MODULES FOR CERTAIN LIE ALGEBRAS OF MAXIMAL CLASS

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0. INTRODUCTION

Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over a field k of characteristic zero. By Ado's Theorem it is known that there exists a faithful $\mathfrak{g}$ - module M of finite dimension. Hence we may consider the following integer valued invariant of $\mathfrak{g}$:

$$\mu(\mathfrak{g}) := \min\{\dim_k M \mid M \text{ is a faithful } \mathfrak{g}\text{- module}\}.$$

Particularly little seems to be known about $\mu(\mathfrak{g})$ if $\mathfrak{g}$ is a nilpotent Lie algebra. From a proof of Ado's Theorem one easily deduces an exponential bound

$$\mu(\mathfrak{g}) \le c_1 \cdot \exp(c_2 \cdot \dim_k \mathfrak{g})$$

with some constants $c_1,\, c_2 > 0$. On the other hand there are classes of Lie algebras $\mathfrak{g}$ for which one has the much better bound

$$\mu(\mathfrak{g}) \le \dim \mathfrak{g} + 1.$$

This holds, for instance, for all nilpotent Lie algebras of class ≤ 3 ([14]) or in low dimensions, for $\mathbb{Z}$ - graded Lie algebras, or for those which posses a nonsingular derivation. Accordingly it is quite difficult to find a nilpotent Lie algebra $\mathfrak{g}$ with $\mu(\mathfrak{g}) > \dim \mathfrak{g} + 1$. The first example of this phenomenon was discovered by Y. Benoist ([2]) :
Let $\mathfrak{a}(r,s,t)$ be the Lie algebra given by the vector space generators $e_1, e_2, e_3, \ \dots$ and the relations

$$(1) \qquad \begin{aligned} [e_1,e_i] &= e_{i+1} \\ [e_2,e_3] &= e_5 \\ [e_2,e_5] &= re_7 + se_8 + te_9 \end{aligned}$$

for $i = 1,2,3, \ \dots$ and $r,s,t \in k$.

Define sets $A_1 := k \setminus \{\frac{9}{10}, 1\}$, $A_2 := k \setminus \{0, \frac{9}{10}, 1, 2, 3\}$, $A := A_2 \setminus A_3$,

where A_3 is the set of zeros of $(5r^2 - 10r + 3)(3r^2 - 2r + 3)$.

Benoist has proved:

LEMMA: *For* $r = \frac{9}{10}, 1$ *the Lie algebra* $\mathfrak{a}(r,s,t)$ *is infinite-dimensional. If* $r \neq \frac{9}{10}, 1$ *then* $\mathfrak{a}(r,s,t)$ *has dimension* 11 *, i.e.* $0 = e_{12} = e_{13} = e_{14} = \dots$ *and hence is nilpotent. If* $r \in A_2$ *, then* $\mathfrak{a}(r,s,t)$ *is of maximal nilpotency class* 10 *, i.e. is a filiform Lie algebra. Two algebras* $\mathfrak{a}(r,s,t)$ *and* $\mathfrak{a}(r',s',t')$ *are isomorphic iff* $r' = r, \;\; s' = \alpha s, \;\; t' = \alpha^2 t$ *for some* $\alpha \neq 0$ *.*

Concerning the invariant μ he has stated that $\mu(\mathfrak{a}(-2,1,t)) > 12$.
The proof in his preprint uses a detailed theory of $\mathfrak{a}(-2,1,t)$ - modules plus heavy computer calculations.
In this paper we analyse faithful $\mathfrak{a}(r,s,t)$ -modules for arbitrary r,s,t . We use an easy combinatorial approach including some computer calculations to establish

THEOREM A: *Let* $s \neq 0$ *and* $r \in A$ *. Then the Lie algebra* $\mathfrak{a}(r,s,t)$ *has no faithful* 12 *-dimensional module.*

Secondly we show

THEOREM B: *Let* $r \in A_2$ *and* s,t *arbitrary. There exists a faithful minimal* $\mathfrak{a}(r,s,t)$ *-module of dimension* 22 *.*

Here a faithful module M is called minimal, if it has no faithful submodule and no faithful quotient.
Problems of the above kind are particularly important in the theory of affine actions of connected nilpotent Lie groups G on affine space $\mathbb{R}^n$. The problem here is to determine which such G act simply transitively and affinely on $\mathbb{R}^n$. This includes the problem, pointed out by Milnor and Auslander ([12] , [1]), whether G always admits a complete left-invariant locally flat affine structure or not (see [5], [10], [7], [8], [14], [13], [15]).
It is well known that once G has such an action then the Lie algebra $\mathfrak{g}$ of G has a faithful module of dimension $\dim \mathfrak{g} + 1$. More precisely, $\mathfrak{g}$ then admits an affine structure, i.e. a faithful linear representation

$$\mathfrak{g} \;\longrightarrow\; \mathfrak{aff}(\mathbb{R}^n) \;\subset\; \mathfrak{gl}(\mathbb{R}^{n+1})$$

of Lie algebras, where $\mathfrak{aff}(\mathbb{R}^n) = \left\{ \left(\begin{smallmatrix} y & a \\ 0 & 0 \end{smallmatrix}\right) \mid y \in \mathfrak{gl}(\mathbb{R}^n), \; a \in \mathbb{R}^n \right\}$ is the Lie algebra of the affine automorphism group $\mathbf{Aff}(\mathbb{R}^n)$ and $n = \dim \mathfrak{g}$ (see [7] , [13]).
Thus the connected nilpotent Lie groups corresponding to the $\mathfrak{a}(r,s,t)$ of Theorem A do not admit such an action.
It should be noted, that the results are contradictory to the articles of Boyom and Nisse ([3] , [11]).

Let k be a field of characteristic zero and $\mathfrak{a}(r,s,t)$ as defined in the introduction.
In the following we consider the filiform algebras $\mathfrak{a}(r,s,t)$ for $r \in A_2$.
They are generated by e_1, e_2 and have one-dimensional center $\mathfrak{z} =< e_{11} >$. Let

$$\varrho : \ \mathfrak{a}(r,s,t) \ \longrightarrow \ \mathfrak{gl}(M)$$

be an $\mathfrak{a}(r,s,t)$ - module. We call M a Δ - module , if the following conditions are satisfied:

a) M is nilpotent, that is every $\varrho(x)$ is a nilpotent endomorphism,
b) M is faithful,
c) $\dim_k M = 12$.

One verifies (see [2]):

LEMMA 1.1 *If* M *is a faithful* $\mathfrak{a}(r,s,t)$ *- module of minimal dimension* m *then one has* $m \geq 11$ *. If there is a faithful* $\mathfrak{a}(r,s,t)$ *- module of dimension* 11 *or* 12 *then there exists also a* Δ *- module.*

We will prove:

THEOREM 1.2. *Let* $r \in A$ *as above and* $s \neq 0$ *. Then there are no* Δ *- modules for* $\mathfrak{a}(r,s,t)$ *.*

As a corollary we obtain Theorem A.

