Weak shocks for a one-dimensional BGK kinetic model for conservation laws

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Abstract

For one-dimensional kinetic BGK models, regarded as relaxation models for scalar conservation laws with genuinely nonlinear fluxes, existence of small amplitude travelling waves is proven. Dynamic stability of these kinetic shock profiles is shown by extending a classical energy method for viscous regularizations of conservation laws.

1 Introduction

In this paper we study small amplitude travelling wave solutions of the following one-dimensional BGK type equation

$$\partial_t f + v\partial_x f = M(\rho_f, v) - f\,, \quad \text{with } t > 0\,,\ x \in \mathbb{R}\,,\ v \in \Omega\,. \tag{1.1}$$

Here $f(t,x,v)$ can be interpreted (in analogy with the Boltzmann equation) as a time dependent phase space density of particles with time t, position x, and velocity v. We shall assume that $\Omega \subset \mathbb{R}$ is the support of a measure $d\mu(v)$. In particular, both continuous velocity distributions as well as discrete velocity models, where (1.1) is a hyperbolic system, are included in our assumptions.

The function $\rho_f(t,x)$ in (1.1) is the macroscopic density corresponding to the distribution f, i.e., the zeroth order velocity moment

$$\rho_f(t,x) = \int f(t,x,v)d\mu(v)\,. \tag{1.2}$$

Here and in the following we omit to write Ω under the integral sign in integrals with respect to the measure $d\mu(v)$. Note that in the case of a discrete velocity model, Ω is a discrete set, and the integral above is a sum. The 'Maxwellian' $M(\rho,v)$ is an equilibrium distribution satisfying the moment conditions

$$\int M(\rho,v)d\mu(v) = \rho, \quad \text{and} \int v\, M(\rho,v)d\mu(v) = a(\rho)\,, \tag{1.3}$$

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for a macroscopic flux function $a(\rho)$ that will be assumed smooth and genuinely nonlinear, actually (without loss of generality) concave: $a''(\rho) < 0$. The properties (1.3) ensure, at least formally, that the macroscopic limit equation (scaling with $(t,x) \to (t/\varepsilon, x/\varepsilon)$ and taking $\varepsilon \to 0$) of (1.1) is the scalar conservation law

$$\partial_t \rho + \partial_x a(\rho) = 0\,. \tag{1.4}$$

It is well-known that initial value problems for equation (1.4) do not possess smooth solutions in general, and that weak solutions are not unique. Uniqueness can be obtained by considering (1.4) as the limit of an appropriately regularized problem. Classically this is done by introducing an artificial viscosity and carrying out the limit $\nu \to 0^+$ in

$$\partial_t \rho + \partial_x a(\rho) = \nu \partial_x^2 \rho\,, \tag{1.5}$$

see, e.g., [10]. In this work, instead of (1.5), the kinetic regularization (1.1) is studied.

Typical weak solutions of (1.4) are shock waves of the form

$$\rho(t,x) = \begin{cases} \rho_- & \text{if } x - st < x_0\,, \\ \\ \rho_+ & \text{if } x - st > x_0\,, \end{cases}$$

where the constants $\rho_\pm$ and the wave speed s are related by the Rankine-Hugoniot condition

$$s = \frac{a(\rho_+) - a(\rho_-)}{\rho_+ - \rho_-}\,. \tag{1.6}$$

The admissibility condition

$$\frac{a(\rho) - a(\rho_-)}{\rho - \rho_-} - s > 0 \quad \text{for all } \rho \in (\min(\rho_+, \rho_-), \max(\rho_+, \rho_-))\,, \tag{1.7}$$

can be derived by constructing viscous profiles, i.e. travelling wave solutions of (1.5). In this framework, (1.7) gives a necessary and sufficient condition for existence of travelling wave solutions connecting the values $\rho_\pm$ at $x - st = \pm\infty$. For the concave flux functions $a(\rho)$ considered here, (1.7) reduces to the condition $\rho_- < \rho_+$. This is called an *entropy condition* since it can also be obtained from the (distributional) entropy inequality

$$\partial_t \phi(\rho) + \partial_x \psi(\rho) \le 0\,,$$

which can be derived for every convex entropy density $\phi(\rho)$ and corresponding entropy flux $\psi(\rho)$ (satisfying $\psi' = \phi' a$) in the limit $\nu \to 0$ from (1.5).

