathfrak{g}$, the Jacobi identity yields

$$0 = [x,[y,c]] + [y,[c,x]] + [c,[x,y]] = [x,[y,c]] + [y,[c,x]],$$

so $[x,[y,c]] = -[y,[c,x]] = [y,[x,c]]$. By an inductive reasoning, we have that

$$[z_1, z_2, \ldots, z_{n-2}, x_1, x_2] = [z_{\sigma(1)}, z_{\sigma(2)}, \ldots, z_{\sigma(n-2)}, x_1, x_2]$$

holds for any permutation σ on $n-2$ letters $\{1, 2, \ldots, n-2\}$. This is what we had to show. $\square$

The lemma easily implies the following identities.

Corollary 11.4.2. *For all $m \geq 2$ and all $1 \leq n \leq m-1$, we have*

$$[x_1, y_n^m] = y_n^{m+1} \quad \textit{and} \quad [x_2, y_n^m] = y_{n+1}^{m+1}.$$

Remark 11.4.3. Take an arbitrary field $\mathbb{F}$, then $\mathfrak{m}_{2,c} = \mathfrak{f}_{2,c}$ holds for all $1 \leq c \leq 4$. For $c \geq 5$, we have that $\dim(\mathfrak{m}_{2,c}) < \dim(\mathfrak{f}_{2,c})$. For example, $[x_2, y_1^4]$ and $[y_1^2, y_1^3]$ are linearly independent in $\mathfrak{f}_{2,c}$, whereas these elements both equal y_2^5 in $\mathfrak{m}_{2,c}$.

Now we can prove the main result of this section.

Proposition 11.4.4. *Let $\mathfrak{m}_{2,c}$ be the free c–step nilpotent and metabelian Lie algebra on two generators over an infinite field $\mathbb{F}$. Then* $\mathrm{AID}(\mathfrak{m}_{2,c}) = \mathrm{Inn}(\mathfrak{m}_{2,c})$.

Proof. We will prove the statement by induction on c.

- Basis step: The abelian Lie algebra $\mathfrak{m}_{2,1}$ and the Heisenberg Lie algebra $\mathfrak{m}_{2,2}$ do not admit non-trivial almost inner derivations.
- Induction step: Take $c \geq 3$ and suppose that the statement holds for $c-1$ (induction hypothesis). Let $D \in \mathrm{AID}(\mathfrak{m}_{2,c})$ be an arbitrary almost inner derivation of $\mathfrak{m}_{2,c}$. The space $I = \langle y_1^c, y_2^c, \ldots, y_{c-1}^c \rangle = \gamma_c(\mathfrak{m}_{2,c}) = Z(\mathfrak{m}_{2,c})$ is an ideal of $\mathfrak{m}_{2,c}$. Hence, D induces an almost inner derivation $\bar{D}$ on

$$\mathfrak{m}_{2,c}/I \cong \mathfrak{m}_{2,c-1}.$$

By the induction hypothesis, $\bar{D}$ is an inner derivation of $\mathfrak{m}_{2,c-1}$. This means that we can alter D by an inner derivation of $\mathfrak{m}_{2,c}$ and assume that

$$D(\mathfrak{m}_{2,c}) \subseteq I = \langle y_1^c, y_2^c, \ldots, y_{c-1}^c \rangle = \gamma_c(\mathfrak{m}_{2,c}).$$

Moreover, since $D \in \mathrm{AID}(\mathfrak{m}_{2,c})$ holds, we must have that $D(x) \in [x, \mathfrak{m}_{2,c}]$. We obtain that

$$D(x_1) \in \langle y_1^c, y_2^c, \ldots, y_{c-2}^c \rangle \quad \text{and} \quad D(x_2) \in \langle y_2^c, y_3^c, \ldots, y_{c-1}^c \rangle.$$

There are parameters $a_1, a_2, \ldots, a_{c-2}, b_2, b_3, \ldots, b_{c-1} \in \mathbb{F}$ such that

$$D(x_1) = a_1 y_1^c + a_2 y_2^c + \cdots + a_{c-2} y_{c-2}^c,$$

$$D(x_2) = b_2 y_2^c + b_3 y_3^c + \cdots + b_{c-1} y_{c-1}^c.$$

By changing D to $D - \mathrm{ad}(a_1 y_1^{c-1} + a_2 y_2^{c-1} + \cdots + a_{c-2} y_{c-2}^{c-1})$ and applying Corollary 11.4.2 several times, we find that $D(x_1) = 0$ and

$$D(x_2) = \beta_2 y_2^c + \beta_3 y_3^c + \cdots + \beta_{c-1} y_{c-1}^c,$$

where $\beta_i := b_i - a_{i-1}$ for all $2 \leq i \leq c-1$.

Take an arbitrary $\lambda \in \mathbb{F}$. On the one hand, we have that

$$D(\lambda x_1 + x_2) = \lambda D(x_1) + D(x_2) = \beta_2 y_2^c + \beta_3 y_3^c + \cdots + \beta_{c-1} y_{c-1}^c. \quad (11.6)$$

On the other hand, we also know that there exists an element $v_\lambda \in \mathfrak{m}_{2,c}$ with

$$D(\lambda x_1 + x_2) = [\lambda x_1 + x_2, v_\lambda].$$

We write

$$v_\lambda = a_1x_1 + a_2x_2 + \sum_{1\le n<m\le c} a_{m,n}y_n^m,$$

where $a_1, a_2, a_{m,n} \in \mathbb{F}$ for all $1 \le n < m \le c$ and find that

$$\begin{aligned}[\lambda x_1 + x_2, v_\lambda] = (a_2\lambda - a_1)y_1^2 + \sum_{1\le n<m\le c-1} \lambda a_{m,n}y_n^{m+1} \\ + \sum_{1\le n<m\le c-1} a_{m,n}y_{n+1}^{m+1} \end{aligned} \tag{11.7}$$

Comparing the coefficients of the basis vectors y_i^c (with $2 \le i \le c-1$) of (11.6) with (11.7), we get the following system of equations:

$$\begin{cases} \lambda a_{c-1,1} & = 0 \\ \lambda a_{c-1,2} + a_{c-1,1} & = \beta_2 \\ \lambda a_{c-1,3} + a_{c-1,2} & = \beta_3 \\ & \vdots \\ \lambda a_{c-1,c-2} + a_{c-1,c-3} & = \beta_{c-2} \\ a_{c-1,c-2} & = \beta_{c-1}. \end{cases}$$

We multiply the different equations with a power of λ and obtain

$$\begin{cases} \lambda a_{c-1,1} & = 0 \\ \lambda^2 a_{c-1,2} + \lambda a_{c-1,1} & = \lambda\beta_2 \\ \lambda^3 a_{c-1,3} + \lambda^2 a_{c-1,2} & = \lambda^2\beta_3 \\ & \vdots \\ \lambda^{c-2} a_{c-1,c-2} + \lambda^{c-3} a_{c-1,c-3} & = \lambda^{c-3}\beta_{c-2} \\ \lambda^{c-2} a_{c-1,c-2} & = \lambda^{c-2}\beta_{c-1}. \end{cases} \tag{11.8}$$

By taking the alternating sum of all these equations, we find that

$$\lambda\beta_2 - \lambda^2\beta_3 + \cdots + (-1)^{c-2}\lambda^{c-3}\beta_{c-2} + (-1)^{c-1}\lambda^{c-2}\beta_{c-1} = 0.$$

This is an equation which has to hold for all possible $\lambda \in \mathbb{F}$. Since $\mathbb{F}$ is infinite, we must have that

$$\beta_2 = \beta_3 = \cdots = \beta_{c-1} = 0.$$

It follows that $D(x_2) = 0$. Together with the fact that $D(x_1) = 0$, this implies that $D = 0$, which means that the original D we started with was an inner derivation.

- Conclusion: By the principle of induction, it follows from the basis and induction step that $\mathrm{AID}(\mathfrak{m}_{2,c}) = \mathrm{Inn}(\mathfrak{m}_{2,c})$ holds for all $c \in \mathbb{N}_0$. □

11.5 General free nilpotent Lie algebras

In this section, we consider a field $\mathbb{F}$ of characteristic zero. We will prove that a free nilpotent Lie algebra over $\mathbb{F}$ does not admit any non-trivial almost inner derivations. The results of this section also appeared in [8]. The following lemma will be very useful.

Lemma 11.5.1. *Let V be a vector space over a field $\mathbb{F}$ of characteristic zero. Consider a sequence $v_0, v_1, v_2, \ldots$ in V. Take $k \in \mathbb{N}$ and suppose that there exist $a_0, a_1, \ldots, a_k \in V$ such that*

$$v_{n+1} - v_n = \sum_{j=0}^{k} n^j a_j$$

for all $n \in \mathbb{N}$. Then there exist vectors $b_0, b_1, \ldots, b_{k+1} \in V$ such that

$$v_n = \sum_{j=0}^{k+1} n^j b_j, \qquad \text{for all } n \in \mathbb{N},$$

$$a_k \neq 0 \Longrightarrow b_{k+1} \neq 0.$$

Proof. We prove this by induction on k.

- Basis step ($k = 0$): Suppose that we have $v_{n+1} - v_n = a_0$ for all $n \in \mathbb{N}$, where $a_0 \in V$ and $v_i \in V$ for all $i \in \mathbb{N}$. We show by induction on $n \in \mathbb{N}$ that $v_n = v_0 + na_0$ holds. This is true for $n \in \{0, 1\}$, so suppose that $v_n = v_0 + na_0$ (induction hypothesis). Then $v_{n+1} = v_n + a_0 = v_0 + (n+1)a_0$ holds. The statement of the lemma follows for $b_0 = v_0$ and $b_1 = a_0$.

- Induction step: We assume that the asssertion holds for k (induction hypothesis). Consider $a_0, a_1, \ldots, a_{k+1} \in V$ such that

$$v_{n+1} - v_n = \sum_{j=0}^{k+1} n^j a_j$$

  for all $n \in \mathbb{N}$. We have to prove that there exist vectors $b_0, b_1, \ldots, b_{k+2} \in V$ such that $v_n = \sum_{j=0}^{k+2} n^j b_j$ holds for all $n \in \mathbb{N}$. Define a new sequence of vectors

$$w_n := v_n - \frac{a_{k+1}}{k+2} n^{k+2},$$

where $n \in \mathbb{N}$. We then have

$$\begin{aligned} w_{n+1} - w_n &= v_{n+1} - v_n - \frac{a_{k+1}}{k+2}((n+1)^{k+2} - n^{k+2}) \\ &= \sum_{j=0}^{k+1} n^j a_j - \frac{a_{k+1}}{k+2} \sum_{j=0}^{k+1} \binom{k+2}{j} n^j \\ &= \sum_{j=0}^{k} n^j \left(a_j - \frac{a_{k+1}}{k+2} \binom{k+2}{j} \right) \end{aligned}$$

for all $n \in \mathbb{N}$. By induction, we find that there exist vectors $b_0, b_1, \dots, b_{k+1} \in V$ such that

$$w_n = \sum_{j=0}^{k+1} n^j b_j \qquad \text{for all } n \in \mathbb{N}.$$

If we now take $b_{k+2} = \frac{a_{k+1}}{k+2}$, we obtain that

$$v_n = w_n + n^{k+2} \frac{a_{k+1}}{k+2} = \sum_{j=0}^{k+2} n^j b_j$$

for all $n \in \mathbb{N}$.

- Conclusion: By the principle of induction, the result follows from the basis and induction step. □

Let $\mathfrak{g} := \mathfrak{f}_2$ be the free Lie algebra on two generators x_1 and x_2. Define $\mathfrak{g}_1$ as the vector space spanned by the generators x_1 and x_2 and $\mathfrak{g}_n$, for $n \geq 2$, as the subspace of $\mathfrak{g}$ generated by all Lie brackets of length n in the generators x_1 and x_2. Denote further $\mathfrak{g}_{i,j}$ for the subspace of $\mathfrak{g}$ generated by all Lie brackets in the generators where the first generator x_1 appears i times and the second one x_2 appears j times. It is clear that

$$\mathfrak{g} = \bigoplus_{n \in \mathbb{N}_0} \mathfrak{g}_n \quad \text{and} \quad \mathfrak{g}_n = \bigoplus_{i=1}^{n-1} \mathfrak{g}_{i,n-i} \ \text{ for } n \geq 2.$$

Moreover, it holds that

$$[\mathfrak{g}_i, \mathfrak{g}_j] \subseteq \mathfrak{g}_{i+j} \quad \text{and} \quad [\mathfrak{g}_{i,j}, \mathfrak{g}_{p,q}] \subseteq \mathfrak{g}_{i+p,j+q}.$$

We are interested in the equation

$$[x_1, x] + [x_2, y] = 0$$

in the variables x and y, which was studied in [71]. Let

$$V = \{(x, y) \in \mathfrak{g} \times \mathfrak{g} \mid [x_1, x] + [x_2, y] = 0\}$$

be the solution space of the equation. Note that V is a vector space. For each $n \in \mathbb{N}_0$, we define $V_n = V \cap (\mathfrak{g}_n \times \mathfrak{g}_n)$. Consider now the maps

$$\varphi_n : V_n \to \mathfrak{g}_n : (x, y) \mapsto x,$$

$$\psi_n : V_n \to \mathfrak{g}_n : (x, y) \mapsto y.$$

Define the set $V_n^x := \varphi_n(V_n)$. We assert that $\tilde{\varphi}_n : V_n \to V_n^x : (x, y) \mapsto \varphi_n(x, y)$ is an isomorphism for all $n \geq 2$. It is obvious that the map $\tilde{\varphi}_n$ is linear. Surjectivity follows by construction. Suppose that $\tilde{\varphi}_n$ is not injective, then there is a solution $(0, 0) \neq (0, y) \in V_n$, which means that $[x_2, y] = 0$. Therefore, $y = 0$ and we have a contradiction. Analogously, also $\tilde{\psi}_n : V_n \to V_n^y : (x, y) \mapsto \psi_n(x, y)$ is an isomorphism, where $V_n^y := \psi_n(V_n)$. Hence, for all $n \geq 2$, there is a vector space isomorphism $\sigma : V_n^x \to V_n^y$ such that

$$[x_1, x] + [x_2, \sigma(x)] = 0 \tag{11.9}$$

for all $x \in V_n^x$. Note that under this isomorphism, we have for all $1 \leq i \leq n-1$ that $\sigma(V_n^x \cap \mathfrak{g}_{i,n-i}) = V_n^y \cap \mathfrak{g}_{i+1,n-i-1}$.