We can compute the Lie brackets for $\mathfrak{a}(r,s,t)$ explicitly using (1) and the Jacobi identity successively. In addition to the relations (1) we will also use:

(R^1) $$[e_2, e_9] = -\frac{5r^3+r^2-7r+3}{2r(r-2)}\, e_{11}$$

(R^2) $$[e_3, e_8] = \frac{(5r^3-7r^2+15r-9)(1-r)}{2r(r-2)(r-3)}\, e_{11}$$

(R^3) $$[e_4, e_7] = \frac{3(5r^2-6r+3)(r-1)^2}{2r(r-2)(r-3)}\, e_{11}$$

(R^4) $$[e_5, e_6] = \frac{3(3-7r)(r-1)^3}{2r(r-2)(r-3)}\, e_{11}$$

(R^6) $$[e_3, e_4] = (1-r)e_7 - se_8 - te_9$$

(R^7) $$[e_3, e_5] = (1-r)e_8 - se_9 - te_{10}$$

Define

$$r_1 := r, \quad r_2 := 2r - 1, \quad r_3 := \frac{5r-3}{3-r}, \quad r_4 := \frac{r(5r-3)}{3-r}, \quad r_5 := \frac{5r^3 + r^2 - 7r + 3}{2r(2-r)} .$$

REMARK 1.3 For some special values of r, s, t there obviously exist 12 -dimensional faithful modules: If $s = t = 0$ there are many modules, e.g. the standard graded and faithful module M_{gr} , defined as follows:
Let $f_1, \dots, f_{12}$ be a basis for M_{gr} . The matrix for the action of e_1 is of type $\{10\}$ (see Definition 3.1) and the action of e_2 is given by

$$\begin{array}{lcllcl}
e_2.f_1 & = & 0 & \qquad e_2.f_7 & = & r f_5 \\
e_2.f_2 & = & 0 & \qquad e_2.f_8 & = & f_6 \\
e_2.f_3 & = & r_5 f_1 & \qquad e_2.f_9 & = & f_7 \\
e_2.f_4 & = & r_4 f_2 & \qquad e_2.f_{10} & = & 0 \\
e_2.f_5 & = & r_3 f_3 & \qquad e_2.f_{11} & = & -f_9 \\
e_2.f_6 & = & r_2 f_4 & \qquad e_2.f_{12} & = & 2 f_{10}
\end{array}$$

One can also construct modules for all $r \in A_3$ (see Remark 4.5 in the case $3r^2 - 2r + 3 = 0$).

REMARK 1.4 If $r \in A$ then the following result can be easily read off from our discussion: For any Δ - module M the associated $\mathfrak{a}(r,0,0)$ -module $\overline{M}$ is isomorphic to M_{gr} or M_{gr}^* . The module $\overline{M}$ is obtained from M by considering the filtration: $M^0 = M,\ M^1 = M,\ M^{i+1} = E_1 M^i + E_2 M^{i-1}$ and forming the associated graded object.

In Theorem B we prove that there exist faithful $\mathfrak{a}(r,s,t)$ modules for $r \in A_2$ of dimension 22 , which are minimal. Such minimal modules are necessarily cyclic. Note, that there are *different* dimensions of minimal faithful $\mathfrak{a}(r,s,t)$ - modules: For $s = t = 0$ the module constructed in Theorem B has dimension 22 , whereas M_{gr} is of dimension 12 .

2. Δ – MODULES

Assume that $\mathfrak{a}(r,s,t)$ possesses a Δ - module. Then there is a basis $f_1, f_2, \dots, f_{12}$ of M such that the matrices of $\varrho(e_1)$ and $\varrho(e_2)$ are as follows:

$$E_1 := \begin{pmatrix} 0 & \lambda_1 & \lambda_{12} & \dots & \lambda_{64} & \lambda_{66} \\ 0 & 0 & \lambda_2 & \dots & \lambda_{62} & \lambda_{65} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \lambda_{10} & \lambda_{21} \\ 0 & 0 & 0 & \dots & 0 & \lambda_{11} \\ 0 & 0 & 0 & \dots & 0 & 0 \end{pmatrix} \quad \text{and} \quad E_2 := \begin{pmatrix} 0 & x_1 & x_{12} & \dots & x_{64} & x_{66} \\ 0 & 0 & x_2 & \dots & x_{62} & x_{65} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & x_{10} & x_{21} \\ 0 & 0 & 0 & \dots & 0 & x_{11} \\ 0 & 0 & 0 & \dots & 0 & 0 \end{pmatrix}$$

where λ_i is 0 or 1 *such that in each row and each column of* E_1 *is at most one nonzero entry.* (It is easy to see that this can be done by base changes of the form $f_i \mapsto \alpha_{1i} f_1 + \ldots + \alpha_{ii} f_i$. This transformation keeps the upper triangularity of E_2). The (i,j) -th coefficient of E_1 is $\lambda_{i+11k-k(k-1)/2}$ ($i<j,\ k:=j-i-1$). Define the *first layer* of E_1 to be the first upperdiagonal containing $\lambda_1, \ldots, \lambda_{11}$, the *second layer* containing $\lambda_{12}, \ldots, \lambda_{21}$ and so on. Since M is a Lie module, the relations (1) hold with E_1, E_2 , that is $E_{i+1} := [E_1, E_i]$ and

$$(N^1) \qquad [E_2, E_3] = E_5$$
$$(N^2) \qquad [E_2, E_5] = rE_7 + sE_8 + tE_9$$

The relations $(R^1), \ldots, (R^7)$ also hold for E_i and M is faithful if and only if

$$E_{11} \neq 0 :$$

The center $\mathfrak{z}$ of $\mathfrak{a}(r,s,t)$ is generated by e_{11} . If ϱ has nonzero kernel then $\ker \varrho$ intersects $\mathfrak{z}$ nontrivially, hence $\varrho(e_{11}) = E_{11} = 0$.

Let

$$N^1 := [E_2, E_3] - E_5, \quad N^2 := [E_2, E_5] - rE_7 - sE_8 - tE_9, \quad N^4 := [E_2, E_{10}]$$

and use the notation R^i for the analoguous matrices corresponding to the (R^i) . Denote the i,j -th entry of $N^k,\ R^k$ by $N^k_{i,j}$ and $R^k_{i,j}$ respectively.
These relations define a system of polynomial equations in r, s, t, λ_i, x_i over k . To solve these equations it is indeed necessary to simplify the form of E_1 as above. Then the equations above are easier. Nevertheless one has two problems – the number of cases for the possible choices of E_1 is large; and secondly, one cannot use computer algorithms for the system of equations for general r , since the equations also contain the large solution varieties for $r = 0, \frac{9}{10}, 1, 2, 3$. At this point one should note, that the calculations are much easier for fixed r .
In order to solve the first problem we need reduction arguments.