An entropy inequality can also be derived for solutions of the kinetic equation (1.1) under an additional structure condition on the equilibrium distribution. We shall assume that the Maxwellian is a smooth and strictly increasing function of ρ:

$$\partial_\rho M(\rho, v) > 0\,. \tag{1.8}$$

Then there exists a function $\theta(f,v)$ such that $f = M(\rho,v)$ is equivalent to $\rho = \theta(f,v)$. With the primitive $\Theta(f,v)$ ($\partial_f\Theta = \theta$), solutions of (1.1) formally satisfy the entropy inequality

$$\begin{aligned}&\partial_t \int \Theta(f,v)d\mu(v) + \partial_x \int v\,\Theta(f,v)\,d\mu(v)\\&= \int (M(\rho_f,v) - f)(\theta(f,v) - \rho_f)\,d\mu(v) \le 0\,.\end{aligned}$$

In the context of relaxation systems, condition (1.8) can be seen as a subcharacteristic condition. It can be used for proving stability results such as a TVD property corresponding to that for entropy solutions of the macroscopic equation (1.4), see [1], [5]. A class of examples of Maxwellians $M(\rho,v)$ satisfying the moment conditions (1.3) as well as (1.8) has been given by the authors in [3]:

$$M(\rho,v) = \int_0^\rho m(v - a'(r))dr\,,$$

where $m(v) > 0$ for $v \in \mathbb{R}$ is an even function satisfying $\int_{-\infty}^{\infty} m(v)d\mu(v) = 1$ ($\Omega = \mathbb{R}$, $d\mu(v) = dv$).

It is our aim to study small amplitude kinetic shock profiles as travelling wave solutions of kinetic models of the form (1.1). Assuming (1.8), we shall prove their existence under the same entropy condition as required for the viscous regularization in Section 3. This is no surprise considering that our constructive existence proof shows asymptotic closeness of viscous and kinetic profiles for small shocks. A main ingredient of the proof is a fluid-kinetic (or micro-macro) decomposition in the spirit of the one introduced by Caflisch and Nicolaenko [2] for the gas dynamics Boltzmann equation.

A well known kinetic model for scalar conservation laws is the Perthame-Tadmor model, see [9]. There the Maxwellian is a discontinuous function. This lack of smoothness is an obstacle for the study of small waves by perturbation arguments as carried out here. Existence of big travelling waves has been studied by compactness arguments in [4]. The same approach has been carried out for (1.1) by the authors of this work [3]. In this parallel, the results of the present study are reviewed, and the existence result is extended to large amplitude waves. As opposed to the results here, the existence proof for large waves is nonconstructive, and their stability is still open.

In Section 4, local dynamic stability of the constructed travelling waves is proven. Again, a micro-macro decomposition (now in the spirit of Liu and Yu [7]) is at the heart of the argument. A classical energy method for proving stability on the macroscopic level is combined with entropy estimates for the kinetic perturbations.

In the remainder of this section, we present the formal asymptotics for the construction of small amplitude waves as well as the energy method for proving stability of viscous profiles.

Formal construction of kinetic shock profiles

We look for solutions of (1.1), whose dependence on x and t is only through the travelling wave variable $\xi = x - st$, with s being the wave speed:

$$(v-s)\partial_\xi f = M(\rho_f, v) - f\,, \qquad \xi \in \mathbb{R},\ v \in \Omega\,, \tag{1.9}$$

subject to the far-field conditions

$$f(\pm\infty, v) = M(\rho^\pm, v)\,, \qquad v \in \Omega\,. \tag{1.10}$$

We are interested in small amplitude waves and assume

$$\rho^+ - \rho^- = \varepsilon \qquad \text{with } 0 < \varepsilon \ll 1\,. \tag{1.11}$$

The positivity of ε reflects the entropy condition (1.7). It turns out that it is appropriate to rescale the travelling wave variable by $\xi \to \xi/\varepsilon$, to get

$$\varepsilon(v-s)\partial_\xi f = M(\rho_f, v) - f \quad \xi \in \mathbb{R},\ v \in \Omega\,. \tag{1.12}$$

The Rankine-Hugoniot condition (1.6) is derived as a necessary condition for existence by integrating equation (1.12) with respect to v and by (1.3).