Theorem 11.5.2. *Let $\mathbb{F}$ be a field of characteristic zero and let $\mathfrak{f}_{r,c}$ be the free c–step nilpotent Lie algebra over $\mathbb{F}$ on r generators. Then $\mathrm{AID}(\mathfrak{f}_{r,c}) = \mathrm{Inn}(\mathfrak{f}_{r,c})$ holds.*

Proof. We prove this theorem by induction on the nilpotency class c.

- Basis step: The case $c = 1$ is clear and the cases $c = 2$ and $c = 3$ were already treated in Section 11.2 and Section 11.3.
- Induction step: Take $c \geq 3$ and assume that the theorem holds for $\mathfrak{f}_{r,c}$ (induction hypothesis). Consider $\mathfrak{f}_{r,c+1}$ with generators $x_1, x_2, \ldots, x_r$. Take an arbitrary $D \in \mathrm{AID}(\mathfrak{f}_{r,c+1})$. We will prove that D is in fact inner. It is clear that D induces an almost inner derivation on

  $$\frac{\mathfrak{f}_{r,c+1}}{\gamma_{c+1}(\mathfrak{f}_{r,c+1})} \cong \mathfrak{f}_{r,c}.$$

  This is an inner derivation by the induction hypothesis. Hence, by changing D up to an inner derivation, we may assume that

  $$D(\mathfrak{f}_{r,c+1}) \subseteq \gamma_{c+1}(\mathfrak{f}_{r,c+1}) = Z(\mathfrak{f}_{r,c+1}),$$

which means that $D \in \mathrm{CAID}(\mathfrak{f}_{r,c+1})$. Hence, there exists $v \in \gamma_c(\mathfrak{f}_{r,c+1})$ such that $D(x_1) = [x_1, v]$. By replacing D by $D + \mathrm{ad}(v)$, we can assume that D is an almost inner derivation of $\mathfrak{f}_{r,c+1}$ with $D(x_1) = 0$. Further, for all $x \in \mathfrak{f}_{r,c+1}$, we have $D(x) = [x, w(x)]$, with $w(x) \in \gamma_c(\mathfrak{f}_{r,c+1})$. It suffices to prove that $D(x_i) = 0$ for all $2 \leq i \leq r$. We first look at x_2. For each $n \in \mathbb{N}$, there exists a $w_n \in \gamma_c(\mathfrak{f}_{r,c+1})$ such that

$$D(nx_1 + x_2) = [nx_1 + x_2, w_n], \tag{11.10}$$

because D is almost inner. We can assume without loss of generality that w_n is a linear combination of Lie brackets of length c in the generators (and does not contain a component using Lie brackets of length $c + 1$). By linearity, we also have that

$$D(nx_1 + x_2) = nD(x_1) + D(x_2) = D(x_2).$$

The two observations above imply that the equation

$$[nx_1 + x_2, w_n] = [mx_1 + x_2, w_m] = [x_2, w_0] \tag{11.11}$$

holds for all $m, n \in \mathbb{N}$. We consider $[nx_1 + x_2, w_n] + [x_2, -w_0] = 0$ as an equation in the free Lie algebra $\mathfrak{f}_r$ on r generators. For $n \neq 0$, define $x_1' := nx_1 + x_2$. It is clear that $x_1', x_2, \ldots, x_r$ is also a free generating set for the free Lie algebra $\mathfrak{f}_r$.

It follows from [71, Section 5] that $w_n, w_0 \in \langle x_1', x_2 \rangle = \langle x_1, x_2 \rangle$ for all $n \in \mathbb{N}_0$, where $\langle x_1, x_2 \rangle$ denotes the Lie algebra generated by x_1 and x_2. This means that w_n can be written as $w_n = \sum_{i=1}^{c-1} v_i(n)$, where $v_i(n)$ is a linear combination of Lie brackets where x_2 and x_1 appear i respectively $c - i$ times. We can assume without loss of generality that we work in $\mathfrak{g} = \mathfrak{f}_2$, the free Lie algebra on two generators x_1 and x_2. This means that $v_i(n) \in \mathfrak{g}_{c-i,i}$ for all $1 \leq i \leq c - 1$, where we use the notations introduced above this theorem. To prove that $D(x_2) = 0$, it suffices by equation (11.10) to show that $w_0 = 0$. Suppose on the contrary that $w_0 \neq 0$. Define

$$k = \max\{i \in \mathbb{N} \mid \exists\, n \in \mathbb{N} \text{ with } v_i(n) \neq 0\},$$

then $w_n = \sum_{i=1}^{k} v_i(n)$ and there exists an $n \in \mathbb{N}$ such that $v_k(n) \neq 0$. It follows from equation (11.11) that $[(n+1)x_1 + x_2, w_{n+1}] = [nx_1 + x_2, w_n]$, which implies that

$$[x_1, (n+1)w_{n+1} - nw_n] + [x_2, w_{n+1} - w_n] = 0, \tag{11.12}$$

with $n \in \mathbb{N}$. In fact, this consists of several equations (one per bi-degree (i, j) with $i + j = c$). We will now prove by induction on p that for all

$0 \leq p \leq k-1$ and all $0 \leq i \leq p$, there exist $b_{p,i} \in \mathfrak{g}_{c-k+p,k-p}$, with $b_{p,p} \neq 0$ such that

$$v_{k-p}(n) = n^p b_{p,p} + n^{p-1} b_{p,p-1} + \cdots + n b_{p,1} + b_{p,0}.$$

– Basis step ($p = 0$): We first consider the component of equation (11.12) with in total $k+1$ appearances of x_2, this is the bi-degree $(c-k, k)$–part. This gives

$$[x_1, 0] + [x_2, v_k(n+1) - v_k(n)] = 0.$$

Hence, $v_k(n+1) - v_k(n) = 0$ holds, which means that $v_k(n)$ is a constant $b_{0,0} \neq 0$ and belongs to $\mathfrak{g}_{c-k,k}$. Therefore, we have that $w_n = \left(\sum_{i=1}^{k-1} v_i(n)\right) + b_{0,0}$.

– Induction step: We assume that the assertion holds for a given $0 \leq p < k-1$ (induction hypothesis). Hence, for all $0 \leq i \leq p$, there exist $b_{p,i} \in \mathfrak{g}_{c-k+p,k-p}$ with $b_{p,p} \neq 0$ such that

$$v_{k-p}(n) = n^p b_{p,p} + n^{p-1} b_{p,p-1} + \cdots + n b_{p,1} + b_{p,0}.$$

From the component of equation (11.12) with $k-p$ appearances of x_2, we have that

$$\begin{aligned} 0 &= [x_1, (n+1) v_{k-p}(n+1) - n v_{k-p}(n)] \\ &\quad + [x_2, v_{k-p-1}(n+1) - v_{k-p-1}(n)]. \end{aligned}$$

It follows from equation (11.9) and the induction hypothesis that

$$\begin{aligned} &v_{k-p-1}(n+1) - v_{k-p-1}(n) \\ &\quad = \sigma\big((n+1) v_{k-p}(n+1) - n v_{k-p}(n)\big) \\ &\quad = \sigma\big((n+1)^{p+1} b_{p,p} + (n+1)^p b_{p,p-1} + \cdots + (n+1) b_{p,0} \\ &\qquad\quad - n^{p+1} b_{p,p} - n^p b_{p,p-1} - \ldots - n b_{p,0}\big) \\ &\quad = (n+1)^{p+1} \sigma(b_{p,p}) + (n+1)^p \sigma(b_{p,p-1}) + \cdots + (n+1) \sigma(b_{p,0}) \\ &\qquad\quad - n^{p+1} \sigma(b_{p,p}) - n^p \sigma(b_{p,p-1}) - \cdots - n \sigma(b_{p,0}). \end{aligned}$$

This expression can be written as the sum of $n^p(p+1)\sigma(b_{p,p})$ and terms of lower degree. Since $b_{p,p} \neq 0$ holds, we also have that $(p+1)\sigma(b_{p,p}) \neq 0$. Note that $\sigma(b_{p,i})$ belongs to $\mathfrak{g}_{c-k+p+1,k-p-1}$

for all $0 \leq i \leq p$. Hence, Lemma 11.5.1 implies that there exist $b_{p+1,p+1}, \ldots, b_{p+1,0} \in \mathfrak{g}_{c-k+p+1,k-p-1}$ with $b_{p+1,p+1} \neq 0$ such that

$$v_{k-p-1}(n) = n^{p+1} b_{p+1,p+1} + n^p b_{p+1,p} + \cdots + b_{p+1,0},$$

which concludes the proof of our claim on the form of $v_{k-p}(n)$, where $0 \leq p \leq k-1$.

The above assertion implies that for all $n \in \mathbb{N}$, the equation

$$v_1(n) = n^{k-1} b_{k-1,k-1} + \cdots + n b_{k-1,1} + b_{k-1,0} \tag{11.13}$$

holds, where $b_{k-1,k-1} \neq 0$ and $b_{k-1,i} \in \mathfrak{g}_{c-1,1}$ for all $0 \leq i \leq k-1$. We now look at the term of equation (11.12) with exactly one factor of x_2. We then have

$$[x_1, (n+1)v_1(n+1) - nv_1(n)] + [x_2, 0] = 0$$

and thus

$$(n+1)v_1(n+1) - nv_1(n) = 0.$$

We obtain from (11.13) that

$$\begin{aligned}
0 &= (n+1)\left((n+1)^{k-1} b_{k-1,k-1} + \cdots + b_{k-1,0}\right) \\
&\quad - n\left(n^{k-1} b_{k-1,k-1} + \cdots + b_{k-1,0}\right) \\
&= (n+1)^k b_{k-1,k-1} + \cdots + (n+1) b_{k-1,0} \\
&\quad - n^k b_{k-1,k-1} - \cdots - n b_{k-1,0} \\
&= k n^{k-1} b_{k-1,k-1} + \sum_{i=2}^{k} \binom{k}{i} n^{k-i} b_{k-1,k-1} \\
&\quad + \sum_{i=1}^{k-1} \binom{k-1}{i} n^{k-1-i} b_{k-1,k-2} + \cdots + b_{k-1,0}.
\end{aligned}$$

Hence, we can write 0 as a sum of $kn^{k-1}b_{k-1,k-1}$ and some terms of lower degree. This equation has to hold for all $n \in \mathbb{N}$, which implies that $kb_{k-1,k-1} = 0$. Since we work in a field of characteristic zero, this gives a contradiction, because $b_{k-1,k-1} \neq 0$. Hence, $w_0 = 0$. It now follows from equation (11.10) that $D(x_2) = 0$. By a similar reasoning, we find that $D(x_i) = 0$ for all $3 \leq i \leq r$. This shows that D was actually an inner derivation.

- Conclusion: By the principle of induction, the result follows from the basis and induction step. □

Chapter 12

Other classes of Lie algebras

In previous chapters, we computed almost inner derivations for low-dimensional Lie algebras and different types of nilpotent Lie algebras. This chapter collects all other results. The first section is about triangular Lie algebras. Further, we also treat Lie algebras with a 1–codimensional abelian subalgebra and Lie algebras with an abelian solvable radical. For each of these classes, we will prove that the only almost inner derivations are the inner ones. The last section contains observations of characteristically nilpotent Lie algebras. Some results already appeared in [7] (Section 12.1) and [8] (Section 12.2 and Section 12.3). In each section, we will specify over which field we work.

12.1 Triangular Lie algebras

Let $\mathbb{F}$ be an arbitrary field. In this section, we consider the almost inner derivations for the triangular Lie algebras. We denote $\mathfrak{t}_n(\mathbb{F})$ for the Lie algebra of all upper triangular $(n \times n)$–matrices over $\mathbb{F}$. Similarly, $\mathfrak{n}_n(\mathbb{F})$ is the Lie algebra of strictly upper triangular $(n \times n)$–matrices over $\mathbb{F}$. We define $e_{i,j}$ as the $(n \times n)$–matrix where the entry on position (i, j) is 1 and where all other entries are zero. Since $\mathfrak{t}_n(\mathbb{F})$ and $\mathfrak{n}_n(\mathbb{F})$ are subalgebras of $\mathfrak{gl}_n(\mathbb{F})$, the Lie bracket of $e_{i,j}$ and $e_{k,l}$ is given by

$$[e_{i,j}, e_{k,l}] = \delta_{j,k}e_{i,l} - \delta_{l,i}e_{k,j} \tag{12.1}$$

for all $1 \leq i, j, k, l \leq n$.

First, we will study the Lie algebra $\mathfrak{t}_n(\mathbb{F})$. This Lie algebra is solvable and not nilpotent when $n \geq 2$. Further, $\mathfrak{t}_n(\mathbb{F})$ has dimension $\frac{n(n+1)}{2}$ and a basis consisting of the matrices $e_{i,j}$, where $1 \leq i \leq j \leq n$. It turns out that the only almost inner derivations are the inner ones.

Proposition 12.1.1. *For all $n \geq 2$, we have that*

$$\mathrm{AID}(\mathfrak{t}_n(\mathbb{F})) = \mathrm{Inn}(\mathfrak{t}_n(\mathbb{F})).$$

Proof. The proof goes by induction on n.

- Basis step: For $n = 2$, we have that $\mathfrak{t}_2(\mathbb{F})$ is 3–dimensional, so the proposition holds.
- Induction step: Take $n \geq 3$ and suppose that the proposition holds for smaller values of n (induction hypothesis). Let $D \in \mathrm{AID}(\mathfrak{t}_n(\mathbb{F}))$ be an arbitrary almost inner derivation. Since

  $$I = \langle e_{1,1}, e_{1,2}, \ldots, e_{1,n} \rangle$$

  is an ideal of $\mathfrak{t}_n(\mathbb{F})$, we have that $D(I) \subseteq I$. Hence, D induces an almost inner derivation $\bar{D}$ on $\mathfrak{t}_n(\mathbb{F})/I \cong \mathfrak{t}_{n-1}(\mathbb{F})$. By the induction hypothesis, $\bar{D}$ is an inner derivation. Hence, there exists $x \in \mathfrak{t}_n(\mathbb{F})$ such that $\bar{D} = \mathrm{ad}(\bar{x})$, where $\bar{x}$ is the projection of x in $\mathfrak{t}_{n-1}(\mathbb{F}) \cong \mathfrak{t}_n(\mathbb{F})/I$.