3. REDUCTION ARGUMENTS

Let M be a Δ - module for $\mathfrak{a}(r,s,t)$, with basis $\{f_i\}$ and

$$I_1 := \{1, \ldots, 11\}, \quad I_2 := \{12, \ldots, 21\}, \quad \ldots \quad I_{10} := \{64, 65\}, \quad I_{11} := \{66\}$$
$$N_1 := \{i \in I_1 \mid \lambda_i = 0\}, \quad N_2 := \{i \in I_2 \mid \lambda_i = 1\}, \ \ldots, \ N_{11} := \{i \in I_{11} \mid \lambda_i = 1\}.$$

DEFINITION 3.1 *Define the* type *of* M *to be*

$$\text{type}(M) := \{N_1 \mid N_2 \mid \ldots \mid N_{11}\}$$

If E_1 is, for instance, of Jordan type (with respect to the basis f_i) with $\lambda_i = 1$ for $i = 1, \ldots, 9$ and $\lambda_i = 0$ else, then $\operatorname{type}(M) = \{10, 11\}$. Empty sets N_i are omitted in this notation. We set $\operatorname{type}(M) = \emptyset$ if E_1 is of full block Jordan form.
If $\operatorname{type}(M) = \{i^1_1, \ldots, i^1_{k_1} \mid i^2_1, \ldots, i^2_{k_2} \mid \ldots \mid i^{11}_{k_{11}}\}$ and M^* is the dual module of M , then it is a simple matter to check that

$$\operatorname{type}(M^*) = \{12 - i^1_1, \ldots, 12 - i^1_{k_1} \mid 33 - i^2_1, \ldots, 33 - i^2_{k_2} \mid \ldots \mid i^{11}_{k_{11}}\}.$$

Computation of E_{11} yields the following formulas: All entries are zero except for

$$E^{11}_{1,11} = \sum_{i=1}^{10} a_i \, \lambda_{i,11} \, x_i \quad , \quad E^{11}_{2,12} = \sum_{i=1}^{10} a_i \, \lambda_{1,i+1} \, x_{i+1}$$

$$E^{11}_{1,12} = \sum_{i=1}^{10} a_i \lambda_{1,i+1} x_{i+11} + \sum_{i=1}^{10} \sum_{j=1}^{i-1} a_i \, \lambda_{i+1,j+1,j} \, \lambda_{j+11} \, x_{i+1} + \sum_{i=1}^{10} \sum_{j=i+1}^{10} a_i \, \lambda_{j+1,j,i} \, \lambda_{j+11} \, x_i$$

where

$$\lambda_{i,j} := \prod_{k=1, k \neq i,j}^{12} \lambda_k, \qquad \lambda_{i,j,k} := \prod_{\ell=1, \ell \neq i,j,k}^{12} \lambda_\ell$$

and $(a_1, a_2, \ldots, a_{10}) = (-1, 9, -36, 84, -126, 126, -84, 36, -9, 1)$.

From this it is obvious that M is faithful only for the following types:

(a)	$\emptyset$		
(b)	$\{i\}$	$i = 1, ..., 11$	
(c)	$\{i, 11 \mid N_{12-i}\}$	$i = 1, ..., 9$	
(d)	$\{1, i \mid N_i\}$	$i = 3, ..., 11$	
(e)	$\{i, i+1\}$	$i = 1, ..., 10$	
(f)	$\{i, i+1 \mid 11+i\}$	$i = 1, ..., 10$	
(g)	$\{i, i+1, j \mid 11+i\}$	$i = 1, ..., 8$	$j > i+2$
(h)	$\{i, j, j+1 \mid 11+j\}$	$j = 3, ..., 10$	$i < j-1$
(k)	$\{i, i+1, i+2 \mid N_2\}$	$i = 1, ..., 9$	

where in the last case N_2 is $\{11+i\}$, $\{12+i\}$ or $\{11+i, 12+i\}$.

We shall reduce this list now

LEMMA 3.2 *If* $\mathfrak{a}(r,s,t)$ *has a* Δ *- module* M *then we may assume that the type of* M *is one of the following:*

(1)	$\emptyset$		
(2)	$\{i\}$	$i = 6, ..., 11$	
(3)	$\{i, i+1\}$	$i = 6, ..., 10$	
(4)	$\{i, i+1 \mid 11+i\}$	$i = 6, ..., 10$	
(5)	$\{i, i+1, j \mid 11+i\}$	$i = 6, 7, 8$	$j > i+2$
(6)	$\{i, j, j+1 \mid 11+j\}$	$j = 6, ..., 10$	$i < j-1$

Proof: If the module M has type $\{i, 11 \mid \ldots\}$ for $i = 1, ..., 9$ then it follows from the formulas for E_{11} that the vector space M_0 generated by $f_1, ..., f_{11}$ is a faithful submodule. Adding a trivial 1-dimensional module we obtain a Δ- module of type $\{10, 11 \mid \ldots\}$. If M is of type $\{1, i \mid \ldots\}$, $i = 3, ..., 11$ then the dual module is of type $\{j, 11 \mid \ldots\}$, $j = 9, ..., 1$. The types $\{i\}$, $\{i, i+1 \mid \ldots\}$, $\{i, i+1, j \mid \ldots\}$ and $\{i, j, j+1 \mid \ldots\}$ are reduced by possibly going to the dual module. Finally one has to look at the case (k). The equation $N^1_{i-1,i+2}$ means $x_i\, x_{i+1} = 0.$
Denote by $f_i \leftrightarrow f_{i+1}$ the base change for M which interchanges f_i and f_{i+1} and fixes the remaining f_j.

First case: $x_i = 0$:
One has $11 + i \in N_2$, otherwise M is not faithful. We may apply the base change $f_i \leftrightarrow f_{i+1}$ since E_2 remains unchanged. Then one obtains a Δ- module of type $\{i-1, i, i+2 \mid \ldots\}$.

Second case: $x_{i+1} = 0$:
If $N_2 = \{11+i\}$ then applying $f_{i+1} \leftrightarrow f_{i+2}$ leads to type $\{i+1, i+2 \mid \ldots\}$. If $N_2 = \{12+i\}$ then one obtains type $\{i, i+1 \mid \ldots\}$ and the case $N_2 = \{11+i, 12+i\}$ leads to type $\{i+1\}$. □

4. PROOF OF THE THEOREMS

Let $i \in \mathbb{N}$ and $x_i, x_{i+1}, x_{i+2}, \ldots$ be unknowns. Set

$$y_{i+3} := x_{i+3} - 3\,x_{i+2} + 3\,x_{i+1} - x_i.$$

Define polynomials $f_i, g_i \in k[x_i, \ldots, x_{i+5}]$ by

$$\begin{aligned} f_i &:= x_{i+3}x_{i+1} - 2x_{i+3}x_i + x_{i+2}x_i + y_{i+3} \\ g_i &:= r\,(y_{i+3} - 2y_{i+4} + y_{i+5}) + y_i x_{i+5} - x_i y_{i+5} \end{aligned}$$

As an example $f_{12} = x_{15}x_{13} - 2x_{15}x_{12} + x_{14}x_{12} + x_{15} - 3x_{14} + 3x_{13} - x_{12}$.