The formal asymptotics below is a variant of the Chapman-Enskog expansion procedure. We start by introducing the decomposition

$$f = M(\rho_f, v) + \varepsilon^2 f^\perp, \quad \text{with } \int f^\perp d\mu(v) = 0\,, \tag{1.13}$$

of the solution into an equilibrium part and a remainder (or into a macroscopic and a microscopic contribution, or into a fluid and a kinetic part). As a next step, the travelling wave equation (1.12) is integrated with respect to v and ξ:

$$\int v f\, d\mu(v) - s\rho_f = a(\rho_-) - s\rho_-\,. \tag{1.14}$$

Essentially, this equation is considered as an equation for the macroscopic density ρ_f, and the full kinetic equation (1.12) should determine $f^\perp$. The smallness of the wave is reflected in the fact that the macroscopic density is everywhere close to its far field value at $\xi = -\infty$ and that the wave speed is close to the characteristic speed there:

$$\rho_f = \rho_- + \varepsilon u, \quad s = a'(\rho_-) + \varepsilon\sigma\,. \tag{1.15}$$

Substitution of this and (1.13) into (1.12) and (1.14), give the leading order ($O(\varepsilon^2)$) term equation

$$f^\perp = -(v - a'(\rho_-))\partial_\rho M(\rho_-, v)\partial_\xi u\,, \tag{1.16}$$

$$-\int v f^\perp d\mu(v) = \frac{a''(\rho_-)}{2}u^2 - \sigma u\,. \tag{1.17}$$

The limiting version of the Rankine-Hugoniot condition (1.6) is given by $\sigma = a''(\rho_-)/2 < 0$. After elimination of $f^\perp$, this becomes the travelling wave equation of the viscous Burgers equation

$$D_0\,\partial_\xi u = -\sigma u(1-u)\,, \tag{1.18}$$

with the diffusivity

$$D_0 = \int (v - a'(\rho_-))^2\,\partial_\rho M(\rho_-, v)\,d\mu(v) > 0\,, \tag{1.19}$$

by $\partial_\rho M > 0$. Obviously, solutions of (1.18) connecting $u = 0$ at $\xi = -\infty$ to $u = 1$ at $\xi = +\infty$ exist. The lack of uniqueness due to the translation invariance of the travelling wave problem will be an issue below.

It is far from obvious how to make this argument rigorous, since the equation (1.16) for $f^\perp$ is a singular limit and, even worse, its solution is a differentiation problem. In the existence proof in Section 3 we adapt an idea from Caflisch and Nicolaenko [2], where existence of weak shock profiles for the Boltzmann equation of gas dynamics has been proven. It is based on a slight modification of the micro-macro decomposition such that the fluid and the kinetic terms satisfy a system of equations with separated derivatives.

Stability of viscous shock profiles

In Section 4 we prove local dynamic stability of small amplitude travelling waves. The idea is to decouple the equation into a macroscopic part and a small microscopic part. Then we use L^2-type energy (actually entropy) methods for the macroscopic equation, which can be extended to also control the microscopic part. Similar techniques have been used by Liu and Yu [7] for the Boltzmann equation. For the Broadwell model, a discrete velocity model for the Boltzmann equation, energy estimates have also been used in [6].

We expand briefly on the ideas behind the L^2-estimates at the macroscopic level. If ϕ is a travelling wave solution of the diffusive regularisation (1.5), then the perturbation $\tilde\rho = \rho - \phi$ satisfies

$$\partial_t\tilde\rho - s\partial_\xi\tilde\rho + \partial_\xi(a(\phi+\tilde\rho) - a(\phi)) = \nu\partial_\xi^2\tilde\rho.$$

Linearizing this equation and testing it with $\tilde\rho$ produces a term with the wrong sign, which is not possible to control. The usual trick (see e.g. [8]) to overcome this problem is to introduce the new macroscopic unknown

$$W(\xi, t) = \int_{-\infty}^{\xi} \tilde\rho(x, t)\,dx\,, \tag{1.20}$$

after choosing the shift in ϕ such that $\int_{\mathbb{R}}(\rho-\phi)d\xi = 0$. Testing the integrated perturbation equation

$$\partial_t W - s\partial_\xi W + a(\phi+\tilde\rho) - a(\phi) = \nu\partial_\xi^2 W\,,$$

with W, gives in particular the term

$$\int_{\mathbb{R}} (a(\phi + \tilde{\rho}) - a(\phi))W \; d\xi = -\frac{1}{2} \int_{\mathbb{R}} \partial_\xi (a'(\phi)) W^2 \, d\xi + n.l.t.$$

(*n.l.t.* stands for *nonlinear terms*). By the monotonicity of the wave profile the term on the right hand side is positive, indicating decay of the L^2-norm of W if the nonlinear terms can be controlled.