  We change D by $D - \mathrm{ad}(x)$ so that we can assume that $D \in \mathrm{AID}(\mathfrak{t}_n(\mathbb{F}))$ is almost inner with $D(\mathfrak{t}_n(\mathbb{F})) \subseteq I$. Take $2 \leq i \leq n$, then

  $$D(e_{i,i}) \in [e_{i,i}, \mathfrak{t}_n(\mathbb{F})] = \langle e_{1,i}, e_{2,i}, \ldots, e_{i-1,i}, e_{i,i+1}, e_{i,i+2}, \ldots, e_{i,n} \rangle$$

  holds by (12.1). Since $D(e_{i,i}) \in I$, there exists a value $\beta_i \in \mathbb{F}$ with $D(e_{i,i}) = \beta_i e_{1,i}$. Define $a := \beta_2 e_{1,2} + \cdots + \beta_n e_{1,n}$, then

  $$\begin{aligned} \mathrm{ad}(a)(e_{i,i}) &= [\beta_2 e_{1,2} + \ldots + \beta_n e_{1,n}, e_{i,i}] \\ &= \beta_i e_{1,i} \\ &= D(e_{i,i}) \end{aligned}$$

  for all $2 \leq i \leq n$. We also have that $\mathrm{ad}(a)(I) \subseteq I$. Hence, we can replace D by $D - \mathrm{ad}(a)$ and obtain that $D(\mathfrak{t}_n(\mathbb{F})) \subseteq I$ with

  $$D(e_{2,2}) = D(e_{3,3}) = \cdots = D(e_{n,n}) = 0.$$

  There also exist $\alpha_2, \alpha_3, \ldots, \alpha_n \in \mathbb{F}$ with

  $$D(e_{1,1}) = \alpha_2 e_{1,2} + \alpha_3 e_{1,3} + \cdots + \alpha_n e_{1,n}.$$

For $2 \leq i \leq n$, we have that $[e_{1,1}, e_{i,i}] = 0$. This means that

$$\begin{aligned} 0 &= D([e_{1,1}, e_{i,i}]) \\ &= [D(e_{1,1}), e_{i,i}] + [e_{1,1}, D(e_{i,i})] \\ &= [\alpha_2 e_{1,2} + \alpha_3 e_{1,3} + \cdots + \alpha_n e_{1,n}, e_{i,i}] + 0 \\ &= \alpha_i e_{1,i}, \end{aligned}$$

so $\alpha_i = 0$ for all $2 \leq i \leq n$. It follows that $D(e_{1,1}) = 0$ as well. Take arbitrary $1 \leq i < j \leq n$ and consider the basis vector $e_{i,j}$. Since

$$D(e_{i,j}) \in [e_{i,j}, \mathfrak{t}_n(\mathbb{F})] \cap I,$$

there exists $\gamma_j \in \mathbb{F}$ such that $D(e_{i,j}) = \gamma_j e_{1,j}$. It follows that

$$\begin{aligned} \gamma_j e_{1,j} &= D([e_{i,i}, e_{i,j}]) \\ &= [D(e_{i,i}), e_{i,j}] + [e_{i,i}, D(e_{i,j})] \\ &= [0, e_{i,j}] + [e_{i,i}, \gamma_j e_{1,j}] = 0, \end{aligned}$$

which implies that $\gamma_j = 0$. Hence, $D(e_{i,j}) = 0$ for all $1 \leq i \leq j \leq n$. This shows that $D = 0$ and the original D is an inner derivation.

- Conclusion: By the principle of induction, it follows from the basis and induction step that $\mathrm{AID}(\mathfrak{t}_n(\mathbb{F})) = \mathrm{Inn}(\mathfrak{t}_n(\mathbb{F}))$ holds for all $n \geq 2$. □

Now, we will consider the Lie algebra $\mathfrak{n}_n(\mathbb{F})$ of all strictly upper triangular $(n \times n)$–matrices over a general field $\mathbb{F}$. This is a nilpotent Lie algebra with nilindex $n - 1$. Further, $\mathfrak{n}_n(\mathbb{F})$ has dimension $\frac{n(n-1)}{2}$ and a basis consisting of the matrices $e_{i,j}$, where $1 \leq i < j \leq n$. We also obtain that every almost inner derivation is inner. For the proof, we can use the same technique as before, but the details differ.

Proposition 12.1.2. *For any $n \geq 2$ we have*

$$\mathrm{AID}(\mathfrak{n}_n(\mathbb{F})) = \mathrm{Inn}(\mathfrak{n}_n(\mathbb{F})).$$

Proof. We will prove the statement by induction on n.

- Basis step: For $n = 2$, we have that $\mathfrak{n}_2(\mathbb{F}) \cong \mathbb{F}$ is abelian, which means that the proposition is trivially true.

- Induction step: Take $n \geq 3$ and assume that the result holds for smaller values of n (induction hypothesis). Take an arbitrary $D \in \mathrm{AID}(\mathfrak{n}_n(\mathbb{F}))$. Note that $I = \langle e_{1,2}, e_{1,3}, \ldots, e_{1,n} \rangle$ is an ideal of $\mathfrak{n}_n(\mathbb{F})$, which implies that $D(I) \subseteq I$. It follows that D induces a derivation $\bar{D}$ of $\mathfrak{n}_n(\mathbb{F})/I \cong \mathfrak{n}_{n-1}(\mathbb{F})$. Of course, we have that $\bar{D} \in \mathrm{AID}(\mathfrak{n}_{n-1}(\mathbb{F}))$. We can conclude from the induction hypothesis that $\bar{D}$ is an inner derivation. Let $x \in \mathfrak{n}_n(\mathbb{F})$ be an element such that $\bar{D} = \mathrm{ad}(\bar{x})$, where $\bar{x}$ denotes the projection of x in $\mathfrak{n}_{n-1}(\mathbb{F}) \cong \mathfrak{n}_n(\mathbb{F})/I$.

  By replacing D by $D - \mathrm{ad}(x)$, we may assume that D is an almost inner derivation of $\mathfrak{n}_n(\mathbb{F})$ with $D(\mathfrak{n}_n(\mathbb{F})) \subseteq I$. Moreover, it follows from (12.1) that
$$D(e_{i,i+1}) \in \langle e_{1,i+1}, e_{2,i+1}, \ldots, e_{i-1,i+1}, e_{i,i+2}, e_{i,i+3}, \ldots, e_{i,n}, \rangle$$
  for all $2 \leq i \leq n-1$. Hence, there exist elements $\beta_3, \beta_4, \ldots, \beta_n \in \mathbb{F}$ such that
$$D(e_{i,i+1}) = \beta_{i+1} e_{1,i+1},$$
  where $2 \leq i \leq n-1$. Define $a := \beta_3 e_{1,2} + \beta_4 e_{1,3} + \cdots + \beta_n e_{1,n-1}$, then we have
$$\begin{aligned} \mathrm{ad}(a)(e_{i,i+1}) &= [\beta_3 e_{1,2} + \beta_4 e_{1,3} + \cdots + \beta_n e_{1,n-1}, e_{i,i+1}] \\ &= [\beta_{i+1} e_{1,i}, e_{i,i+1}] \\ &= \beta_{i+1} e_{1,i+1} \\ &= D(e_{i,i+1}) \end{aligned}$$
  for all $2 \leq i \leq n-1$. So, by replacing D by $D - \mathrm{ad}(a)$, we may assume that
$$D(e_{2,3}) = D(e_{3,4}) = \cdots = D(e_{n-1,n}) = 0.$$
  Note that $\mathrm{ad}(a)(I) \subseteq I$, which means that also after we modify D, we still have that $D(\mathfrak{n}_n(\mathbb{F})) \subseteq I$. There also exist $\alpha_3, \alpha_4, \ldots, \alpha_n \in \mathbb{F}$ with
$$D(e_{1,2}) = \alpha_3 e_{1,3} + \alpha_4 e_{1,4} + \cdots + \alpha_n e_{1,n}.$$
  For $3 \leq i \leq n-1$, we have $[e_{1,2}, e_{i,i+1}] = 0$, so that
$$\begin{aligned} 0 &= D([e_{1,2}, e_{i,i+1}]) \\ &= [D(e_{1,2}), e_{i,i+1}] + [e_{1,2}, D(e_{i,i+1})] \\ &= [\alpha_3 e_{1,3} + \alpha_4 e_{1,4} + \cdots + \alpha_n e_{1,n}, e_{i,i+1}] + 0 \\ &= \alpha_i e_{1,i+1}. \end{aligned}$$

It follows that $\alpha_i = 0$ for all $3 \leq i \leq n-1$, so that

$$D(e_{1,2}) = \alpha_n e_{1,n} = \mathrm{ad}(-\alpha_n e_{2,n})(e_{1,2}).$$

Note that for $2 \leq i \leq n-1$, we have $\mathrm{ad}(-\alpha_n e_{2,n})(e_{i,i+1}) = 0$. By finally replacing D by $D + \mathrm{ad}(\alpha_n e_{2,n})$, we find that $D(e_{i,i+1}) = 0$ holds for all $1 \leq i \leq n-1$.

Take arbitrary $1 \leq i \leq n-1$ and $1 \leq k \leq n-i$ and consider the basis vector $e_{i,i+k}$. We will show by induction on k that $D(e_{i,i+k}) = 0$.

- Basis step: For $k = 1$, we established the proof above.
- Induction step: Take $2 \leq k \leq n-i$ and assume that the result holds for smaller values of k (induction hypothesis). We have

$$\begin{aligned} D(e_{i,i+k}) &= D([e_{i,i+1}, e_{i+1,i+k}]) \\ &= [D(e_{i,i+1}), e_{i+1,i+k}] + [e_{i,i+1}, D(e_{i+1,i+k})] \\ &= [0, e_{i+1,i+k}] + [e_{i,i+1}, 0], \end{aligned}$$

  where we use the basis step and the induction hypothesis for the last equality. This implies that $D(e_{i,i+k}) = 0$.
- Conclusion: By the basis and induction step, it follows from the principle of induction that $D(e_{i,i+k}) = 0$ for all $1 \leq i \leq n-1$ and $1 \leq k \leq n-i$.

Since $D(e_{i,j}) = 0$ for all $1 \leq i < j \leq n$, this implies that $D = 0$, so that the original D is an inner derivation.

- Conclusion: By the basis and induction step, it follows from the principle of induction that when $n \geq 2$, the only almost inner derivations of $\mathfrak{n}_n(\mathbb{F})$ are the inner ones. □

12.2 Almost abelian Lie algebras

We work in this section over an arbitrary field $\mathbb{F}$. For an abelian Lie algebra $\mathfrak{g}$ over $\mathbb{F}$, it is clear that there are no (almost) inner derivations, since $\mathrm{ad}(x) = 0$ for all $x \in \mathfrak{g}$. In this section, we will study Lie algebras which are almost abelian. This class of Lie algebras has no unique definition in the literature. A common convention is that a Lie algebra $\mathfrak{g}$ is 'almost abelian' if it contains a 1–codimensional abelian ideal. Note that $\mathfrak{g}$ is metabelian, but does not have to be nilpotent. It is enough to require that $\mathfrak{g}$ contains an abelian subalgebra of codimension one.

Proposition 12.2.1 ([6])**.** *Let $\mathfrak{g}$ be a Lie algebra over a field $\mathbb{F}$. If $\mathfrak{g}$ has a 1–codimensional abelian subalgebra, then $\mathfrak{g}$ has an abelian ideal of codimension 1 which can be constructed explicitly.*

The result in [6] is stated for Lie algebras over a field of characteristic zero, but the proof is also valid in general. We will show that, as for the abelian Lie algebras, all almost inner derivations are inner.

Theorem 12.2.2. *Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over a field $\mathbb{F}$ containing an abelian subalgebra of codimension one. Then $\mathrm{AID}(\mathfrak{g}) = \mathrm{Inn}(\mathfrak{g})$ holds.*

Proof. It follows from the previous result that $\mathfrak{g}$ has an abelian ideal of codimension one. Hence, there exists a Lie algebra morphism $\varphi : \mathbb{F} \to \mathfrak{gl}_n(\mathbb{F})$ such that $\mathfrak{g} \cong \mathbb{F}^n \rtimes_\varphi \mathbb{F}$. We use t to denote a basis vector of $\mathbb{F}$. With respect to a suitable basis of $\mathbb{F}^n$, we may assume that $\varphi(t)$ is in rational canonical form. This means that there is a basis $e_{i,j}$ (with $1 \leq i \leq r$ and $1 \leq j \leq k_i$) of $\mathbb{F}^n$ such that

$$\varphi(t) = \begin{pmatrix} C_1 & 0 & \cdots & 0 \\ 0 & C_2 & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & C_r \end{pmatrix}$$

is a blocked diagonal matrix. Each block C_i (with $1 \leq i \leq r$) is a companion matrix

$$C_i = \begin{pmatrix} 0 & 0 & \cdots & 0 & -\alpha_0 \\ 1 & 0 & \cdots & 0 & -\alpha_1 \\ 0 & 1 & & 0 & -\alpha_2 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & & 1 & -\alpha_{k_i-1} \end{pmatrix}$$

of a polynomial $q(x)^m = \alpha_0 + \alpha_1 x + \cdots + \alpha_{k_i-1}x^{k_i-1} + x^{k_i}$, where $q(x)$ is irreducible. Since $q(x)$ is irreducible, it holds that either $\alpha_0 \neq 0$ or $q(x)^m = x^{k_i}$ and hence $\alpha_0 = \alpha_1 = \cdots = \alpha_{k_i-1} = 0$. For more information on the rational canonical form, see for instance [47, Section 6.7].

Take an arbitrary $D \in \mathrm{AID}(\mathbb{F}^n \rtimes_\varphi \mathbb{F})$, then there exists an element $v \in \mathbb{F}^n \rtimes_\varphi \mathbb{F}$ with $D(t) = [t, v]$. By replacing D by $D + \mathrm{ad}(v)$, we may assume that $D(t) = 0$.

For any vector $e \in \mathbb{F}^n$, there exists a scalar $\alpha(e) \in \mathbb{F}$ such that

$$D(e) = [e, \alpha(e)t].$$

Take different basis vectors $e_{i,j}$ (with $1 \leq i \leq r$ and $1 \leq j \leq k_i$) and $e_{p,q}$ (with $1 \leq p \leq r$ and $1 \leq q \leq k_p$) and suppose that both $e_{i,j}, e_{p,q} \notin C_{\mathfrak{g}}(t)$. Our aim is to show that $\alpha(e_{i,j}) = \alpha(e_{p,q})$. Since we assume that $e_{i,j}, e_{p,q} \notin C_{\mathfrak{g}}(t)$, it holds that

$$[t, e_{i,j}] = C_i e_{i,j} \neq 0 \quad \text{and} \quad [t, e_{p,q}] = C_p e_{p,q} \neq 0.$$

We use $e_{i,j}$ (with $1 \leq i \leq r$ and $1 \leq j \leq k_i$) to denote the $(k_i \times 1)$–column vector with 1 on position i and 0 on the other entries. Hence, $C_i e_{i,j}$ is a matrix multiplication. By considering several cases, we can see that $C_i e_{i,j}$ and $C_p e_{p,q}$ are linearly independent.