LEMMA 4.1 *If* $r \in A_1$ *then the system of equations*

$$\begin{aligned} f_{12} = 0, \ldots, f_{18} = 0 \\ g_{12} = 0, \ldots, g_{16} = 0 \end{aligned}$$

in the unknows $x_{12},\dots,x_{21}$ *has only the solution* $x_{12}=x_{13}=\dots=x_{21}$.

LEMMA 4.2 *Let* f_i, g_i *be defined as above and* $r\in A$. *The system of equations*

$$\begin{array}{ll} f_i \quad = 0 & g_i \quad = 0 \\ f_{i+1} = 0 & g_{i+1} = 0 \\ f_{i+2} = 0 & x_{i+6} = 2x_{i+5} - 1 \\ f_{i+3} = 0 & x_{i+5} = (r-1) + 3x_{i+4} - 2x_{i+3} \end{array}$$

has only the "standard" solution:

$$x_{i+4} = r_1, \quad x_{i+3} = r_2, \quad x_{i+2} = r_3, \quad x_{i+1} = r_4, \quad x_i = r_5 \ .$$

Proof of Lemma 4.2 : Substituting the terms for x_{i+6} and x_{i+5} one obtains six polynomial equations $f_i=0,\dots,g_{i+1}=0$ denoted by $(1),\dots,(6)$. The linear combination $(6)-(4)+3\cdot(3)+3\cdot(2)$ yields

$$(r-3)(x_{i+4}-4x_{i+3}+5x_{i+2}-2x_{i+1}+1) = r(3r-5).$$

(Hence $r\neq 3$). Using this equation one eliminates x_{i+1} . By similar procedures, one eliminates other variables and computes then resultants assuming that we have not the standard solution. It leads to:

$$(10r-9)(3r-1)(r-1)^8(r-2)^3(r-3)^6(7r^2-26r+23)(3r^2-2r+3)(5r^2-10r+3) = 0$$

All factors except the two last factors are contradictory to the remaining equations. It is also easy to see that the last two factors (i.e $r\in A_2$) lead to one further solution. (For $3r^2-2r+3$ this is, for instance, $x_i=-1$, $x_{i+1}=-1$, $x_{i+2}=-r$, $x_{i+3}=-(2r-1)$, $x_{i+4}=3(1-r)/2$ and for $5r^2-10r+3$ one has $x_i,\dots,x_{i+3}$ as before and $x_{i+4}=5(1-2r)/3$.) □

Proof of Lemma 4.1: The computations are harder than in the preceding lemma, but similar.
If $y_{i+3}-2y_{i+4}+y_{i+5}\neq 0$, then it follows that there is no solution with $r\neq \frac{9}{10},1$. Otherwise eliminating and taking resultants gives the following condition :

$$(x_{i+2}x_{i+3}+5x_{i+3}-5x_{i+2}-1)(5x_{i+3}-5x_{i+2}-2)^2(x_{i+3}-x_{i+2})^3(x_{i+3}+1)^2(x_{i+3}-1)x_{i+2} = 0.$$

Now one has to deal with these subcases. In fact, the case $x_{i+3}=x_{i+2}$ leads to the general solution.
For $r=\frac{9}{10},1$ the equations have many solutions. After cutting out these solution varieties (by suitable eliminations) one can check the result by computer algorithms. □

REMARK 4.3 For $s=t=0$ one has

$$[e_2,e_i] = r_{i-4}e_{i+2} \quad i=5,\dots,9$$

The coefficients r_i involved in the above lemma are precisely those from $\operatorname{ad} e_2$ for the graded algebra $\mathfrak{a}(r,0,0)$.

Let M be a module satisfying (2) given by E_1 and E_2 . We call M *normal* if $x_1 x_2 \neq 0$.

LEMMA 4.4 *Let* $r \in A$ *. There is no normal* Δ *- module for* $\mathfrak{a}(r,s,t)$ *.*

Proof: We will prove the Lemma for types (5) , (6) and $\{6\}$, $\{6,7 \mid \ldots\}$ (see Lemma 3.2) later in the general context.
Hence assume that there exists a normal Δ - module such that $\lambda_1 = \ldots = \lambda_6 = 1, \quad \lambda_{12} = \ldots = \lambda_{16} = 0$ and $\lambda_{22} = \ldots = \lambda_{66} = 0$.
Set $x_2 = \alpha x_1$ with $\alpha \neq 0$. We will show $\alpha = 1$. The equations

$$N^1_{i,i+3}: \qquad x_{i+2}\,(i+1-i\alpha) = \alpha x_1 \qquad i = 1,\ldots,4$$

imply $z := (\alpha-2)(2\alpha-3)(3\alpha-4)(4\alpha-5) \neq 0$ and $x_{i+2} = (\alpha x_1/(i+1-i\alpha)$. Then substitute x_{14} , x_{15} , x_{16} in $N^1_{1,5}$, $N^1_{2,6}$, $N^1_{3,7}$. It follows $N^2_{1,7}$: $z(\alpha-1)^5(10r-9) = 0$ and therefore $\alpha = 1$.

It is $\lambda_7 = 1$. Otherwise the equations $N^1_{5,8}$, $N^1_{4,8}, N^1_{5,9}$ imply $x_7 = x_{17} = 0$ and $x_{18} = \lambda_{18} x_1$. If $\lambda_{18} = 0$ then $E_{11} = 0$, hence $\lambda_{18} = 1, \lambda_8 = 0$.
By the same argument $\lambda_9 = 1, \lambda_{19} = \lambda_{20} = 0$ and $N^1_{6,10}$: $\lambda_{10} x_1 = x_{10}$. Now $\lambda_{10} = 1$ because of faithfulness, so $\lambda_{21} = 0$ and $N^1_{9,12}$: $\lambda_{11} x_1 = x_{11}$. But then $E_{11} = 0$, a contradiction.
Now the equations imply $x_7 = x_1$ and $x_{17} = 5x_{13} - 4x_{12}$. In the same way we have $\lambda_8 = 1,\ x_8 = x_1$ and $x_{18} = 6x_{13} - 5x_{12}$ (use the equations one level higher). Repeating this step one obtains

$$\begin{array}{lcll}
\lambda_i & = & 1 & i = 1,\ldots,11 \\
\lambda_{i+11} & = & 0 & i = 1,\ldots,11 \\
x_i & = & x_1 & i = 1,\ldots,11 \\
x_{i+11} & = & (i-1)x_{13} - (i-2)x_{12} & i = 1,\ldots,10
\end{array}$$

Then $E_{11} = 0$, contradiction. □

Proof of Theorem 1.2 :

Assume that there exist a Δ - module for $\mathfrak{a}(r,s,t)$. We prove the result by direct computation for the types listed in Lemma 3.2 . The equations are either linear or quadratic (like the f_i, g_i from above). We can always solve the equations, very often by direct application of Lemma 4.2 . We divide the cases into three parts, depending on how many zeros are contained in the first layer of E_1 (the more zeros the easier the computations).