The idea now consists of combining the energy estimates for W and for $\tilde{\rho} = \partial_\xi W$ to get an estimate on the H^1-norm of W. Clearly in both cases the contribution of the diffusion term has the good sign. This way we can also control the term with the wrong sign for $\tilde{\rho}$ by the term coming from diffusion for W. The basic estimate reads

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}} \left(W^2 + \gamma (\partial_\xi W)^2 \right) \, d\xi \leq - \left(\nu - C_0 \sup |W| - \gamma C_1 \right) \int_{\mathbb{R}} (\partial_\xi W)^2 \, d\xi \,, \tag{1.21}$$

for some arbitrary $\gamma > 0$. Here C_0 and C_1 are positive constants depending on pointwise bounds for the density ρ which are a consequence of the maximum principle. The supremum norm of W is controlled by the H^1-norm in one dimension. Hence, starting with initial data such that $\sup |W(t=0)|$ is small enough, and choosing γ small enough, the right hand side of (1.21) is negative initially and remains so. This implies global existence for $W \in H^1(\mathbb{R})$, as well as stability of macroscopic travelling waves. To achieve an analogous result for the kinetic equation, the argument will be similar, but one has to take care of the contribution of the microscopic part, which shall, however, be small by assumption (1.11). Additional difficulties will rise from the absence of a maximum principle requiring estimates for $\tilde{\rho}$ in H^1 (W in H^2) for pointwise control of $\tilde{\rho}$, as well as from the fact that the monotonicity of the macroscopic density of the kinetic travelling wave is not obvious.

2 Notation and assumptions

Since we shall linearise around the state $M(\rho_-, v)$, we introduce the notation $F(v) := \partial_\rho M(\rho_-, v)$ for simplicity. We shall sometimes skip the dependence on v in the function M, and write $M'(\rho)$ instead of $\partial_\rho M(\rho, v)$ (i.e. $F = M'(\rho_-)$).

We shall work in the weighted Hilbert space L^2_v of functions of the velocity, defined by the scalar product

$$\langle f, g \rangle_v = \int \frac{fg}{F} d\mu(v) \,,$$

where $\|\cdot\|_v$ denotes the induced norm. We also consider the L^2- and H^k-norms for functions of ξ. We write these spaces as L^2_ξ and H^k_ξ, and their norms as $\|\cdot\|_\xi$ and $\|\cdot\|_{H^k}$, respectively.

The Hilbert space $L^2_{\xi,v}$ is then naturally defined by the scalar product

$$\langle f, g \rangle_{\xi,v} = \int_{\mathbb{R}} \langle f, g \rangle_v d\xi \,,$$

with the induced norm $\|\cdot\|_{\xi,v}$. Similarly, we shall denote by $H^k_\xi(L^2_v)$ the space of functions with derivatives with respect to ξ up to order k in L^2_v, and the corresponding norm

$$\|f\|_{H^k_\xi(L^2_v)} = \left(\|f\|^2_{\xi,v} + \ldots + \|\partial^k_\xi f\|^2_{\xi,v}\right)^{1/2} . \tag{2.1}$$

The linearisation of the collision operator on the right hand side of (1.1) around $M(\rho_-, v)$ is given by

$$\mathcal{L}f := F\rho_f - f . \tag{2.2}$$

It is symmetric and negative semidefinite in L^2_v. These properties are easily seen from the identity

$$\langle \mathcal{L}f_1, f_2\rangle_v = -\frac{1}{2}\iint FF'\left(\frac{f_1}{F} - \frac{f_1'}{F'}\right)\left(\frac{f_2}{F} - \frac{f_2'}{F'}\right) d\mu(v)\, d\mu(v') ,$$

where $'$ denotes evaluation at v'. The entropy inequality $\langle \mathcal{L}f, f\rangle_v \le 0$ is a straightforward consequence.

Apart from the essential requirements (1.3) and (1.8), our existence and stability proofs rely on additional technical assumptions on M. For fixed $v \in \Omega$, we assume that $M(\rho, v)$ is a C^3-function of ρ. Moreover, for a given ρ, up to fourth order moments of the derivatives exist, i.e.,

$$\int \left|v^m \partial^k_\rho M(\rho, v)\right| d\mu(v) < \infty \qquad \text{for } k \le 3,\ m \le 4 . \tag{2.3}$$

For the second and third order derivatives of M with respect to ρ we require that for given ρ_1 and ρ_2:

$$\int |v|^m \frac{(\partial^k_\rho M(\rho_1, v))^2}{\partial_\rho M(\rho_2, v)}\, d\mu(v) < \infty \qquad \text{for } k \le 3,\ m \le 4 . \tag{2.4}$$