- Suppose that $i \neq p$, then $C_i e_{i,j} \in \langle e_{i,1}, e_{i,2}, \ldots, e_{i,k_i} \rangle$ holds, whereas we have $C_p e_{p,q} \in \langle e_{p,1}, e_{p,2}, \ldots, e_{p,k_p} \rangle$. This shows that these vectors are linearly independent.
- When $i = p$, we may assume that $1 \leq j < q \leq k_i$.
  - If $q < k_i$, then $C_i e_{i,j} = e_{i,j+1}$ and $C_i e_{i,q} = e_{i,q+1}$ are clearly linearly independent.
  - When $q = k_i$, we have that $C_i e_{i,k_i} = -\alpha_0 e_{i,1} - \alpha_1 e_{i,2} - \cdots - \alpha_{k_i-1} e_{i,k_i}$. Note that $\alpha_0 \neq 0$, since if $\alpha_0 = 0$, then also $\alpha_1 = \cdots = \alpha_{k_i-1} = 0$ and $e_{i,k_i} \in C_{\mathfrak{g}}(t)$. Hence, we obtain that $C_i e_{i,j}$ and $C_i e_{q,k_i}$ are linearly independent in this case as well.

We find that

$$\begin{aligned} D(e_{i,j} + e_{p,q}) &= [e_{i,j} + e_{p,q}, \alpha(e_{i,j} + e_{p,q})t] \\ &= -\alpha(e_{i,j} + e_{p,q}) C_i e_{i,j} - \alpha(e_{i,j} + e_{p,q}) C_p e_{p,q}. \end{aligned} \tag{12.2}$$

Further, we also have

$$\begin{aligned} D(e_{i,j}) + D(e_{p,q}) &= [e_{i,j}, \alpha(e_{i,j})t] + [e_{p,q}, \alpha(e_{p,q})t] \\ &= -\alpha(e_{i,j}) C_i e_{i,j} - \alpha(e_{p,q}) C_p e_{p,q}. \end{aligned} \tag{12.3}$$

Using the facts that (12.2) and (12.3) must coincide and that $C_i e_{i,j}$ and $C_p e_{p,q}$ are linearly independent, we finally find that

$$\alpha(e_{i,j}) = \alpha(e_{i,j} + e_{p,q}) = \alpha(e_{p,q}).$$

Since $e_{i,j}$ and $e_{p,q}$ were arbitrarily chosen, there exists a fixed value $\alpha \in \mathbb{F}$ such that when $e_{i,j} \notin C_{\mathfrak{g}}(t)$, we have that $D(e_{i,j}) = [e_{i,j}, \alpha t]$ for all $1 \leq i \leq r$ and $1 \leq j \leq k_i$. Since $0 = D(t) = [t, \alpha t]$, it follows that D coincides with $\mathrm{ad}(-\alpha t)$ on all basis vectors. Hence, $D = \mathrm{ad}(-\alpha t) \in \mathrm{Inn}(\mathfrak{g})$ is an inner derivation. $\square$

Note that a standard graded filiform Lie algebra is almost abelian, so in fact, Proposition 10.1.2 is a special case of the last result. In some ways, this is the most general property we can get: when we slightly change the conditions, there exist Lie algebras with non-trivial almost inner derivations. For instance, the result cannot be extended to Lie algebras $\mathfrak{g}$ of the form $\mathfrak{g} \cong \mathbb{F}^n \rtimes \mathbb{F}^2$.

Example 12.2.3. Let $n \geq 3$ and consider the Lie algebra $\mathfrak{g}$ over $\mathbb{F}$ with basis $\{e_1, e_2, \ldots, e_n, s, t\}$ and non-vanishing Lie brackets

$$[s, e_i] = e_{i+1}, \quad 1 \leq i \leq n-1,$$

$$[t, e_i] = e_{i+2}, \quad 1 \leq i \leq n-2.$$

Then we have $\mathfrak{g} = \mathbb{F}^n \rtimes \mathbb{F}^2$. Let $D \in \mathrm{Der}(\mathfrak{g})$ be the derivation which maps $a_1 e_1 + \cdots + a_n e_n + bs + ct$ to ce_n. Define the map $\varphi_D : \mathfrak{g} \to \mathfrak{g}$ as

$$a_1 e_1 + \cdots + a_n e_n + bs + ct \mapsto \begin{cases} \frac{c}{b} e_{n-1} & \text{if } b \neq 0, \\ e_{n-2} & \text{if } b = 0. \end{cases}$$

For all $x \in \mathfrak{g}$, we have that $D(x) = [x, \varphi_D(x)]$, showing that $D \in \mathrm{AID}(\mathfrak{g})$. It is easy to see that $D \notin \mathrm{Inn}(\mathfrak{g})$. Hence we have $\mathrm{AID}(\mathfrak{g}) \neq \mathrm{Inn}(\mathfrak{g})$.

The result of Theorem 12.2.2 can also not be generalised to Lie algebras of the form $\mathfrak{g} \cong \mathfrak{f}_{r,c} \rtimes \mathbb{F}$ where $\mathfrak{f}_{r,c}$ is a free nilpotent Lie algebra on r generators and of class $c > 1$.

Example 12.2.4. Let $\mathfrak{f}_{3,2}$ be the free 2–step nilpotent Lie algebra on 3 generators, then $\mathfrak{f}_{3,2}$ has a basis $\{x_1, x_2, x_3, y_1, y_2, y_3\}$ and non-trivial brackets

$$[x_1, x_2] = y_1, \qquad [x_1, x_3] = y_2 \quad \text{and} \quad [x_2, x_3] = y_3.$$

Add one more generator t and one extra non-trivial bracket

$$[t, x_1] = y_3$$

to obtain a 7-dimensional Lie algebra $\mathfrak{g} := \mathfrak{f}_{3,2} \rtimes \mathbb{F}$. Define $D : \mathfrak{g} \to \mathfrak{g}$ by

$$a_1 x_1 + a_2 x_2 + a_3 x_3 + b_1 y_1 + b_2 y_2 + b_3 y_3 + ct \mapsto a_1(y_1 + y_2).$$

Again, it is obvious that D is a derivation of $\mathfrak{g}$. Define $\varphi_D : \mathfrak{g} \to \mathfrak{g}$ by

$$a_1 x_1 + a_2 x_2 + a_3 x_3 + b_1 y_1 + b_2 y_2 + b_3 y_3 + ct \mapsto \begin{cases} x_2 + x_3 + \frac{a_2 - a_3}{a_1} t & \text{if } a_1 \neq 0, \\ 0 & \text{if } a_1 = 0. \end{cases}$$

Then $D(x) = [x, \varphi_D(x)]$ for all $x \in \mathfrak{g}$, showing that $D \in \mathrm{AID}(\mathfrak{g})$. It is easy to see that $D \notin \mathrm{Inn}(\mathfrak{g})$, and so also in this case we have that $\mathrm{AID}(\mathfrak{g}) \neq \mathrm{Inn}(\mathfrak{g})$.

12.3 Lie algebras whose solvable radical is abelian

In this section, we consider an algebraically closed field $\mathbb{F}$ of characteristic zero. Let $\mathfrak{g}$ be a Lie algebra $\mathfrak{g}$ over $\mathbb{F}$ whose solvable radical $\mathfrak{a} := \mathrm{Rad}(\mathfrak{g})$ is abelian. We will show that $\mathrm{AID}(\mathfrak{g}) = \mathrm{Inn}(\mathfrak{g})$ holds. The results of this section also appeared in [8].

By the Levi decomposition, we can write $\mathfrak{g} = \mathfrak{a} \rtimes_\rho \mathfrak{s}$, where $\mathfrak{s}$ is a semisimple Lie algebra and $\rho : \mathfrak{s} \to \mathfrak{gl}(\mathfrak{a})$ is a representation of $\mathfrak{s}$ on $\mathfrak{a}$. Take arbitrary $a_1, a_2 \in \mathfrak{a}$ and $s_1, s_2 \in \mathfrak{s}$. We will denote by $s_1 \cdot a_1 = \rho(s_1)(a_1) = [s_1, a_1]$ the $\mathfrak{s}$–module structure of $\mathfrak{a}$. Then the Lie bracket in $\mathfrak{g}$ is given by

$$[(a_1, s_1), (a_2, s_2)] = (s_1 \cdot a_2 - s_2 \cdot a_1, [s_1, s_2]),$$

since $\mathfrak{a}$ is abelian. We will illustrate the above notions with an example.

Example 12.3.1. Denote $\mathfrak{sl}_2(\mathbb{C}) = \{A \in M_2(\mathbb{C}) \mid \mathrm{tr}(A) = 0\}$ for the Lie subalgebra of $\mathfrak{gl}_2(\mathbb{C})$. Define the matrices

$$x_1 := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad x_2 := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad x_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then $\{x_1, x_2, x_3\}$ is a basis for $\mathfrak{sl}_2(\mathbb{C})$ and the non-zero Lie brackets are given by

$$[x_1, x_2] = x_3, \qquad [x_1, x_3] = -2x_1 \quad \text{and} \quad [x_2, x_3] = 2x_2. \tag{12.4}$$

Note that $\mathfrak{sl}_2(\mathbb{C})$ is a semisimple Lie algebra.

Take $n \geq 2$ and let $\mathfrak{a}_n = \langle y_1, \dots, y_{n+1} \rangle$ be the abelian Lie algebra of dimension $n + 1$. Define the linear map

$$\rho : \mathfrak{sl}_2(\mathbb{C}) \to \mathfrak{gl}(\mathfrak{a}_n),$$

where ρ is determined by

$$\rho(x_1) = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 2 & & 0 \\ \vdots & \vdots & \ddots & \ddots & \\ 0 & 0 & 0 & \ddots & n \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \qquad \rho(x_2) = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ n & 0 & \cdots & 0 & 0 \\ 0 & n-1 & \ddots & 0 & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & 0 & & 1 & 0 \end{pmatrix},$$

$$\rho(x_3) = \begin{pmatrix} n & 0 & \cdots & 0 & 0 \\ 0 & n-2 & & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & -n+2 & 0 \\ 0 & 0 & \cdots & 0 & -n \end{pmatrix}.$$

It can be shown that ρ is a Lie algebra homomorphism and thus also a representation of $\mathfrak{sl}_2(\mathbb{C})$ on $\mathfrak{a}_n$. Consider the Lie algebra $\mathfrak{g}_n := \mathfrak{a}_n \rtimes_\rho \mathfrak{sl}_2(\mathbb{C})$. In addition to the Lie brackets from (12.4), we also have

$$\begin{aligned} x_1 \cdot y_i &= (i-1)y_{i-1}, && 2 \le i \le n+1, \\ x_2 \cdot y_i &= (n+1-i)y_{i+1}, && 1 \le i \le n, \\ x_3 \cdot y_i &= (n+2-2i)y_i, && 1 \le i \le n+1. \end{aligned}$$

Note that $[\mathfrak{g}_n, \mathfrak{g}_n] = \mathfrak{g}_n$, which means that $\mathfrak{g}_n$ is a perfect Lie algebra. Further, the solvable radical $\mathfrak{a}_n$ is abelian.

In the sequel, we will use $\mathrm{End}_\mathfrak{s}(\mathfrak{a})$ to denote

$$\{\varphi : \mathfrak{a} \to \mathfrak{a} \mid \varphi \text{ is linear and } \varphi(s \cdot a) = s \cdot \varphi(a), \text{ for all } s \in \mathfrak{s} \text{ and all } a \in \mathfrak{a}\},$$

the space of $\mathfrak{s}$–endomorphisms of $\mathfrak{a}$. For any $\varphi \in \mathrm{End}_\mathfrak{s}(\mathfrak{a})$, we define

$$D_\varphi : \mathfrak{a} \rtimes_\rho \mathfrak{s} \to \mathfrak{a} \rtimes_\rho \mathfrak{s} : (a, s) \mapsto (\varphi(a), 0).$$

We further introduce the set $\mathfrak{D} := \{D_\varphi \mid \varphi \in \mathrm{End}_\mathfrak{s}(\mathfrak{a})\}$.

Lemma 12.3.2. *With the notations from above, we have* $\mathfrak{D} \subseteq \mathrm{Der}(\mathfrak{a} \rtimes_\rho \mathfrak{s})$.

Proof. Take an arbitrary $D_\varphi \in \mathfrak{D}$, so $\varphi \in \mathrm{End}_\mathfrak{s}(\mathfrak{a})$. Let $a_1, a_2 \in \mathfrak{a}$ and $s_1, s_2 \in \mathfrak{s}$. On the one hand, we have that

$$\begin{aligned} D_\varphi([(a_1, s_1), (a_2, s_2)]) &= D_\varphi(s_1 \cdot a_2 - s_2 \cdot a_1, [s_1, s_2]) \\ &= (\varphi(s_1 \cdot a_2 - s_2 \cdot a_1), 0), \end{aligned} \tag{12.5}$$

while on the other hand

$$\begin{aligned} &[D_\varphi(a_1, s_1), (a_2, s_2)] + [(a_1, s_1), D_\varphi(a_2, s_2)] \\ &\quad = [(\varphi(a_1), 0), (a_2, s_2)] + [(a_1, s_1), (\varphi(a_2), 0)] \\ &\quad = (-s_2 \cdot \varphi(a_1), 0) + (s_1 \cdot \varphi(a_2), 0) \\ &\quad = (-s_2 \cdot \varphi(a_1) + s_1 \cdot \varphi(a_2), 0). \end{aligned} \tag{12.6}$$

Since $\varphi \in \mathrm{End}_\mathfrak{s}(\mathfrak{a})$, it holds that (12.5) equals (12.6). Hence, we have that $D_\varphi \in \mathrm{Der}(\mathfrak{a} \rtimes_\rho \mathfrak{s})$ and this finishes the proof. □

Proposition 12.3.3. *As vector spaces, we have that*

$$\mathrm{Der}(\mathfrak{a} \rtimes_\rho \mathfrak{s}) = \mathrm{Inn}(\mathfrak{a} \rtimes_\rho \mathfrak{s}) \oplus \mathfrak{D}.$$