I. Three zeros in the first layer:

If $\lambda_1 = 1$ then the computations are almost trivial. The typical computation goes as follows:

Type $\{3, 9, 10 \mid 20\}$ *:*

Since M is faithful, the formulas for E_{11} imply $x_3 \neq 0$, we may assume $x_3 = 1$. It follows $N^1_{1,4} : x_2 = 2x_1$, $N^1_{2,5} : x_4 = -x_2$, $N^1_{3,5} : 2x_5 = x_4$, $N^2_{2,7} : 3x_6 = -x_2$, $N^4_{1,12} : 7x_{11} = -2x_1$, $R^1_{1,11} : 3x_{20} = -x_1$, $R^1_{1,10} : x_9 = 0$, $N^1_{2,6} : x_1 x_{14} = -x_{15} - 3$, $N^1_{3,7} : 3x_{16} = x_{15} - 3$. Then $R^6_{1,7} :\ r = 9/10$, a contradiction.

The types $\{1, i, i+1 \mid i+11\}$ are a little bit longer. As an example we prove :

Type $\{1, 10, 11 \mid 21\}$ *:*

By faithfulness $x_1 = 1$. If $x_{11} = 0$ then we could apply $f_{11} \leftrightarrow f_{12}$ to obtain a module of type $\{1, 10\}$. Thus $x_{11} \neq 0$ and $x_{10} = 0$ by $N^2_{7,12}$. Then $x_{20} = 0,\ 2x_3 = x_2,\ 4x_5 = 6x_4 - x_2$ by $R^1_{1,11},\ N^1_{1,4},\ N^2_{1,6}$.
Case a: $x_2 \neq 0$ *:* It is immediate that $3x_4 = 5x_6 = 6x_7 = 7x_8 = 8x_9 = 9x_{21} = x_2$. Then $N^1_{1,5} :\ x_2 x_{12} = 6(x_{13} - 2x_{14} - 1)$, $N^1_{2,6} :\ 18x_{15} = 15x_{14} - 2x_{13} - 3$, $N^1_{3,7} :\ 40x_{16} = 42x_{15} - 9x_{14} - 3$ and $R^6_{1,7} :\ r = 9/10$

Case b: $x_2 = 0$ *:* One has $x_4 = x_6 = x_7 = x_8 = x_9 = 0$ and $2x_{14} = x_{13} - 1$. The equations $N^1_{2,7},\ N^1_{3,8},\ N^1_{4,9},\ N^1_{5,10},\ N^2_{3,10},\ R^7_{2,10}$ have the solution $x_{15} = -r_1,\ x_{16} = -r_2, \ldots, x_{19} = -r_5$ (and $x_{13} = x_{14} = -1$) for $r \in A$. This follows (after slight modification) from Lemma 4.2 . Then $N^1_{7,12} :\ (5r^2 - 10r + 3)(3r^2 - 2r + 3)x_{21} = 0$. Since $r \in A$ one has $x_{21} = 0$. Now $N^1_{1,6},\ N^1_{6,12},\ N^2_{4,12}$ imply $(10r - 9)(r-1)^5(r-2) = 0$, a contradiction.

II. Two zeros in the first layer:

Most of the computations for the types (4) and (3) can be done simultaneously. Moreover we need not compute all cases, since some of them can be reduced to others. We show that for the following types:

Type $\{10, 11 \mid 21\}$, *Type* $\{10, 11\}$ *:*

Assume that there exists a Δ - module of type $\{10, 11 \mid 21\}$. Then $N^4_{2,12} :\ x_{10} x_{11} = 0$. If $x_{11} = 0$ then we may apply $f_{11} \leftrightarrow f_{12}$ to obtain a module of type $\{11\}$. If $x_{10} = 0$ then $f_{10} \leftrightarrow f_{11}$ is admissible and leads to type $\{9, 10 \mid 20\}$.

Assume that there exists a module of type $\{10, 11\}$.
It is faithful iff x_{10} or x_{21} is nonzero. By Lemma 4.4 we have $x_1 x_2 = 0$. We may assume $x_{10} = 0,\ x_{21} \neq 0$ and $x_{11} = 1$: The case $x_{10} \neq 0,\ x_{21} = 0$ goes similarly and if $x_{21} \neq 0$, one may apply the base change $\widehat{f_{11}} = f_{11} - \frac{x_{10}}{x_{21}} f_{12}$ to get $\widehat{x_{10}} = 0$. x_{11}, x_{21} can be chosen to be 1. It follows $x_2 = \ldots = x_8 = 0$ using elementary equations from the relations $(N^1),\ (N^2),\ (R^3),\ (R^4),\ (R^7)$. Then

$R^7_{4,12}: \ x_{18} = 3x_{17} - 2x_{16} + r - 1, \quad N^1_{7,12}: \ x_{29} = 2x_{18} - x_{19} - 1, \quad N^1_{8,12}: \ x_{20} = -x_9.$

Now we distinguish two cases:

Case a: $x_1 = 0$.

The equations $N^1_{2,7}, N^1_{3,8}, N^1_{4,9}, N^1_{5,10} + N^1_{5,11}, N^2_{2,9}, N^2_{3,10} + N^2_{3,11}$ are precisely the equations f_i, g_i of Lemma 4.2 with $i = 13$, hence $x_{17} = r, \ x_{16} = 2r - 1, \ \ldots, \ x_{13} = r_5$.
$\Rightarrow \ N^1_{1,6}: \ x_{12} = (20r^4 - 28r^3 + 27r^2 - 24r + 9)/r(5r^2 - 12r + 3)(r - 2)$ and $N^1_{6,10}$: $x_9(r - 1) = 0$, hence $x_9 = 0$. Now $N^2_{1,8}: \ (10r - 9)(r - 1)^5 = 0$, a contradiction.

Case b: $x_1 \neq 0$.