As a consequence of (2.3), up to second order moments of velocity distributions can be bound by their L^2_v-norm:

$$\left|\int (v-s)^m f\, d\mu(v)\right| \le \left(\int |v-s|^{2m} F\, d\mu(v)\right)^{1/2} \|f\|_v \qquad \text{for } m \le 2 . \tag{2.5}$$

This will be used repeatedly in the following. For simplicity we also adopt the notation

$$\hat{D} := \int (v-s)^2 F\, d\mu(v) . \tag{2.6}$$

Finally, we assume that for fixed ρ, $\partial_\rho M(\rho, v)$ is a continuous function of $v \in \Omega$.

3 Existence of small amplitude travelling waves

3.1 An approximate solution

In this section we prove existence of solutions of (1.12) subject to (1.10) for $\varepsilon \ll 1$. We start by returning to the problem of constructing a formal approximation. Instead of formally passing to the limit as in Section 1, we avoid expansion errors whereever possible and produce a residual whose v-integral vanishes. We start with the ansatz

$$f_{as} := M(\rho, v) + \varepsilon^2 f^\perp \,, \tag{3.1}$$

formally resembling (1.13). The residual is then given by

$$\varepsilon^3 h := M(\rho_{as}) - f_{as} - \varepsilon(v-s)\partial_\xi f_{as} \,, \qquad \text{with } \rho_{as} = \rho + \varepsilon^2 \rho^\perp \,. \tag{3.2}$$

Recalling (1.16), we eliminate two terms in the right hand side by the choice

$$f^\perp := -\frac{1}{\varepsilon}(v-s)\partial_\xi M(\rho, v) \,. \tag{3.3}$$

Finally, the requirement that the v-integral of h vanishes and that f_{as} satisfies the far-field conditions (1.10) leads to an ordinary differential equation for ρ:

$$\frac{a(\rho) - a(\rho_-) - s(\rho - \rho_-)}{\varepsilon^2} = \frac{1}{\varepsilon}\left(\int (v-s)^2 M'(\rho, v) d\mu(v)\right) \partial_\xi \rho \,, \tag{3.4}$$

subject to

$$\rho(-\infty) = \rho_- \quad \text{and} \quad \rho(+\infty) = \rho_+ \,. \tag{3.5}$$

With $\rho = \rho_- + \varepsilon u$ the problem for u formally tends to (1.18) in the limit $\varepsilon \to 0$. Actually, since the diffusivity $D(\rho) := \int (v-s)^2 M'(\rho, v)\, d\mu(v)$ is obviously positive, (3.4) has the same qualitative properties as (1.18), and a solution of (3.4), (3.5) exists, which is determined uniquely by the condition

$$\rho(0) = \frac{\rho_- + \rho_+}{2} \,. \tag{3.6}$$

It is easily shown that u and $\partial_\xi u$ are uniformly bounded as $\varepsilon \to 0$ and for $\xi \in \mathbb{R}$, and, therefore, the same holds for ρ and $\partial_\xi \rho/\varepsilon$. As a consequence, $D(\rho)$ is uniformly bounded away from zero. Division of (3.4) by $D(\rho)$ and differentiation shows that also $\partial_\xi^k \rho/\varepsilon$ is uniformly bounded for $k = 2, 3$ (here assumption (2.3) is used). Furthermore, the convergence of all these terms as $\xi \to \pm\infty$ is exponential. Recalling $s = a'(\rho_-) + O(\varepsilon)$,

$$\rho^\perp = -\frac{1}{\varepsilon}(a'(\rho) - s)\partial_\xi \rho = O(\varepsilon)$$

holds uniformly for $\xi \in \mathbb{R}$. This shows that the scaling of the residual in (3.2) has been chosen correctly in terms of the sup-norm:

$$h = \frac{M(\rho + \varepsilon^2 \rho^\perp) - M(\rho)}{\varepsilon^3} - (v-s)\partial_\xi f^\perp \tag{3.7}$$

is uniformly bounded in ε and ξ and decays exponentially as $\xi \to \pm\infty$. However, we shall need this result also in other norms:

Lemma 3.1 *Let the assumptions (2.3) and (2.4) be satisfied and let f_{as} be determined by (3.1) and (3.3)–(3.5). Then f_{as} satisfies the far-field conditions (1.10), and the travelling wave equation (1.12) up to the residual $\varepsilon^3 h$, where h is in $H^1_\xi(L^2_v)$ uniformly in ε, and $\int h\, d\mu(v) = 0$.*

Proof. The far-field conditions and the last statement are a direct consequence of the construction of f_{as}.