Proof. Take an arbitrary $D_\varphi \in \mathfrak{D}$, so $\varphi \in \mathrm{End}_{\mathfrak{s}}(\mathfrak{a})$ and suppose that D_φ is an inner derivation. Then there exists $(a_1, s_1) \in \mathfrak{a} \rtimes_\rho \mathfrak{s}$ such that $D_\varphi = \mathrm{ad}((a_1, s_1))$. For arbitrary $(a_2, s_2) \in \mathfrak{a} \rtimes_\rho \mathfrak{s}$, we have

$$\begin{aligned}(\varphi(a_2), 0) &= D_\varphi(a_2, s_2)\\ &= [(a_1, s_1), (a_2, s_2)]\\ &= (s_1 \cdot a_2 - s_2 \cdot a_1, [s_1, s_2])\end{aligned}$$

and this means that $\varphi : \mathfrak{a} \to \mathfrak{a}$ has to be the zero map. This implies that $\mathrm{Inn}(\mathfrak{a} \rtimes_\rho \mathfrak{s}) \cap \mathfrak{D} = \{0\}$, so we have to show that $\mathrm{Der}(\mathfrak{a} \rtimes_\rho \mathfrak{s}) = \mathrm{Inn}(\mathfrak{a} \rtimes_\rho \mathfrak{s}) + \mathfrak{D}$. Consider any $D \in \mathrm{Der}(\mathfrak{a} \rtimes_\rho \mathfrak{s})$. The derivation D induces a derivation on $\mathfrak{s}$, which is an inner derivation, since $\mathfrak{s}$ is semisimple. So, after changing D up to an inner derivation, we may assume that D induces the zero map on $\mathfrak{s}$. It follows that there exists a linear map $f : \mathfrak{s} \to \mathfrak{a}$ such that $D(0, s) = (f(s), 0)$ for all $s \in \mathfrak{s}$. Using this observation and the fact that D is a derivation, we have that

$$\begin{aligned}(f([s_1, s_2]), 0) &= D(0, [s_1, s_2])\\ &= D([(0, s_1), (0, s_2)])\\ &= [D(0, s_1), (0, s_2)] + [(0, s_1), D(0, s_2)]\\ &= [(f(s_1), 0), (0, s_2)] + [(0, s_1), (f(s_2), 0)]\\ &= [-s_2 \cdot f(s_1), 0] + [s_1 \cdot f(s_2), 0],\end{aligned}$$

where we also use the definition of the Lie brackets. This implies that

$$f([s_1, s_2]) = s_1 \cdot f(s_2) - s_2 \cdot f(s_1).$$

Hence, $f \in Z^1(\mathfrak{s}, \mathfrak{a})$ is a 1–cocycle. As $\mathfrak{s}$ is semisimple, we have that

$$H^1(\mathfrak{s}, \mathfrak{a}) = 0$$

by the first Whitehead lemma and so there exists an element $a_0 \in \mathfrak{a}$ such that $f(s) = s \cdot a_0$ for all $s \in \mathfrak{s}$. Moreover, we have that

$$(D + \mathrm{ad}((a_0, 0)))(0, s) = (f(s), 0) + [(a_0, 0), (0, s)] = (0, 0).$$

This means that, after changing D with an inner derivation, we can assume that $D(\mathfrak{s}) = 0$. Thus, there is a linear map $\varphi : \mathfrak{a} \to \mathfrak{a}$ such that $D(a, s) = (\varphi(a), 0)$

holds for all $a \in \mathfrak{a}$ and all $s \in \mathfrak{s}$. Since D is a derivation, we must have

$$\begin{aligned} D(-s \cdot a, 0) &= D([(a,0),(0,s)]) \\ &= [D(a,0),(0,s)] + [(a,0), D(0,s)] \\ &= [(\varphi(a),0),(0,s)] + (0,0) \\ &= (-s \cdot \varphi(a), 0), \end{aligned}$$

which implies that $\varphi(s \cdot a) = s \cdot \varphi(a)$. This shows that, after changing D up to an inner derivation, we have that $D = D_\varphi \in \mathfrak{D}$, which finishes the proof. ☐

We are now ready to prove the main result of this section.

Theorem 12.3.4. *Let $\mathfrak{g}$ be a Lie algebra over an algebraically closed field $\mathbb{F}$ of characteristic zero whose solvable radical is abelian. Then* $\operatorname{AID}(\mathfrak{g}) = \operatorname{Inn}(\mathfrak{g})$.

Proof. As before, we can write $\mathfrak{g} = \mathfrak{a} \rtimes_\rho \mathfrak{s}$, where $\mathfrak{a}$ is the abelian radical and $\mathfrak{s}$ is semisimple. It follows from the last proposition that $\operatorname{Der}(\mathfrak{g}) = \operatorname{Inn}(\mathfrak{g}) \oplus \mathfrak{D}$. Consider a nonzero $D \in \mathfrak{D}$. In order to prove the result, we have to show that D is not an almost inner derivation. By definition, we have $D = D_\varphi$ for some nonzero $\varphi \in \operatorname{End}_{\mathfrak{s}}(\mathfrak{a})$. Let $V = \varphi(\mathfrak{a})$ be the image of φ. Then V is a nonzero $\mathfrak{s}$–submodule of $\mathfrak{a}$. It follows from for example [81, Section 35] that the Lie algebra $\mathfrak{s}$ contains a so-called 'distinguished' element, which is a nilpotent element s_0 such that $C_{\mathfrak{s}}(s_0)$ consists entirely of nilpotent elements. Engel's theorem implies that $C_{\mathfrak{s}}(s_0)$ is also nilpotent as a Lie algebra. Consider the map $\psi : \mathfrak{s} \to \operatorname{End}(V) : s \mapsto \psi(s)$, where $\psi(s)(v) = s \cdot v$. By definition, ψ is a representation of Lie algebras and since $\mathfrak{s}$ is semisimple, ψ maps nilpotent elements to nilpotent elements. It follows that $\psi(C_{\mathfrak{s}}(s_0))$ consists of nilpotent endomorphisms. In particular, $\psi(C_{\mathfrak{s}}(s_0))(V) = C_{\mathfrak{s}}(s_0) \cdot V$ is strictly contained in V. Let $v_0 \in V \backslash (C_{\mathfrak{s}}(s_0) \cdot V)$ and pick an $a_0 \in \mathfrak{a}$ with $\varphi(a_0) = v_0$. Note that since $\mathfrak{s}$ is semisimple, we can find a complementary $\mathfrak{s}$–submodule W of V in $\mathfrak{a}$ such that $\mathfrak{a}$ decomposes as a direct sum $\mathfrak{a} = V \oplus W$ of $\mathfrak{s}$–modules. Hence, we also find that $v_0 \in \mathfrak{a} \setminus C_{\mathfrak{s}}(s_0) \cdot \mathfrak{a}$.

We will prove by contradiction that $D_\varphi(a_0, s_0) \notin [(a_0, s_0), \mathfrak{g}]$. Assume that $D_\varphi(a_0, s_0) = [(a_0, s_0), (a, s)]$ for some $a \in \mathfrak{a}$ and $s \in \mathfrak{s}$. Then we have that

$$(\varphi(a_0), 0) = [(a_0, s_0), (a, s)] = (s_0 \cdot a - s \cdot a_0, [s_0, s]).$$

This shows that $[s_0, s] = 0$ and so $s \in C_{\mathfrak{s}}(s_0)$. However, this implies that

$$v_0 = \varphi(a_0) = s_0 \cdot a - s \cdot a_0 \in C_{\mathfrak{s}}(s_0) \cdot \mathfrak{a},$$

which contradicts the fact that we have chosen v_0 such that $v_0 \in \mathfrak{a} \setminus C_{\mathfrak{s}}(s_0) \cdot \mathfrak{a}$. Hence, we have that $D_\varphi(a_0, s_0) \notin [(a_0, s_0), \mathfrak{g}]$, so D_φ is not an almost inner derivation. □

Example 12.3.5. Consider the Lie algebra $\mathfrak{g}_n := \mathfrak{a}_n \rtimes_\rho \mathfrak{sl}_2(\mathbb{C})$ with $n \geq 2$. It is clear that $\mathrm{Id} : \mathfrak{a}_n \to \mathfrak{a}_n$ is a $\mathfrak{sl}_2(\mathbb{C})$–equivariant linear map. The only invariant subspaces of $\mathfrak{a}_n$ are the zero space and $\mathfrak{a}_n$ itself. Hence, $\mathfrak{a}_n$ is an irreducible $\mathfrak{sl}_2(\mathbb{C})$–module. Schur's lemma implies that if $\varphi : \mathfrak{a}_n \to \mathfrak{a}_n$ is an endomorphism, then φ is a scalar multiple of Id. This shows that

$$\mathrm{Der}(\mathfrak{g}_n) = \mathrm{Inn}(\mathfrak{g}_n) \oplus \langle D_{\mathrm{Id}} \rangle,$$

where $D_{\mathrm{Id}} : \mathfrak{a}_n \rtimes_\rho \mathfrak{sl}_2(\mathbb{C}) \to \mathfrak{a}_n \rtimes_\rho \mathfrak{sl}_2(\mathbb{C}) : (a, s) \mapsto (a, 0)$. To prove that the only almost inner derivations are the inner ones, it suffices to show that D_{Id} is not almost inner. Note that $s_0 := x_1$ is nilpotent with $C_{\mathfrak{s}}(s_0) = \langle x_1 \rangle$. We further have that $V = \varphi(\mathfrak{a}) = \mathfrak{a}$ and this means that

$$\psi(C_{\mathfrak{s}}(s_0))(V) = C_{\mathfrak{s}}(s_0) \cdot V = \langle y_1, \ldots, y_n \rangle.$$

We can take $v_0 = a_0 = y_{n+1}$. Suppose that D_{Id} is almost inner. Take an arbitrary $(a, s) \in \mathfrak{a} \rtimes_\rho \mathfrak{s}$, then

$$[(y_{n+1}, x_1), (a, s)] = (x_1 \cdot a - s \cdot y_{n+1}, [x_1, s]).$$

Since $D_{\mathrm{Id}}(y_{n+1}, x_1) = (y_{n+1}, 0)$, we must have that $s \in \langle x_1 \rangle$. However, this leads to a contradiction, because $y_{n+1} \notin [x_1, \mathfrak{a}]$. As a result, D_{Id} is not almost inner and $\mathrm{AID}(\mathfrak{g}_n) = \mathrm{Inn}(\mathfrak{g}_n)$.

12.4 Characteristically nilpotent Lie algebras

Over a field of characteristic zero, a Lie algebra with a nonsingular derivation is nilpotent. This result was found by Jacobson ([50]) and he conjectured that the opposite is also true. However, Dixmier and Lister ([24]) gave a counterexample of dimension 8, which was already studied in detail in Chapter 5. Their discovery led to the study of a new type of Lie algebras, which are called 'characteristically nilpotent'.

Definition 12.4.1 (CNLA)**.** A Lie algebra whose derivations all are nilpotent is called a characteristically nilpotent Lie algebra (CNLA).

Since then, a lot of properties about characteristically nilpotent Lie algebras have been studied. For instance, a Lie algebra is characteristically nilpotent if and only if it has at least dimension two and its derivation algebra is nilpotent

([58]). If all derivations are nilpotent, then the Lie algebra itself is also nilpotent. Hence, a CNLA has to be nilpotent. Further, a direct sum of characteristically nilpotent Lie algebras is again a CNLA, which is useful for constructing different examples. Moreover, the nilindex of a CNLA must be at least three ([58]). This occurs for the Dixmier-Lister Lie algebra. More interesting facts can be found in the survey [2].

Let $\mathfrak{g}$ be a CNLA. One of the objectives of this thesis was to find which of the inclusions

$$\mathrm{Inn}(\mathfrak{g}) \subseteq \mathrm{CAID}(\mathfrak{g}) \subseteq \mathrm{AID}(\mathfrak{g}) \subseteq \mathrm{Der}(\mathfrak{g}). \tag{12.7}$$

are equalities for different classes of Lie algebras. Throughout this thesis, we discussed a few possibilities for characteristically nilpotent Lie algebras. For the Dixmier-Lister algebra $\mathfrak{g}$ over a field $\mathbb{F}$, we found that the dimension of $\mathrm{AID}(\mathfrak{g})$ depends on the number of different roots of $X^3 - 1$ over $\mathbb{F}$. It turned out that if this polynomial splits over $\mathbb{F}$, the only almost inner derivations are the inner ones. The opposite case is where all derivations are almost inner. Suppose that $\mathfrak{g}$ is nilpotent. Proposition 4.1.8 implies that all $D \in \mathrm{AID}(\mathfrak{g})$ are nilpotent as well. This means that $\mathrm{Der}(\mathfrak{g}) = \mathrm{AID}(\mathfrak{g})$ only can happen when $\mathfrak{g}$ is a CNLA. In Section 10.3, we studied a class of filiform and characteristically nilpotent Lie algebras $\mathfrak{f}_n$ (for $n \geq 13$) where this is the case. In this section, we will give an example of a CNLA for which all of the sets from (12.7) are different. This Lie algebra was first studied by Favre ([27]), thereby showing that there exists a CNLA of dimension 7.

Example 12.4.2. Let $\mathbb{F}$ be an arbitrary field. Consider the Lie algebra $\mathfrak{g}$ over $\mathbb{F}$ with basis $\{e_1, \dots, e_7\}$ and non-zero Lie brackets

$$\begin{aligned}
&[e_1, e_2] = e_3, \quad &[e_1, e_3] = e_4, \quad &[e_1, e_4] = e_5,\\
&[e_1, e_5] = e_6, \quad &[e_1, e_6] = e_7, \quad &[e_2, e_3] = -e_6,\\
&[e_2, e_4] = -e_7, \quad &[e_2, e_5] = -e_7, \quad &[e_3, e_4] = e_7.
\end{aligned} \tag{12.8}$$

An arbitrary derivation for $\mathfrak{g}$ is given by

$$D = a_1\mathrm{ad}(e_1) + \cdots + a_6\mathrm{ad}(e_6) + d_1 D_1 + \cdots + d_4 D_4$$

and has matrix form

$$\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0\\
-2d_3 & 0 & 0 & 0 & 0 & 0 & 0\\
-a_2 & a_1 & 0 & 0 & 0 & 0 & 0\\
-a_3 & d_3 & a_1 & 0 & 0 & 0 & 0\\
-a_4 & d_4 & e_1 & a_1 & 0 & 0 & 0\\
-a_5 & a_3 + d_1 & d_4 - a_2 & 3d_3 & a_1 & 0 & 0\\
-a_6 & a_4 + a_5 + d_2 & -a_4 + d_1 & a_3 - a_2 + d_4 & -a_2 + 5d_3 & a_1 + 2d_3 & 0
\end{pmatrix},$$

which means that $\mathrm{Der}(\mathfrak{g}) = \langle \mathrm{ad}(e_1), \dots, \mathrm{ad}(e_6), D_1, \dots, D_4 \rangle$. We will show that $\mathrm{AID}(\mathfrak{g}) = \mathrm{Inn}(\mathfrak{g}) \oplus \langle D_1, D_2 \rangle$ holds. Consider the map $\varphi_{D_1} : \mathfrak{g} \to \mathfrak{g}$, where the image of $x = \sum_{i=1}^{7} x_i e_i$ is given by

$$\varphi_{D_1}(x) = \begin{cases} \frac{1}{x_1}(x_2 e_5 + \frac{(x_2^2 + x_1 x_3)}{x_1} e_6) & \text{if } x_1 \neq 0, \\ \frac{1}{x_2}(-x_2 e_3 + (x_4 - x_3) e_5) & \text{if } x_1 = 0 \text{ and } x_2 \neq 0, \\ e_4 & \text{if } x_1 = x_2 = 0. \end{cases}$$

It is an easy computation to see that $D_1(x) = [x, \varphi_{D_1}(x)]$ for all $x \in \mathfrak{g}$. Consider the map $\varphi_{D_2} : \mathfrak{g} \to \mathfrak{g}$, where $x = \sum_{i=1}^{7} x_i e_i$ is mapped to

$$\varphi_{D_2}(x) = \begin{cases} \frac{x_2}{x_1} e_6 & \text{if } x_1 \neq 0, \\ -e_5 & \text{if } x_1 = 0. \end{cases}$$

A calculation shows that $D_2(x) = [x, \varphi_{D_2}(x)]$ for all $x \in \mathfrak{g}$. Let $a, b \in \mathbb{F}$ be arbitrary and define $D := aD_3 + bD_4$. Suppose that D is almost inner. We have that $D(e_2) = ae_4 + be_5$. Since $[e_2, \mathfrak{g}] \subseteq \langle e_3, e_6, e_7 \rangle$, it is clear that $a = b = 0$. Hence, no non-zero linear combination of D_3 and D_4 is almost inner. This gives an example of a Lie algebra $\mathfrak{g}$ with

$$\mathrm{Inn}(\mathfrak{g}) \subsetneq \mathrm{CAID}(\mathfrak{g}) \subsetneq \mathrm{AID}(\mathfrak{g}) \subsetneq \mathrm{Der}(\mathfrak{g}),$$

since $D_2 \in \mathrm{CAID}(\mathfrak{g})$ and $D_1 \in \mathrm{AID}(\mathfrak{g}) \setminus \mathrm{CAID}(\mathfrak{g})$.