Then $N^2_{1,7}, R^4_{1,10}, N^1_{1,5}$ say $x_9 = 0, x_{16} = (3x_{15} - x_{14} + 1 - r)/2$ and $x_{14} = (x_{13} - 1)/2$. Consider $N^1_{2,7}, \ N^1_{3,8}, N^1_{4,9}, N^2_{2,9}, \ R^5_{3,9}, \ R^6_{2,9}$. If $x_{13} = -1$ then we may apply Lemma 4.2 to these equations with some modification and the result is ($N^1_{2,7}, \ N^1_{3,8}, \ N^1_{5,10}$):

$$x_{15} = -r, \ x_{17} = -r_3, \ x_{19} = -r_5.$$

But then $N^1_{5,11}: \ (r - 2)(3r^2 - 2r + 3)(5r^2 - 10r + 3) = 0$, contradiction.
For $x_{13} \neq -1$ one can eliminate $x_{15}, \ x_{17}$ (using $N^1_{2,7}, \ N^1_{3,8}$). The resultant of $N^1_{4,9}, \ N^2_{2,9}$ with respect to x_{13} must be zero, that is

$$(5r^2 - 10r + 3)(3r^2 - 2r + 3)(r^2 + 4r - 1)(r^2 - 4r + 31) = 0 .$$

But all factors are nonzero: the first two by assumtion, the last two would contradict the preceding equations. □

We also prove:

Type $\{9, 10 \mid 20\}$, *Type* $\{9, 10\}$ *:*

One has $N^4_{1,11}: \ x_9 x_{10} = 0$. If $x_9 = 0$ we are in the case $\{8, 9 \mid 19\}$ (apply $f_9 \leftrightarrow f_{10}$). Hence $x_{10} = 0, x_9 \neq 0$. Let λ_{20} be 1 or 0. We have $x_1 = \ldots x_8 = 0$ by elementary equations and may assume $x_9 = 1$ (Set $\widehat{f_{10}} = x_9^{-1} f_{10}$). By $N^1_{6,10}, N^2_{4,10}$ we have $x_{18} = 2x_{17} - 1$ and $x_{17} = 3x_{16} - 2x_{15} + r - 1$. The equations $N^1_{1,6}, N^1_{2,7}, N^1_{3,8}, N^1_{4,9}, N^2_{1,8}, N^2_{2,9}$ are precisely those from Lemma 4.2 , hence $x_{16} = r_1, \ldots, x_{12} = r_5$. Then $R^7_{4,12}$: $(2x_{20} + \lambda_{20} x_{11})(r - 1) = 0$ Note that $r \neq 1$. $\Rightarrow$ The only nonzero entry of E_{11} is $11\lambda_{20} x_{11}$. Therefore the module is not faithful for $\lambda_{20} = 0$ and we have proved the result for type $\{9, 10\}$.
For $\lambda_{20} = 1$ we get $N^1_{8,12}: \ 2x_{21} = x^2_{11}$ and $x_1 1 \neq 0$. Furthermore $N^1_{7,12}$: $2x_{29} = -x_{19} x_{11}$. From $N^1_{5,10}, \ N^1_{2,8}$ we may eliminate x_{26}, x_{27}. Now the equations $N^1_{1,7}, \ N^1_{3,9}, \ N^1_{4,11}, \ N^1_{6,12}, \ N^1_{4,10}, \ N^2_{1,9}, \ N^2_{3,10}, \ R^6_{4,12}$, enforce $r = 1$, a contradiction. □

REMARK 4.5: The assumption $r \in A$ is necessary. In fact, otherwise there are many Δ - modules of type $\{10, 11\}$. We will give an example:
Let $3r^2 - 2r + 3 = 0, \ s = 0$, t arbitrary and the action of E_2 as follows:

$$
\begin{aligned}
e_2.f_1 &= 0\\
e_2.f_2 &= f_1\\
e_2.f_3 &= 0\\
e_2.f_4 &= -f_2\\
e_2.f_5 &= -f_3\\
e_2.f_6 &= -rf_4\\
e_2.f_7 &= (1-2r)f_5\\
e_2.f_8 &= -tf_4 - \tfrac{5r-3}{3-r}f_6\\
e_2.f_9 &= -2tf_5 - \tfrac{r-15}{3(3-r)}f_7\\
e_2.f_{10} &= -tf_6 + \tfrac{41r+6}{3(4r+3)}f_8\\
e_2.f_{11} &= 0\\
e_2.f_{12} &= \tfrac{27t^2(51-241r)}{4(3485r-3351)}f_6 + \tfrac{9t(51-241r)}{4(296r-471)}f_8 + f_{10} + f_{11}
\end{aligned}
$$

III. At most one zero in the first layer:

Type $\emptyset$ *:*

Assume that there is an Δ - module of type $\emptyset$. By Lemma 4.4 $x_1x_2 = 0$. We have $x_3 = 0$, otherwise $N^1_{1,4}$ implies $x_1 = x_2 = 0$ and

$N^1_{2,5}:\ x_4 = 0,\quad N^1_{1,5}:\ x_{12} = 3,\quad N^1_{3,6}:\ x_5 = 0,\quad N^1_{2,6}:\ x_{15} = -3,\quad N^1_{1,6}:\ x_{15} = -3/5$, a contradiction.

If $x_2 \neq 0$, we similarly obtain $x_1 = 0,\ x_4 = \ldots = x_9 = 0,\ x_{14} = -3,\ x_{15} = -2,\ x_{16} = -5r/3,\ x_{17} = (3-10r)/5$ by
$N^1_{1,4},\ N^1_{2,5},\ N^2_{1,6},\ N^2_{2,7},\ N^1_{6,9},\ R^3_{1,10},\ R^4_{1,10},\ N^1_{1,5}, N^1_{2,6},\ N^2_{1,7},\ N^1_{3,8}.$
Then $N^2_{2,8}:\ 10r - 9 = 0.$

Hence $x_2 = 0$. In this way it is easy to see that $x_1 = \ldots = x_{11} = 0$. Consider the equations

$$
\begin{aligned}
f_i = {} & N^1_{i-11,i-6} \qquad i = 12, \ldots 18\\
g_i = {} & N^2_{i-11,i-4} \qquad i = 12, \ldots 16
\end{aligned}
$$

These are exactly the equations from Lemma 3.1 , hence $x_{12} = \ldots = x_{21}$. From the formulas for E_{11} it is clear that M is faithful iff

$$x_{21} - 9x_{20} + 36x_{19} - 84x_{18} + 126x_{17} - 126x_{16} + 84x_{15} - 36x_{14} + 9x_{13} - x_{12} \neq 0.$$

But obviously this condition now is contradicted. □

Type $\{10\}$ *:*

The condition for faithfulness of such a module is $x_{21} \neq 9x_{20}$.

Case a: $x_{20} = 0$.

We may assume $x_{21} = 1$. From $N^1_{8,12}, N^2_{6,12}, N^1_{6,9}, N^1_{7,10}$ it follows easily $x_6 = \ldots = x_9 = 0$. Also $x_2 = \ldots = x_5 = 0$ by $R^4_{2,12}, R^3_{2,12}, N^1_{2,5}, N^1_{3,6}$. Applying $\widehat{f_{10}} = f_{10} + \beta f_{11}$ one obtains $\widehat{x_{11}} = x_{11} - \beta$. Hence we may assume $x_{11} = 0$. With $N^1_{7,12} : x_{19} = 2x_{18} - 1$ and $N^2_{5,12} : x_{18} = r - 1 + 3x_{17} - 2x_{16}$ we are lead once more to the standard system of Lemma 4.2 ($i = 13, N^1_{2,7}, \ldots, N^2_{3,10}, x_{17} = r, x_{16} = r_2, \ldots, x_{13} = r_5$) and $N^1_{1,5}$: $x_1(5r^2 - 10r + 3)(3r^2 - 2r + 3) = 0$ enforces $x_1 = 0$. The equations $N^1_{1,6}, N^2_{1,8}$ are polynomials in r and x_{12} , which must be zero. Hence their resultant with respect to x_{12} is also zero. The condition is $(10r - 9)(r - 1)^5(r - 2) = 0$, a contradiction.