The boundedness of the first term on the right hand side of (3.7) in $H^1_\xi(L^2_v)$ is a consequence of our observations above and of (2.4) ($m = 0$, $k = 2$). For the second term (2.4) is used with $m = 4$, $k = 2$. Of course the exponential decay of all terms suffices for integrability with respect to ξ. □

3.2 The micro-macro decomposition of the correction term

In terms of the correction term $\varepsilon^2 g = f - f_{as}$, the travelling wave problem reads

$$\begin{aligned} &\varepsilon(v-s)\partial_\xi g - \mathcal{L}g \\ &= (M'(\rho_{as}) - F)\rho_g + \frac{M(\rho_{as} + \varepsilon^2\rho_g) - M(\rho_{as}) - \varepsilon^2 M'(\rho_{as})\rho_g}{\varepsilon^2} + \varepsilon h\,, \end{aligned} \tag{3.8}$$

subject to

$$g(\pm\infty, v) = 0, \quad \text{for all } v \in \Omega\,. \tag{3.9}$$

The left hand side of (3.8) is the linearization of the travelling wave equation (1.12) around $M(\rho_-)$ with the linearized collision operator $\mathcal{L}$ defined in (2.2). The right hand side contains an $O(\varepsilon)$ linear correction (since we should actually linearize around $M(\rho_{as})$), an $O(\varepsilon^2\rho_g^2)$ nonlinear term, and the residual. The homogeneous far-field conditions and Lemma 3.1 imply, after integration of (3.8) with respect to ξ, that the flux of g vanishes:

$$\int (v-s) g\, d\mu(v) = 0\,. \tag{3.10}$$

The problem (3.8), (3.9) will be solved in several steps. First, we introduce a splitting of g into a macroscopic part and a microscopic part. Then, in the following two subsections, we solve the linear equations associated to the decomposition of g, and finally solve the nonlinear problem.

In the first step, two ideas from the work by Caflisch and Nicolaenko [2] on the Boltzmann equation will be adapted to the present situation. The first one is a special micro-macro decomposition defined by

$$g(\xi, v) = z(\xi)\Phi(v) + \varepsilon w(\xi, v)\,, \tag{3.11}$$

where $\Phi := F\left(1 + \varepsilon\frac{\sigma}{D}(v-s)\right)$, and the orthogonality condition $\langle (v-s)\Phi, w\rangle_v = 0$ holds.

The choice of the coefficient $\sigma/\hat{D}$ (see (1.15) and (2.6) for the definition of the constants) in front of the correction term in Φ guarantees that Φ shares the property (3.10) with g:

$$\int (v-s)\Phi\, d\mu(v) = 0\,. \tag{3.12}$$

This and the definition of the decomposition imply several properties of z and w.

Lemma 3.2 *If g satisfies (3.8), (3.9), then*

$$w(\pm\infty, v) \equiv 0,\;\; z(\pm\infty) = 0\,, \tag{3.13}$$

and

$$\int (v-s)w(\xi,v)\, d\mu(v) = 0, \qquad \int (v-s)^2 w(\xi,v)\, d\mu(v) = 0 \quad \textit{for all } \xi \in \mathbb{R}\,. \tag{3.14}$$

Substitution of (3.11) into (3.8) and division by ε gives

$$(v-s)\Phi\,\partial_\xi z + \varepsilon(v-s)\partial_\xi w - \mathcal{L}w - \Lambda z = \varepsilon\Gamma\rho_w + \varepsilon R(\rho_g) + h\,, \tag{3.15}$$

where again the right hand side contains a linear correction, the nonlinearity, and the residual, with

$$\Lambda := \frac{M'(\rho_{as})\rho_\Phi - \Phi}{\varepsilon}\,, \qquad \Gamma := \frac{M'(\rho_{as}) - F}{\varepsilon}\,,$$
$$R(\rho_g) := \frac{1}{\varepsilon^4}\Big[M(\rho_{as} + \varepsilon^2\rho_g) - M(\rho_{as}) - \varepsilon^2 M'(\rho_{as})\rho_g\Big]\,.$$

These terms are formally $O(1)$ such that the ε-powers in (3.15) reflect the expected orders of magnitude.