By studying the low-dimensional nilpotent Lie algebras from 8.1.3, it turns out that, when $\mathrm{char}(\mathbb{F}) \neq 2$, there are no examples of a CNLA over $\mathbb{F}$ of dimension at most six. Hence, the example of Favre is a CNLA of smallest dimension. One year after Favre, Bratzlavky ([4]) found a 6–dimensional Lie algebra $\mathfrak{g}$ over a field $\mathbb{F}$ which is characteristically nilpotent if and only if $\mathrm{char}(\mathbb{F}) = 2$. This Lie algebra has basis $\{e_1, \dots, e_6\}$ and non-zero Lie brackets

$$\begin{aligned} &[e_1, e_2] = e_3, \quad &[e_1, e_3] = e_4, \quad &[e_1, e_4] = e_5, \\ &[e_1, e_5] = e_6, \quad &[e_2, e_3] = e_5 + e_6, \quad &[e_2, e_4] = e_6. \end{aligned}$$

With the notation from 8.1.3, we have that $\mathfrak{g} = \mathfrak{g}_{6,2}^{(2)}$. For this Lie algebra, we already found that $\mathrm{AID}(\mathfrak{g}) = \mathrm{Inn}(\mathfrak{g}) \oplus \langle E_{6,2} \rangle$. When $\mathrm{char}(\mathbb{F}) \neq 2$, we have that $\mathfrak{g}$ is isomorphic to R_5 from Example 10.1.4, which is denoted in Chapter 8 as $\mathfrak{g}_{6,15}$. Hence, $\mathfrak{g}$ is a CNLA if and only if $\mathrm{char}(\mathbb{F}) = 2$.

Appendix

Appendix A

Tables

In this appendix, we list several low-dimensional Lie algebras. Most of them are denoted with $\mathfrak{g}_{i,j}$, where i is the dimension of the Lie algebra and j is the number in the classification used. Some Lie algebras also have parameters, which are described between parentheses. Each time, the first table contains all non-vanishing Lie brackets. The second table gives an overview of some properties. Let $\mathfrak{g}$ be a Lie algebra from the classification, then $c(\mathfrak{g})$ denotes the nilpotency class of $\mathfrak{g}$ and $d(\mathfrak{g})$ stands for the derived length (when these notions are well-defined). Further, we will write $I(\mathfrak{g})$ and $C(\mathfrak{g})$ instead of $\dim(\mathrm{Inn}(\mathfrak{g}))$ respectively $\dim(\mathrm{CAID}(\mathfrak{g}))$. Similarly, $\dim(\mathrm{AID}(\mathfrak{g}))$ and $\dim(\mathrm{Der}(\mathfrak{g}))$ are denoted with $A(\mathfrak{g})$ and $D(\mathfrak{g})$. If the entry in the column with 'D' is non-zero, it gives examples of almost inner derivations, which together with the inner derivations generate $\mathrm{AID}(\mathfrak{g})$.

Name	Non-vanishing Lie brackets
$\mathfrak{g}_{1,1}$	$-$
$\mathfrak{g}_{2,1}$	$-$
$\mathfrak{g}_{2,2}$	$[e_1, e_2] = e_2$
$\mathfrak{g}_{3,1}$	$-$
$\mathfrak{g}_{3,2}$	$[e_1, e_2] = e_2,\ [e_1, e_3] = e_3$
$\mathfrak{g}_{3,3}(\varepsilon)$	$[e_1, e_2] = e_3,\ [e_1, e_3] = \varepsilon e_2 + e_3$
$\mathfrak{g}_{3,4}(\varepsilon)$	$[e_1, e_2] = e_3,\ [e_1, e_3] = \varepsilon e_2$
$\mathfrak{g}(\alpha, \beta)$	$[e_1, e_2] = e_3,\ [e_1, e_3] = -\beta e_2,\ [e_2, e_3] = \alpha e_1$

Table A.1: Lie algebras of dimension at most 3 over an arbitrary field $\mathbb{F}$, where $\varepsilon \in \mathbb{F}$ and $\alpha, \beta \in \mathbb{F}^*$.

Name	$c(\mathfrak{g})$	$d(\mathfrak{g})$	$I(\mathfrak{g})$	$C(\mathfrak{g})$	$A(\mathfrak{g})$	$D(\mathfrak{g})$	D
$\mathfrak{g}_{1,1}$	1	1	0	0	0	1	0
$\mathfrak{g}_{2,1}$	1	1	0	0	0	4	0
$\mathfrak{g}_{2,2}$	-	2	2	2	2	2	0
$\mathfrak{g}_{3,1}$	1	1	0	0	0	9	0
$\mathfrak{g}_{3,2}$	-	2	3	3	3	6	0
$\mathfrak{g}_{3,3}(0)$	-	2	2	2	2	4	0
$\mathfrak{g}_{3,3}(\varepsilon^*)$	-	2	3	3	3	4	0
$\mathfrak{g}_{3,4}(0)$	2	2	2	2	2	6	0
$\mathfrak{g}_{3,4}(\varepsilon^*)$	-	2	3	3	3	4/5	0
$\mathfrak{g}(\alpha,\beta)$	-	-	3	3	3	3/5	0

Table A.2: Results for Lie algebras of dimension at most 3 over an arbitrary field $\mathbb{F}$, where $\varepsilon^*, \alpha, \beta \in \mathbb{F}^*$. Remark 8.1.8 contains comments on this table.

Name	Non-vanishing Lie brackets
$\mathfrak{g}_{4,1}$	$-$
$\mathfrak{g}_{4,2}$	$[e_1,e_2]=e_2,\ [e_1,e_3]=e_3,\ [e_1,e_4]=e_4$
$\mathfrak{g}_{4,3}(\varepsilon)$	$[e_1,e_2]=e_2,\ [e_1,e_3]=e_4,\ [e_1,e_4]=-\varepsilon e_3+(\varepsilon+1)e_4$
$\mathfrak{g}_{4,4}$	$[e_1,e_2]=e_3,\ [e_1,e_3]=e_3$
$\mathfrak{g}_{4,5}$	$[e_1,e_2]=e_3$
$\mathfrak{g}_{4,6}(\varepsilon,\delta)$	$[e_1,e_2]=e_3,\ [e_1,e_3]=e_4,\ [e_1,e_4]=\varepsilon e_2+\delta e_3+e_4$
$\mathfrak{g}_{4,7}(\varepsilon,\delta)$	$[e_1,e_2]=e_3,\ [e_1,e_3]=e_4,\ [e_1,e_4]=\varepsilon e_2+\delta e_3$
$\mathfrak{g}_{4,8}$	$[e_1,e_2]=e_2,\ [e_3,e_4]=e_4$
$\mathfrak{g}_{4,9}(\varepsilon)$	$[e_1,e_3]=e_3+\varepsilon e_4,\ [e_1,e_4]=e_3,\ [e_2,e_3]=e_3,$ $[e_2,e_4]=e_4$
$\mathfrak{g}_{4,10}(\varepsilon)$	$[e_1,e_3]=e_4,\ [e_1,e_4]=\varepsilon e_3,\ [e_2,e_3]=e_3,\ [e_2,e_4]=e_4$
$\mathfrak{g}_{4,11}(\varepsilon,\delta)$	$[e_1,e_2]=(1+\delta)e_2,\ [e_1,e_3]=\delta e_3,\ [e_1,e_4]=e_4,$ $[e_2,e_4]=e_3,\ [e_2,e_3]=\varepsilon e_4$
$\mathfrak{g}_{4,12}$	$[e_1,e_2]=e_2,\ [e_1,e_3]=e_3, [e_1,e_4]=2e_4,\ [e_2,e_3]=-e_4$
$\mathfrak{g}_{4,13}(\varepsilon)$	$[e_1,e_2]=e_3,\ [e_1,e_3]=\varepsilon e_2+e_3,\ [e_1,e_4]=e_4,$ $[e_2,e_3]=e_4$
$\mathfrak{g}_{4,14}(\varepsilon)$	$[e_1,e_2]=e_3,\ [e_1,e_3]=\varepsilon e_2,\ [e_2,e_3]=e_4$

Table A.3: Solvable Lie algebras of dimension 4 over an arbitrary field $\mathbb{F}$. The conditions on $\mathbb{F}$ and on the parameters are stated in Remark 8.1.10.

Name	$c(\mathfrak{g})$	$d(\mathfrak{g})$	$I(\mathfrak{g})$	$C(\mathfrak{g})$	$A(\mathfrak{g})$	$D(\mathfrak{g})$	D
$\mathfrak{g}_{4,1}$	1	1	0	0	0	16	0
$\mathfrak{g}_{4,2}$	-	2	4	4	4	12	0
$\mathfrak{g}_{4,3}(0)$	-	2	3	3	3	8	0
$\mathfrak{g}_{4,3}(\varepsilon)$	-	2	4	4	4	8	0
$\mathfrak{g}_{4,4}$	-	2	2	2	2	8	0
$\mathfrak{g}_{4,5}$	2	2	2	2	2	10	0
$\mathfrak{g}_{4,6}(0,\delta)$	-	2	3	3	3	6	0
$\mathfrak{g}_{4,6}(\varepsilon,\delta)$	-	2	4	4	4	6	0
$\mathfrak{g}_{4,7}(0,0)$	2	2	3	3	3	7	0
$\mathfrak{g}_{4,7}(0,\delta)$	-	2	3	3	3	6/7	0
$\mathfrak{g}_{4,7}(\varepsilon,0)$	-	2	4	4	4	6/7	0
$\mathfrak{g}_{4,7}(\varepsilon,\delta)$	-	2	4	4	4	6	0
$\mathfrak{g}_{4,8}$	-	2	4	4	4	4	0
$\mathfrak{g}_{4,9}(\varepsilon)$	-	2	4	4	4	4/5	0
$\mathfrak{g}_{4,10}(\varepsilon)$	-	2	4	4	4	6	0
$\mathfrak{g}_{4,11}(\varepsilon,\delta)$	-	3	4	4	4	5	0
$\mathfrak{g}_{4,11}(\varepsilon,\delta)$	-	3	4	5	5	5	$E_{3,3}+E_{4,4}$
$\mathfrak{g}_{4,12}$	-	3	3/4	3/4	3/4	7	0
$\mathfrak{g}_{4,13}(0)$	-	2	4	4	4	5/6	0
$\mathfrak{g}_{4,13}(\varepsilon)$	-	3	4	4	4	5	0
$\mathfrak{g}_{4,14}(\varepsilon)$	-	3	3	3	3	5/6	0

Table A.4: Results for the solvable Lie algebras of dimension 4 over an arbitrary field. Remark 8.1.12 contains comments on this table.