Case b: $x_{20} \neq 0$.

If $x_{21} = 0$ then $x_1 = \ldots x_8 = 0$ and $x_{18} = 3, x_{17} = 2, x_{15} = (10r - 3)/5$ by $N^4_{1,12}, R^1_{2,12}, R^1_{1,11}, N^1_{3,6}, R^7_{5,12}, R^7_{4,11}, N^1_{8,12}, N^1_{7,11}$ and $N^1_{7,12}, N^1_{6,12}, N^1_{4,8}$. Then $N^2_{4,11}$: $10r = 9$. Hence we may assume $x_{21} = 2, x_{11} = 0$ (apply $\widehat{f_{10}} = f_{10} + \beta f_{11}$). It follows easily $x_1 = \ldots = x_8 = 0$ and then $x_{18} = 2x_{17} - 1, \ x_{17} = (r - 1) + 3x_{16} - x_{15}$ from $N^1_{6,11}, N^2_{4,11}$. Now we are ready to apply the standard system of Lemma 4.2 ($\Rightarrow x_{16} = r, x_{15} = 2r - 1, \ldots, x_{12} = r_5$). We obtain $x_{19} = 0, x_{20} = -1$ by $N^1_{5,10}, N^2_{7,12}$ and

$$
\begin{aligned}
N^1_{3,9} &: \quad x_{29} = (2x_{28} - 5x_{27})/2 \\
N^1_{4,10} &: \quad x_{28} = (2rx_{27} + 3x_{26} - 4x_{27})/3(r - 1) \\
N^1_{5,11} &: \quad x_{27} = (11r - 10)(r - 1) = 0
\end{aligned}
$$

If $r = 10/11$ then $s = 0, t = 0$ by $N^2_{3,12}$. Otherwise $x_{27} = 0$ and

$$
\begin{aligned}
N^2_{4,12} &: \quad x_{26} = (7x_{25} - 8s)/7 \\
N^1_{3,11} &: \quad x_{25}(32rs - 7rx_{24} - 48s + 21x_{24})/14r \\
N^2_{3,10} &: \quad s(r - 1) = 0
\end{aligned}
$$

This implies $s = 0$ (we may also deduce $t = 0$), which we have excluded. For $s = t = 0$ however the remainig equations can be fulfilled, there are several modules of type $\{10\}$, see Remark 1.3 . □

Type $\{11\}$ *:*

Assume that there is a Δ - module of type $\{11\}$. The nonzero coefficients of E_{11} are $x_{10}, \ x_{21}$ and $\sum_{i=1}^{10} a_i x_i$. We have $x_{11} = 0$ by $N^3_{1,12}$. The case $x_{21} \neq 0$ reduces to the case of type $\{10, 11\}$; the computation is very similar. So let us assume $x_{21} = 0$. Moreover $x_1 x_2 = 0$ by Lemma 4.4 . It is easy to see that $x_4 = \ldots = x_8 = 0$.

Case a: $x_{10} \neq 0$.

It follows $x_2 = 0, \ x_{19} = 2x_{18} - 1, \ x_{18} = 3x_{17} - 2x_{16} + r - 1$ by $N^4_{1,11}, N^1_{7,11}, N^2_{5,11}$. This leads to the "standard system" of Lemma 4.2 (with $N^1_{2,7}, N^1_{3,8}, N^1_{4,8}, N^1_{5,10}, N^2_{2,9}, N^2_{3,10}$ and

$i=13$), hence $x_{17}=r_1,\ x_{16}=r_2,\dots,x_{13}=r_5$. $N^1_{1,5}:\ x_1(5r^2-10r+3)(3r^2-2r+3)=0,\ \Rightarrow\ x_1=0$. By $N^1_{1,6}, N^2_{1,8}$ we have $(10r-9)(r-1)^5$ just as in *case a* of type $\{10,11\}$.

Case b: $x_{10}=0$.

One has $x_2=0$, otherwise $N^1_{2,6}, N^2_{2,8}, N^1_{1,5}, N^2_{1,7}$ would imply $N^1_{3,8}:\ 10r-9=0$. The module is faithful iff $x_1\neq 0$. Now the situation is the same as in *case b:* of type $\{10,11\}$, i.e $x_{13}=-1, x_{17}=-r_3, x_{18}=-r_4$ and $x_{19}=-r_5$ as before. After replacing x_{20} by $20r^4-28r^3+27r^2-24r+9/(2-r)(5r^2-12r+3)$ ($N^1_{6,11}$) we get $N^2_{4,11}$: $x_1(10r-9)(r-1)^5=0$, a contradiction. □

The remainig cases can be proved by the same methods. They are shorter than the above types. As a final example of such a computation we will prove

Type $\{6\}$ *:*

Assume that there exists a such a module. From $N^3_{1,12}$ one has $x_6=0$. E_{11} is zero iff $x_{17}=x_{16}$.

Case a: $x_{16}\neq 0$.

Then we may assume $x_{16}=1$ and $x_5=0$ (set $\hat{f}_6:=f_6+\alpha f_7$ and $\hat{f}_i=f_i$ for $i\neq 6$, by a diagonal base change one obtains $x_{16}=1$).
Now $N^1_{3,7}, N^1_{3,6}$ mean $x_3=x_4=0$ and $N^1_{1,4},\ R^6_{1,7}$ say $x_1=x_2=0$. The module is faithful iff $x_{17}\neq 1$. $N^1_{4,8},\ R^7_{2,9}, N^2_{4,10}$ imply $x_7=x_8=x_9=0$ and $R^4_{1,11},\ R^1_{2,12}$: $x_{10}=x_{11}=0$. It is $x_{17}\neq 0$, otherwise $x_{14}=3,\ x_{13}=2,\ x_{18}=-3$ by $N^1_{3,8}, N^1_{2,7}, N^1_{4,9}$ and then $N^2_{2,9}$: $10r=9$. Substituting $x_{14}=2x_{13}-1$ and $x_{20}=(x_{19}-1)/2$ $(N^1_{2,7}, N^1_{6,11})$ yields the following system of equations:

$$
\begin{array}{lcl}
N^1_{3,8} & : & x_{17}(x_{15}-4x_{13}+3)+2(x_{13}-2)=0\\
N^1_{5,10} & : & x_{17}(x_{19}-x_{15}+6)-2(x_{19}+2)=0\\
N^2_{2,9} & : & x_{17}(5r-3x_{13}+x_{15}-3)-10r+3x_{13}+3=0\\
N^2_{3,10} & : & x_{17}(10r+x_{19}-6x_{13}+3)-10r-3x_{19}+2x_{13}-1=0\\
N^2_{4,11} & : & x_{17}(20r+3x_{19}-2x_{15}-3)-10r-3x_{19}+3=0
\end{array}
$$

Eliminating quadratic terms one easily gets $(10r-9)(x_{17}-1)=0$, a contradiction.