Observe that, in terms of z and w, the nonlinearity should be written as $R(\rho_g) = R(z\rho_\Phi + \varepsilon\rho_w)$ with $\rho_\Phi = 1 - \varepsilon^2\sigma^2/\hat{D}$ (a constant). The identities $\int \Lambda\, d\mu(v) = \int \Gamma\, d\mu(v) = \int R(\rho_g)\, d\mu(v) = 0$ hold.

In order to get an equation for z (the macroscopic equation), we apply an approximation of the macroscopic projection to (3.15), i.e. we multiply equation (3.15) by $(v-s)$ and integrate with respect to v:

$$\tilde{D}\,\partial_\xi z - r(\xi)z = \varepsilon\frac{a'(\rho_{as}) - s}{\varepsilon}\rho_w + \varepsilon\int (v-s)R(\rho_g)\, d\mu(v) + \int (v-s)h\, d\mu(v)\,, \tag{3.16}$$

with

$$\tilde{D} \;:=\; \int (v-s)^2\Phi\, d\mu(v) = D_0 + O(\varepsilon) > 0\,, \tag{3.17}$$

$$r(\xi) \;:=\; \int (v-s)\Lambda(\xi,v)\, d\mu(v) = \frac{a'(\rho_{as}(\xi)) - s}{\varepsilon}\rho_\Phi\,. \tag{3.18}$$

Here we have used (3.12) and Lemma 3.2. This equation already reveals the magic of the micro-macro decomposition (3.11). It does not contain derivatives of w, and actually becomes independent of w as $\varepsilon \to 0$. The formal limit is the linearization of the viscous Burgers travelling wave equation (1.18). In particular, $r(\xi) = \sigma(2u-1) + O(\varepsilon)$ and, consequently, there exist $\gamma, \bar\xi > 0$ such that

$$r(\xi) \le -\gamma \quad \text{for } \xi \ge \bar\xi\,, \qquad r(\xi) \ge \gamma \quad \text{for } \xi \le -\bar\xi\,. \tag{3.19}$$

Now an equation for w (the microscopic equation) is derived by substituting (3.16) into (3.15), which actually amounts to applying the projection

$$Pf := f - \frac{(v-s)\Phi}{\tilde D}\int (v-s) f\, d\mu(v) \tag{3.20}$$

to (3.15):

$$\varepsilon(v-s)\partial_\xi w - \mathcal{L}w - P\Lambda z = \varepsilon\tilde\Gamma\rho_w + \varepsilon P R(\rho_g) + Ph\,, \tag{3.21}$$

with

$$\tilde\Gamma = P\Gamma - \frac{(v-s)\Phi}{\tilde D}\int (v-s)F\, d\mu(v) = \Gamma - \frac{(v-s)\Phi}{\tilde D}(a'(\rho_{as}) - s)\,.$$

We make one more manipulation to get a final equation for w. This corresponds to the second idea from [2]. As we observed in Section 2 the operator $\mathcal{L}$ is symmetric negative semidefinite, but not strictly negative. We introduce a new operator $\mathcal{M}$, which is strictly negative and coincides with $\mathcal{L}$ on the set of functions w satisfying the property (3.14):

$$\mathcal{M}w := \mathcal{L}w - (v-s)^2 F\int (v-s)^2 w\, d\mu(v)\,.$$

Lemma 3.3 *The operator $\mathcal{M}$ is symmetric and negative definite in L^2_v, i.e., there exists a $\kappa > 0$ such that*

$$-\langle \mathcal{M}w, w\rangle_v > \kappa \|w\|_v^2 \qquad \textit{for all } w \in L^2_v\,.$$

Proof. The symmetry follows from the symmetry of $\mathcal{L}$ and from

$$\langle \mathcal{M}w_1, w_2\rangle_v = \langle \mathcal{L}w_1, w_2\rangle_v - \int (v-s)^2 w_1\, d\mu(v)\int (v-s)^2 w_2\, d\mu(v)\,.$$

To prove that $\mathcal{M}$ is negative definite, we write $w = F\rho_w + w^\perp$ and observe that $\mathcal{L}w = -w^\perp$:

$$\begin{aligned} -\langle \mathcal{M}w, w\rangle_v &= \|w^\perp\|_v^2 + \left(D_0\,\rho_w + \int (v-s)^2 w^\perp d\mu(v)\right)^2 \\ &= \|w^\perp\|_v^2 + \gamma D_0^2\,\rho_w^2 + (1-\gamma)D_0^2\,\rho_w^2 \\ &\qquad + 2D_0\,\rho_w \int (v-s)^2 w^\perp\, d\mu(v) + \left(\int (v-s)^2 w^\perp\, d\mu(v)\right)^2\,, \end{aligned}$$