Name	Non-vanishing Lie brackets
$\mathfrak{g}_{5,1} - \mathfrak{g}_{6,1}$	$-$
$\mathfrak{g}_{5,2} - \mathfrak{g}_{6,2}$	$[e_1,e_2]=e_3$
$\mathfrak{g}_{5,3} - \mathfrak{g}_{6,3}$	$[e_1,e_2]=e_3,\ [e_1,e_3]=e_4$
$\mathfrak{g}_{5,4} - \mathfrak{g}_{6,4}$	$[e_1,e_2]=e_5,\ [e_3,e_4]=e_5$
$\mathfrak{g}_{5,5} - \mathfrak{g}_{6,5}$	$[e_1,e_2]=e_3,\ [e_1,e_3]=e_5,\ [e_2,e_4]=e_5$
$\mathfrak{g}_{5,6} - \mathfrak{g}_{6,6}$	$[e_1,e_2]=e_3,\ [e_1,e_3]=e_4,\ [e_1,e_4]=e_5,\ [e_2,e_3]=e_5$
$\mathfrak{g}_{5,7} - \mathfrak{g}_{6,7}$	$[e_1,e_2]=e_3,\ [e_1,e_3]=e_4,\ [e_1,e_4]=e_5$
$\mathfrak{g}_{5,8} - \mathfrak{g}_{6,8}$	$[e_1,e_2]=e_4,\ [e_1,e_3]=e_5$
$\mathfrak{g}_{5,9} - \mathfrak{g}_{6,9}$	$[e_1,e_2]=e_3,\ [e_1,e_3]=e_4,\ [e_2,e_3]=e_5$
$\mathfrak{g}_{6,10}$	$[e_1,e_2]=e_3,\ [e_1,e_3]=e_6,\ [e_4,e_5]=e_6$
$\mathfrak{g}_{6,11}$	$[e_1,e_2]=e_3,\ [e_1,e_3]=e_4,\ [e_1,e_4]=e_6,\ [e_2,e_3]=e_6,$ $[e_2,e_5]=e_6$
$\mathfrak{g}_{6,12}$	$[e_1,e_2]=e_3,\ [e_1,e_3]=e_4,\ [e_1,e_4]=e_6,\ [e_2,e_5]=e_6$
$\mathfrak{g}_{6,13}$	$[e_1,e_2]=e_3,\ [e_1,e_3]=e_5,\ [e_1,e_5]=e_6,\ [e_2,e_4]=e_5,$ $[e_3,e_4]=e_6$
$\mathfrak{g}_{6,14}$	$[e_1,e_2]=e_3,\ [e_1,e_3]=e_4,\ [e_1,e_4]=e_5,\ [e_2,e_3]=e_5,$ $[e_2,e_5]=e_6,\ [e_3,e_4]=-e_6$
$\mathfrak{g}_{6,15}$	$[e_1,e_2]=e_3,\ [e_1,e_3]=e_4,\ [e_1,e_4]=e_5,\ [e_1,e_5]=e_6,$ $[e_2,e_3]=e_5,\ [e_2,e_4]=e_6$
$\mathfrak{g}_{6,16}$	$[e_1,e_2]=e_3,\ [e_1,e_3]=e_4,\ [e_1,e_4]=e_5,$ $[e_2,e_5]=e_6,\ [e_3,e_4]=-e_6$
$\mathfrak{g}_{6,17}$	$[e_1,e_2]=e_3,\ [e_1,e_3]=e_4,\ [e_1,e_4]=e_5,\ [e_1,e_5]=e_6,$ $[e_2,e_3]=e_6$
$\mathfrak{g}_{6,18}$	$[e_1,e_2]=e_3,\ [e_1,e_3]=e_4,\ [e_1,e_4]=e_5,\ [e_1,e_5]=e_6$
$\mathfrak{g}_{6,19}(\varepsilon_1)$	$[e_1,e_2]=e_4,\ [e_1,e_3]=e_5,\ [e_1,e_5]=e_6,\ [e_2,e_4]=e_6,$ $[e_3,e_5]=\varepsilon_1 e_6$
$\mathfrak{g}_{6,20}$	$[e_1,e_2]=e_4,\ [e_1,e_3]=e_5,\ [e_1,e_5]=e_6,\ [e_2,e_4]=e_6$
$\mathfrak{g}_{6,21}(\varepsilon_1)$	$[e_1,e_2]=e_3,\ [e_1,e_3]=e_4,\ [e_1,e_4]=e_6,\ [e_2,e_3]=e_5,$ $[e_2,e_5]=\varepsilon_1 e_6$
$\mathfrak{g}_{6,22}(\varepsilon_2)$	$[e_1,e_2]=e_5,\ [e_1,e_3]=e_6,\ [e_2,e_4]=\varepsilon_2 e_6,\ [e_3,e_4]=e_5$
$\mathfrak{g}_{6,23}$	$[e_1,e_2]=e_3,\ [e_1,e_3]=e_5,\ [e_1,e_4]=e_6,\ [e_2,e_4]=e_5$
$\mathfrak{g}_{6,24}(\varepsilon_2)$	$[e_1,e_2]=e_3,\ [e_1,e_3]=e_5,\ [e_1,e_4]=\varepsilon_2 e_6, [e_2,e_3]=e_6,$ $[e_2,e_4]=e_5$
$\mathfrak{g}_{6,25}$	$[e_1,e_2]=e_3,\ [e_1,e_3]=e_5,\ [e_1,e_4]=e_6$
$\mathfrak{g}_{6,26}$	$[e_1,e_2]=e_4,\ [e_1,e_3]=e_5,\ [e_2,e_3]=e_6$
$\mathfrak{g}_{6,27}$	$[e_1,e_2]=e_3,\ [e_1,e_3]=e_5,\ [e_2,e_4]=e_6$
$\mathfrak{g}_{6,28}$	$[e_1,e_2]=e_3,\ [e_1,e_3]=e_4,\ [e_1,e_4]=e_5,\ [e_2,e_3]=e_6$

Table A.5: Nilpotent Lie algebras of dimension 5 and 6 over a field $\mathbb{F}$, where $\varepsilon_1 \in \mathbb{F}^*/(\overset{*}{\sim})$ and $\varepsilon_2 \in \mathbb{F}/(\overset{*}{\sim})$ when $\mathrm{char}(\mathbb{F}) \neq 2$ and $\varepsilon_2 \in \mathbb{F}/(\overset{*+}{\sim})$ for $\mathrm{char}(\mathbb{F}) = 2$.

Name ($\mathbb{F}=\mathbb{C}$)	$c(\mathfrak{g})$	$d(\mathfrak{g})$	$I(\mathfrak{g})$	$C(\mathfrak{g})$	$A(\mathfrak{g})$	$D(\mathfrak{g})$	D	$\mathbb{F}=\mathbb{C}$
$\mathfrak{g}_{5,1}-\mathfrak{g}_{6,1}$	1	1	0	0	0	25 - 36	0	$\mathbb{C}^6$
$\mathfrak{g}_{5,2}-\mathfrak{g}_{6,2}$	2	2	2	2	2	16 - 24	0	$\mathfrak{n}_3\oplus\mathbb{C}^3$
$\mathfrak{g}_{5,3}-\mathfrak{g}_{6,3}$	3	2	3	3	3	11 - 17	0	$\mathfrak{n}_4\oplus\mathbb{C}^2$
$\mathfrak{g}_{5,4}-\mathfrak{g}_{6,4}$	2	2	4	4	4	15 - 21	0	$\mathcal{G}_{5,1}\oplus\mathbb{C}$
$\mathfrak{g}_{5,5}-\mathfrak{g}_{6,5}$	3	2	4	5	5	10 - 15	$E_{5,4}$	$\mathcal{G}_{5,3}\oplus\mathbb{C}$
$\mathfrak{g}_{5,6}-\mathfrak{g}_{6,6}$	4	2	4	5	5	8 - 12	$E_{5,2}$	$\mathcal{G}_{5,6}\oplus\mathbb{C}$
$\mathfrak{g}_{5,7}-\mathfrak{g}_{6,7}$	4	2	4	4	4	9 - 13	0	$\mathcal{G}_{5,5}\oplus\mathbb{C}$
$\mathfrak{g}_{5,8}-\mathfrak{g}_{6,8}$	2	2	3	3	3	13 - 19	0	$\mathcal{G}_{5,2}\oplus\mathbb{C}$
$\mathfrak{g}_{5,9}-\mathfrak{g}_{6,9}$	3	2	3	3	3	10 - 15	0	$\mathcal{G}_{5,4}\oplus\mathbb{C}$
$\mathfrak{g}_{6,10}$	3	2	5	5	5	14	0	$\mathcal{G}_{6,2}$
$\mathfrak{g}_{6,11}$	4	2	5	5	5	11	0	$\mathcal{G}_{6,12}$
$\mathfrak{g}_{6,12}$	4	2	5	6	6	12	$E_{6,5}$	$\mathcal{G}_{6,11}$
$\mathfrak{g}_{6,13}$	4	2	5	6	6	10/11	$E_{6,4}$	$\mathcal{G}_{6,13}$
$\mathfrak{g}_{6,14}$	5	3	5	5	6	8/9	$E_{5,2}$	$\mathcal{G}_{6,20}$
$\mathfrak{g}_{6,15}$	5	2	5	6	6	9/10	$E_{6,2}$	$\mathcal{G}_{6,19}$
$\mathfrak{g}_{6,16}$	5	3	5	5	5	9/10	0	$\mathcal{G}_{6,18}$
$\mathfrak{g}_{6,17}$	5	2	5	6	6	10	$E_{6,2}$	$\mathcal{G}_{6,17}$
$\mathfrak{g}_{6,18}$	5	2	5	5	5	11	0	$\mathcal{G}_{6,16}$
$\mathfrak{g}_{6,19}(\varepsilon_1)$	3	2	5	5	5	11/12	0	$\mathcal{G}_{6,9}$
$\mathfrak{g}_{6,20}$	3	2	5	5	5	12	0	$\mathcal{G}_{6,10}$
$\mathfrak{g}_{6,21}(\varepsilon_1)$	4	2	5	5	5	10/11	0	$\mathcal{G}_{6,15}$
$\mathfrak{g}_{6,22}(0)$	2	2	4	6	6	17/18	$E_{6,1}, E_{6,3}$	$\mathcal{G}_{6,1}$
$\mathfrak{g}_{6,22}(\varepsilon_2)$	2	2	4	4	4	16/18	0	$\mathfrak{n}_3\oplus\mathfrak{n}_3$
$\mathfrak{g}_{6,22}(\varepsilon_2)$	2	2	4	8	8	16/18	$E_{6,1}, E_{6,2}, E_{6,3}, E_{6,4}$	-
$\mathfrak{g}_{6,23}$	3	2	4	6	6	14	$E_{6,1}, E_{5,4}$	$\mathcal{G}_{6,7}$
$\mathfrak{g}_{6,24}(0)$	3	2	4	6	6	13/14	$E_{6,2}, E_{5,4}$	$\mathcal{G}_{6,8}$
$\mathfrak{g}_{6,24}(\varepsilon_2)$	3	2	4	4	4	12/14	0	$\mathcal{G}_{6,5}$
$\mathfrak{g}_{6,24}(\varepsilon_2)$	3	2	4	8	8	12/14	$E_{6,1}, E_{6,2}, E_{6,3}, E_{6,4}$	-
$\mathfrak{g}_{6,25}$	3	2	4	4	4	15	0	$\mathcal{G}_{6,6}$
$\mathfrak{g}_{6,26}$	2	2	3	3	3	18	0	$\mathcal{G}_{6,3}$
$\mathfrak{g}_{6,27}$	3	2	4	4	4	13	0	$\mathcal{G}_{6,4}$
$\mathfrak{g}_{6,28}$	4	2	4	4	4	11	0	$\mathcal{G}_{6,14}$

Table A.6: Results for the nilpotent Lie algebras of dimension 5 and 6 over an arbitrary field $\mathbb{F}$, where $\varepsilon_1 \in \mathbb{F}^*/(\overset{*}{\sim})$ and $\varepsilon_2 \in \mathbb{F}^*/(\overset{*}{\sim})$ when $\mathrm{char}(\mathbb{F}) \neq 2$ and $\varepsilon_2 \in \mathbb{F}^*/(\overset{*+}{\sim})$ for $\mathrm{char}(\mathbb{F}) = 2$. Remark 8.1.17 contains comments on this table.

Name	Non-vanishing Lie brackets
$\mathfrak{g}_{6,1}^{(2)}$	$[e_1,e_2]=e_3,\ [e_1,e_3]=e_5,\ [e_1,e_5]=e_6,$ $[e_2,e_4]=e_5+e_6,\ [e_3,e_4]=e_6$
$\mathfrak{g}_{6,2}^{(2)}$	$[e_1,e_2]=e_3,\ [e_1,e_3]=e_4,\ [e_1,e_4]=e_5,\ [e_1,e_5]=e_6,$ $[e_2,e_3]=e_5+e_6,\ [e_2,e_4]=e_6$
$\mathfrak{g}_{6,3}^{(2)}(\varepsilon_3)$	$[e_1,e_2]=e_3,\ [e_1,e_3]=e_4,\ [e_1,e_4]=e_5,$ $[e_2,e_3]=e_5+\varepsilon_3e_6,\ [e_2,e_5]=e_6,\ [e_3,e_4]=e_6$
$\mathfrak{g}_{6,4}^{(2)}(\varepsilon_3)$	$[e_1,e_2]=e_3,\ [e_1,e_3]=e_4,\ [e_1,e_4]=e_5,\ [e_2,e_3]=\varepsilon_3e_6,$ $[e_2,e_5]=e_6,\ [e_3,e_4]=e_6$
$\mathfrak{g}_{6,5}^{(2)}$	$[e_1,e_2]=e_4,\ [e_1,e_3]=e_5,\ [e_2,e_5]=e_6,\ [e_3,e_4]=e_6$
$\mathfrak{g}_{6,6}^{(2)}$	$[e_1,e_2]=e_3,\ [e_1,e_3]=e_4,\ [e_1,e_5]=e_6,\ [e_2,e_3]=e_5,$ $[e_2,e_4]=e_6$
$\mathfrak{g}_{6,7}^{(2)}(\varepsilon_4)$	$[e_1,e_2]=e_5,\ [e_1,e_3]=e_6,\ [e_2,e_4]=\varepsilon_4e_6,$ $[e_3,e_4]=e_5+e_6$
$\mathfrak{g}_{6,8}^{(2)}(\varepsilon_4)$	$[e_1,e_2]=e_3,\ [e_1,e_3]=e_5,\ [e_1,e_4]=\varepsilon_4e_6,\ [e_2,e_3]=e_6,$ $[e_2,e_4]=e_5+e_6$

Table A.7: Additional nilpotent Lie algebras of dimension 6 over fields $\mathbb{F}$ with $\operatorname{char}(\mathbb{F})=2$, where $\varepsilon_3\in\mathbb{F}^*/(\overset{*+}{\sim})$ and $\varepsilon_4\in\mathbb{F}/(\overset{\psi}{\sim})$.

Name	$c(\mathfrak{g})$	$d(\mathfrak{g})$	$I(\mathfrak{g})$	$C(\mathfrak{g})$	$A(\mathfrak{g})$	$D(\mathfrak{g})$	D
$\mathfrak{g}_{6,1}^{(2)}$	4	2	5	6	6	10	$E_{6,4}$
$\mathfrak{g}_{6,2}^{(2)}$	5	2	5	6	6	9	$E_{6,2}$
$\mathfrak{g}_{6,3}^{(2)}(\varepsilon_3)$	5	3	5	5	6	9	$E_{5,2}$
$\mathfrak{g}_{6,4}^{(2)}(\varepsilon_3)$	5	3	5	5	5	9	0
$\mathfrak{g}_{6,5}^{(2)}$	3	2	5	5	5	13	0
$\mathfrak{g}_{6,6}^{(2)}$	3	2	5	5	5	12	0
$\mathfrak{g}_{6,7}^{(2)}(\varepsilon_4)$	2	2	4	4	4	16	0
$\mathfrak{g}_{6,7}^{(2)}(\varepsilon_4)$	2	2	4	8	8	16	$E_{6,1},E_{6,2},$ $E_{6,3},E_{6,4}$
$\mathfrak{g}_{6,8}^{(2)}(\varepsilon_4)$	3	2	4	4	4	12	0
$\mathfrak{g}_{6,8}^{(2)}(\varepsilon_4)$	3	2	4	8	8	12	$E_{6,1},E_{6,2},$ $E_{6,3},E_{6,4}$

Table A.8: Results for the additional nilpotent Lie algebras of dimension 6 over fields $\mathbb{F}$ with $\operatorname{char}(\mathbb{F})=2$, where $\varepsilon_3\in\mathbb{F}^*/(\overset{*+}{\sim})$ and $\varepsilon_4\in\mathbb{F}/(\overset{\psi}{\sim})$. A comment on this table can be found in Remark 8.1.17.