Case b: $x_{16}=0$. This case is reduced to *case a:* by duality. □

We will now prove Theorem B:
The following Birkhoff Embedding Theorem is a special case of Ado's Theorem:

THEOREM: *Let* $\mathfrak{g}$ *be a nilpotent Lie algebra over* k *. Then there is a finite-dimensional vectorspace* V *together with a faithful representation* $\varrho:\mathfrak{g}\longrightarrow\mathfrak{gl}(V)$ *, such that* $\varrho(X)$ *is nilpotent for all* $X\in\mathfrak{g}$.

The construction goes as follows (see [6]):
Let $\mathfrak{g}$ be k-step nilpotent, $\mathfrak{g}^{(1)} = \mathfrak{g}$ and $\mathfrak{g}^{(i+1)} = [\mathfrak{g}, \mathfrak{g}^{(i)}]$. Choose a basis $X_1, \dots, X_n$ of $\mathfrak{g}$ such that the first n_1 elements span $\mathfrak{g}^{(k)}$, the first n_2 elements span $\mathfrak{g}^{(k-1)}$ and so on. We will construct V as a quotient of the universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$. By the Poincaré-Birkhoff-Witt Theorem the ordered monomials

$$X^\alpha = X_1^{\alpha_1} \dots X_n^{\alpha_n}, \quad \alpha = (\alpha_1, \dots \alpha_n) \in \mathbb{Z}_+^n$$

form a basis for $U(\mathfrak{g})$. Let $T = \sum_\alpha c_\alpha X^\alpha$ be an element of $U(\mathfrak{g})$ (with only finitely many nonzero c_α). Define an order function as follows:

$$\begin{array}{lll} \operatorname{ord}(X_j) & := & \max\{m : X_j \in \mathfrak{g}^{(m)}\} \qquad \operatorname{ord}(X^\alpha) := \sum_{j=1}^n \alpha_j \operatorname{ord}(X_j) \\ \operatorname{ord}(T) & := & \min\{\operatorname{ord}(X^\alpha) : c_\alpha \neq 0\} \qquad \operatorname{ord}(1_{U(\mathfrak{g})}) = 0, \ \operatorname{ord}(0) = \infty \end{array}$$

One can show that the order function satisfies:

$$\begin{array}{lll} \operatorname{ord}(T_1 + \dots + T_j) & \geq & \min\{\operatorname{ord}(T_1), \dots, \operatorname{ord}(T_j)\} \\ \operatorname{ord}(T_1 \dots T_j) & \geq & \operatorname{ord}(T_1) + \dots + \operatorname{ord}(T_j) \end{array}$$

Now let

$$U^m(\mathfrak{g}) = \{T \in U(\mathfrak{g}) : \operatorname{ord}(T) \geq m\}$$

From the above it is clear that $U^m(\mathfrak{g})$ is an ideal of $U(\mathfrak{g})$ having finite codimension. Define

$$V = U(\mathfrak{g})/U^m(\mathfrak{g}).$$

Choose a basis $\{T_1, \dots, T_l\}$ of V such that $T_1, \dots, T_{l_1}$ span $U^{m-1}(\mathfrak{g})/U^m(\mathfrak{g})$, $T_1, \dots, T_{l_2}$ span $U^{m-2}(\mathfrak{g})/U^m(\mathfrak{g})$ and so on. Then it is easy to check that the desired representation of $\mathfrak{g}$ is obtained by setting

$$\varrho(X)(T_j) = XT_j \pmod{U^m(\mathfrak{g})}.$$

If $m > k$ then $\varrho(X) \cdot 1_{U(\mathfrak{g})} = X \neq 0$ for all $X \in \mathfrak{g}$, so that ϱ is faithful.

Now let $\mathfrak{g} = \mathfrak{a}(r, s, t)$: Take $\{X_1, \dots, X_n\} = \{e_{11}, \dots, e_1\}$, $e^\alpha = e_{11}^{\alpha_{11}} \cdots e_1^{\alpha_1}$. One has $\operatorname{ord}(e_1) = \operatorname{ord}(e_2) = 1$ and $\operatorname{ord}(e_i) = i - 1$ for $i > 2$. The module V described above has the vector space basis ($k = 10$, choose $m = 11$):

$$\{e_{11}^{\alpha_{11}} \cdots e_1^{\alpha_1} \mid 10\alpha_{11} + 9\alpha_{10} + \dots + 2\alpha_3 + \alpha_2 + \alpha_1 \leq 10\}$$

The elements e_i of $\mathfrak{g}$ act on V by $e_i e_j = [e_i, e_j] + e_j e_i$ for $i < j$ (otherwise the monomial $e_i e_j$ is already in the right order, i.e is element of V). We may factor out any proper submodule of V not containing e_{11} to obtain a faithful $\mathfrak{g}$-module of smaller dimension. It is preferable to factor out only monomials, not linear combinations of monomials. If one factors out as many monomials as possible it is not difficult to see that one is led to a quotient module $\widehat{V}$ of V with the following remainig monomials as a vector space base for $\widehat{V}$:

$$\{e_{11}, e_{10}, e_9, e_5^2, e_8, e_5e_4, e_4e_3^2, e_7, e_5e_3, e_5e_2^2, e_4^2, e_4e_3e_2, e_4e_2^3, e_3^3, e_3^2e_2^2, e_6,$$
$$e_5e_2, e_4e_3, e_4e_2^2, e_3^2e_2, e_3e_2^3, e_2^5, e_5, e_4e_2, e_3^2, e_3e_2^2, e_2^4, e_4, e_3e_2, e_2^3, e_3, e_2^2, e_2, 1\}$$

We have constructed a faithful $\mathfrak{a}(r,s,t)$ - module $\widehat{V}$ of dimension 34 ; the action of e_1, e_2 can be written down explicitly. This module has a seven-dimensional center Z containing e_{11} . Factor out a subspace of Z complementary to the vector space generated by e_{11} . The quotient is of dimension 28 and has a four-dimensional center. Repeat the forgoing step to obtain a faithful module which has also a four-dimensional center. The next quotient W finally has one-dimensional center e_{11} . Every proper submodule of W intersects this center nontrivially, i.e contains e_{11} .
The computation of the centers is much simpler for fixed r (take for example $r = 1/2$ or $r = -2$). The dimension of the centers does not depend on r, s, t as long as $r \in A$. The dimension of W is $34 - 6 - 3 - 3 = 22$ and W is cyclic, generated by 1 . $\square$

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