Name	Non-vanishing Lie brackets
$C_{5,1}$	$[e_1, e_2] = e_1,\ [e_3, e_4] = 2e_4,\ [e_3, e_5] = -2e_5,$ $[e_4, e_5] = e_3$
$C_{5,2}$	$[e_1, e_3] = -e_1,\ [e_1, e_5] = e_2,\ [e_2, e_3] = e_2,\ [e_2, e_4] = e_1,$ $[e_3, e_4] = 2e_4,\ [e_3, e_5] = -2e_5,\ [e_4, e_5] = e_3$
$C_{5,3}$	$[e_3, e_4] = 2e_4,\ [e_3, e_5] = -2e_5,\ [e_4, e_5] = e_3$
$C_{5,4}$	$[e_1, e_4] = e_1,\ [e_1, e_5] = e_1,\ [e_2, e_3] = e_1,\ [e_2, e_4] = e_2,$ $[e_3, e_5] = e_3$
$C_{5,5}(p, q, r)$	$[e_1, e_5] = (q - r)pe_1,\ [e_2, e_4] = e_1 + pe_2,$ $[e_2, e_5] = (r - q)e_1,\ [e_3, e_4] = e_2 + qe_3,$ $[e_3, e_5] = e_1 + re_2 + r(q - p)e_3$
$C_{5,6}(p, q)$	$[e_1, e_5] = -pe_1,\ [e_2, e_4] = e_1 + pe_2,\ [e_2, e_5] = e_1,$ $[e_3, e_4] = e_2 + qe_3,\ [e_3, e_5] = e_2 + (q - p)e_3$
$C_{5,7}$	$[e_2, e_4] = e_1 + e_2,\ [e_3, e_4] = e_2 + 2e_3,$ $[e_3, e_5] = e_1 + 2e_2 + 2e_3,\ [e_4, e_5] = e_3$
$C_{5,8}$	$[e_1, e_4] = e_1,\ [e_2, e_4] = e_2,\ [e_3, e_5] = e_3$
$C_{5,9}(p, q)$	$[e_1, e_5] = (2p + q)e_3,\ [e_2, e_3] = e_1,\ [e_2, e_5] = (p + q)e_2,$ $[e_3, e_4] = e_2,\ [e_3, e_5] = pe_3 + e_4,\ [e_4, e_5] = e_1 + qe_4$
$C_{5,10}$	$[e_1, e_5] = 3e_1,\ [e_2, e_3] = e_1,\ [e_2, e_5] = 2e_2,\ [e_3, e_4] = e_2,$ $[e_3, e_5] = e_3,\ [e_4, e_5] = e_1 + e_4$
$C_{5,11}$	$[e_1, e_5] = e_1,\ [e_2, e_3] = e_1,\ [e_2, e_5] = e_2,\ [e_3, e_4] = e_2,$ $[e_3, e_5] = e_4,\ [e_4, e_5] = e_4$
$C_{5,12}(p, q, r)$	$[e_1, e_5] = pe_1 + e_2,\ [e_2, e_5] = (q + r)e_2,\ [e_3, e_4] = e_2,$ $[e_3, e_5] = e_1 + qe_3,\ [e_4, e_5] = e_3 + re_4$
$C_{5,13}(p, q)$	$[e_1, e_5] = (p + q)e_1,\ [e_2, e_5] = (p + q)e_2,\ [e_3, e_4] = e_2,$ $[e_3, e_5] = e_1 + pe_3,\ [e_4, e_5] = e_3 + qe_4$
$C_{5,14}(p, q)$	$[e_1, e_5] = pe_1 + e_2,\ [e_2, e_5] = (p + q)e_2,\ [e_3, e_4] = e_2,$ $[e_3, e_5] = e_1 + qe_3 + e_4,\ [e_4, e_5] = pe_4$
$C_{5,15}(p, q)$	$[e_1, e_5] = pe_1 + e_2,\ [e_2, e_5] = 2qe_2,\ [e_3, e_4] = e_2,$ $[e_3, e_5] = e_1 + qe_3,\ [e_4, e_5] = qe_4$
$C_{5,16}$	$[e_1, e_5] = 2e_1,\ [e_2, e_5] = 2e_2,\ [e_3, e_4] = e_2,$ $[e_3, e_5] = e_1 + e_3,\ [e_4, e_5] = e_4$
$C_{5,17}$	$[e_1, e_5] = e_1 + e_2,\ [e_2, e_5] = 2e_2,\ [e_3, e_4] = e_2,$ $[e_3, e_5] = e_3,\ [e_4, e_5] = e_4$
$C_{5,18}$	$[e_1, e_5] = e_1,\ [e_2, e_5] = e_2,\ [e_3, e_4] = e_2,$ $[e_3, e_5] = e_1 + e_4,\ [e_4, e_5] = e_4$
$C_{5,19}$	$[e_1, e_5] = e_2,\ [e_3, e_4] = e_2$
$C_{5,20}(p, q, r, s)$	$[e_1, e_5] = pe_1,\ [e_2, e_5] = e_1 + qe_2,\ [e_3, e_5] = e_2 + re_3,$ $[e_4, e_5] = e_3 + se_4$

Table A.9: Lie algebras of dimension 5 over $\mathbb{C}$, where $p, q, r, s \in \mathbb{C}$.

Name	Non-vanishing Lie brackets
$C_{5,21}(p,q,r)$	$[e_1,e_5]=pe_1,\ [e_2,e_5]=pe_2,\ [e_3,e_5]=e_2+qe_3,$ $[e_4,e_5]=e_3+re_4$
$C_{5,22}(p,q)$	$[e_1,e_5]=pe_1,\ [e_2,e_5]=e_1+qe_2,\ [e_3,e_5]=pe_3,$ $[e_4,e_5]=e_3+qe_4$
$C_{5,23}(p,q)$	$[e_1,e_5]=pe_1,\ [e_2,e_5]=pe_2,\ [e_3,e_5]=pe_3,$ $[e_4,e_5]=e_3+qe_4$
$C_{5,24}$	$[e_1,e_5]=e_1,\ [e_2,e_5]=e_2,\ [e_3,e_5]=e_3,\ [e_4,e_5]=e_4$

Table A.9: Lie algebras of dimension 5 over $\mathbb{C}$, where $p,q,r,s\in\mathbb{C}$.

Name	Non-vanishing Lie brackets
$R_{5,7}(u,v,w)$	$[e_1,e_5]=e_1,\ [e_2,e_5]=ue_2,\ [e_3,e_5]=ve_3,$ $[e_4,e_5]=we_4$
$R_{5,8}(u)$	$[e_2,e_5]=e_1,\ [e_3,e_5]=e_3,\ [e_4,e_5]=ue_4$
$R_{5,9}(u,v)$	$[e_1,e_5]=e_1,\ [e_2,e_5]=e_1+e_2,\ [e_3,e_5]=ue_3,$ $[e_4,e_5]=ve_4$
$R_{5,10}$	$[e_2,e_5]=e_1,\ [e_3,e_5]=e_2,\ [e_4,e_5]=e_4$
$R_{5,11}(u)$	$[e_1,e_5]=e_1,\ [e_2,e_5]=e_1+e_2,\ [e_3,e_5]=e_2+e_3,$ $[e_4,e_5]=ue_4$
$R_{5,12}$	$[e_1,e_5]=e_1,\ [e_2,e_5]=e_1+e_2,\ [e_3,e_5]=e_2+e_3,$ $[e_4,e_5]=e_3+e_4$
$R_{5,13}(u,v,w)$	$[e_1,e_5]=e_1,\ [e_2,e_5]=ue_2,\ [e_3,e_5]=ve_3-we_4,$ $[e_4,e_5]=we_3+ve_4$
$R_{5,14}(u)$	$[e_2,e_5]=e_1,\ [e_3,e_5]=ue_3-e_4,\ [e_4,e_5]=e_3+ue_4$
$R_{5,15}(u)$	$[e_1,e_5]=e_1,\ [e_2,e_5]=e_1+e_2,\ [e_3,e_5]=ue_3,$ $[e_4,e_5]=e_3+ue_4$
$R_{5,16}(u,v)$	$[e_1,e_5]=e_1,\ [e_2,e_5]=e_1+e_2,\ [e_3,e_5]=ue_3-ve_4,$ $[e_4,e_5]=ve_3+ue_4$
$R_{5,17}(u,v,w)$	$[e_1,e_5]=ue_1-e_2,\ [e_2,e_5]=e_1+ue_2,$ $[e_3,e_5]=ve_3-we_4,\ [e_4,e_5]=we_3+e_4$
$R_{5,18}(u)$	$[e_1,e_5]=ue_1-e_2,\ [e_2,e_5]=e_1+ue_2,$ $[e_3,e_5]=e_1+ue_3-e_4,\ [e_4,e_5]=e_2+e_3+ue_4$
$R_{5,19}(u,v)$	$[e_1,e_5]=ue_1,\ [e_2,e_3]=e_1,\ [e_2,e_5]=e_2,$ $[e_3,e_5]=(u-1)e_3,\ [e_4,e_5]=ve_4$
$R_{5,20}(u)$	$[e_1,e_5]=ue_1,\ [e_2,e_3]=e_1,\ [e_2,e_5]=e_2,$ $[e_3,e_5]=(u-1)e_3,\ [e_4,e_5]=e_1+ue_4$
$R_{5,21}$	$[e_1,e_5]=2e_1,\ [e_2,e_3]=e_1,\ [e_2,e_5]=e_2+e_3,$ $[e_3,e_5]=e_3+e_4,\ [e_4,e_5]=e_4$

Table A.10: Non-decomposable non-nilpotent Lie algebras of dimension 5 over $\mathbb{R}$, where $u,v,w\in\mathbb{R}$ and $\varepsilon\in\{1,-1\}$.

Name	Non-vanishing Lie brackets
$R_{5,22}$	$[e_2,e_3]=e_1,\ [e_2,e_5]=e_3,\ [e_4,e_5]=e_4$
$R_{5,23}(u)$	$[e_1,e_5]=2e_1,\ [e_2,e_3]=e_1,\ [e_2,e_5]=e_2+e_3,$ $[e_3,e_5]=e_3,\ [e_4,e_5]=ue_4$
$R_{5,24}(\varepsilon)$	$[e_1,e_5]=2e_1,\ [e_2,e_3]=e_1,\ [e_2,e_5]=e_2+e_3,$ $[e_3,e_5]=e_3,\ [e_4,e_5]=2e_4+\varepsilon e_1$
$R_{5,25}(u,v)$	$[e_1,e_5]=2ve_1,\ [e_2,e_3]=e_1,\ [e_2,e_5]=ve_2+e_3,$ $[e_3,e_5]=ve_3-e_2,\ [e_4,e_5]=ue_4$
$R_{5,26}(u,\varepsilon)$	$[e_1,e_5]=2ue_1,\ [e_2,e_3]=e_1,\ [e_2,e_5]=ue_2+e_3,$ $[e_3,e_5]=-e_2+ue_3,\ [e_4,e_5]=\varepsilon e_1+2ue_4$
$R_{5,27}$	$[e_1,e_5]=e_1,\ [e_2,e_3]=e_1,\ [e_3,e_5]=e_3+e_4,$ $[e_4,e_5]=e_1+e_4$
$R_{5,28}(u)$	$[e_1,e_5]=ue_1,\ [e_2,e_3]=e_1,\ [e_2,e_5]=(u-1)e_2,$ $[e_3,e_5]=e_3+e_4,\ [e_4,e_5]=e_4$
$R_{5,29}$	$[e_1,e_5]=e_1,\ [e_2,e_4]=e_1,\ [e_2,e_5]=e_2,\ [e_4,e_5]=e_3$
$R_{5,30}(u)$	$[e_1,e_5]=(u+1)e_1,\ [e_2,e_4]=e_1,\ [e_2,e_5]=ue_2,$ $[e_3,e_4]=e_2,\ [e_3,e_5]=(u-1)e_3$
$R_{5,31}$	$[e_1,e_5]=3e_1,\ [e_2,e_4]=e_1,\ [e_2,e_5]=2e_2,\ [e_3,e_4]=e_2,$ $[e_3,e_5]=e_3,\ [e_4,e_5]=e_3+e_4$
$R_{5,32}(u)$	$[e_1,e_5]=e_1,\ [e_2,e_4]=e_1,\ [e_2,e_5]=e_2,\ [e_3,e_4]=e_2,$ $[e_3,e_5]=ue_1+e_3$
$R_{5,33}(u)$	$[e_1,e_4]=e_1,\ [e_2,e_5]=e_2,\ [e_3,e_4]=ve_3,\ [e_3,e_5]=ue_3$
$R_{5,34}$	$[e_1,e_4]=ue_1,\ [e_1,e_5]=e_1,\ [e_2,e_4]=e_2,\ [e_3,e_4]=e_3,$ $[e_3,e_5]=e_2$
$R_{5,35}(u,v)$	$[e_1,e_4]=ve_1,\ [e_1,e_5]=ue_1,\ [e_2,e_4]=e_2,$ $[e_2,e_5]=-e_3,\ [e_3,e_4]=e_3,\ [e_3,e_5]=e_2$
$R_{5,36}$	$[e_1,e_4]=e_1,\ [e_2,e_3]=e_1,\ [e_2,e_4]=e_2,\ [e_2,e_5]=-e_2,$ $[e_3,e_5]=e_3$
$R_{5,37}$	$[e_1,e_4]=2e_1,\ [e_2,e_3]=e_1,\ [e_2,e_4]=e_2,$ $[e_2,e_5]=-e_3,\ [e_3,e_4]=e_3,\ [e_3,e_5]=e_2$
$R_{5,38}$	$[e_1,e_4]=e_1,\ [e_2,e_5]=e_2,\ [e_4,e_5]=e_3$
$R_{5,39}$	$[e_1,e_4]=e_1,\ [e_1,e_5]=-e_2,\ [e_2,e_4]=e_2,\ [e_2,e_5]=e_1,$ $[e_4,e_5]=e_3$
$R_{5,40}$	$[e_1,e_2]=2e_1,\ [e_1,e_3]=-e_2,\ [e_1,e_4]=e_5,$ $[e_2,e_3]=2e_3,\ [e_2,e_4]=e_4,\ [e_2,e_5]=-e_5,\ [e_3,e_5]=e_4$

Table A.10: Non-decomposable non-nilpotent Lie algebras of dimension 5 over $\mathbb{R}$, where $u,v,w\in\mathbb{R}$ and $\varepsilon\in\{1,-1\}$.

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