for $\gamma \in (0,1)$. Hence

$$
\begin{aligned}
&-\langle \mathcal{M}w, w\rangle_v = \|w^\perp\|_v^2 + \gamma D_0^2\rho_w^2 \\
&+(1-\gamma)\left[D_0\,\rho_w + \frac{1}{1-\gamma}\int (v-s)^2 w^\perp\, d\mu(v)\right]^2 - \frac{\gamma}{1-\gamma}\left(\int (v-s)^2 w^\perp\, d\mu(v)\right)^2 \\
&\ge \gamma D_0^2\,\rho_w^2 + \|w^\perp\|_v^2\left(1 - \frac{\gamma}{1-\gamma}\|(v-s)^2 F\|_v^2\right) \ge \kappa(\rho_w^2 + \|w^\perp\|_v^2) = \kappa\|w\|_v^2\,,
\end{aligned}
$$

with $\kappa > 0$ for γ small enough. Here we have used (2.5) with $m = 2$. $\square$

We shall prove existence of solutions of equations (3.16) and

$$
\varepsilon(v-s)\partial_\xi w - \mathcal{M}w - zP\Lambda = \varepsilon\tilde\Gamma\rho_w + \varepsilon PR(\rho_g) + Ph\,, \tag{3.22}
$$

subject to (3.13). The relation to the original problem (3.8), (3.9) is not obvious.

Lemma 3.4 *The function $g = z\Phi + \varepsilon w$ is a solution of (3.8), (3.9) iff z and w solve (3.16), (3.22) subject to (3.13).*

Proof. The problem (3.16), (3.22), (3.13) has been derived from (3.8), (3.9) using the property (3.14) of solutions of the latter. The proof relies on showing that (3.14) also holds for solutions of (3.16), (3.22), (3.13) without requiring it as a side condition.

The properties of Φ imply that for a $f(v)$ satisfying $\int f\, d\mu(v) = 0$, also $\int Pf\, d\mu(v) = 0$ and $\int (v-s)Pf\, d\mu(v) = 0$ hold. Since the v integrals of Λ, Γ, $R(\rho_g)$, and h vanish, these terms do note contribute, when we integrate (3.22) and its product with $v-s$ with respect to v:

$$
\begin{aligned}
&\varepsilon\partial_\xi \int (v-s)w\, d\mu(v) = -D_0\int (v-s)^2 w\, d\mu(v)\,, \\
&\varepsilon\partial_\xi \int (v-s)^2 w\, d\mu(v) = -\int (v-s)w\, d\mu(v) - \int (v-s)^3 F\, d\mu(v)\,\int (v-s)^2 w\, d\mu(v)\,.
\end{aligned}
$$

This is a system of linear ODEs with constant coefficients for the unknowns $\int (v-s)w\, d\mu(v)$ and $\int (v-s)^2 w\, d\mu(v)$. The decay of w at $\xi = \pm\infty$ implies homogeneous far field conditions for these quantities and, thus, $\int (v-s)w\, d\mu(v) \equiv \int (v-s)^2 w\, d\mu(v) \equiv 0$. $\square$

3.3 The linear problem

In this section we prove solvability of the equations (3.16) and (3.22) regarding the right hand sides as given inhomogeneities. In particular, we look for solutions of

$$
\varepsilon(v-s)\partial_\xi w - \mathcal{M}w = h_w \qquad \text{with } h_w \in H_\xi^1(L_v^2)\,, \tag{3.23}
$$

and

$$
\partial_\xi z - r(\xi)z = h_z \qquad \text{with } h_z \in L_\xi^2\,, \tag{3.24}
$$

We shall look for solutions in the same spaces as the inhomogeneities. This will replace homogeneous far-field conditions in the following. Whereas this requirement provides uniqueness for the the solution of (3.23), it permits a one parameter set of solutions of (3.24). This reflects the arbitrary shift in travelling wave solutions. Uniqueness will be guaranteed by the additional requirement

$$z(0) = z_0 \,, \tag{3.25}$$

where $z_0 \in \mathbb{R}$ parametrizes the set of solutions.

Lemma 3.5 *Let z be the solution of (3.24), (3.25) with r bounded and satisfying (3.19). Then there exists a positive constant C, such that*

$$\|z\|_{H^1_\xi} \le C(|z_0| + \|h_z\|_\xi) \,.$$

Proof. The solution of (3.24), (3.25) is given by

$$z(\xi) = E(\xi,0)z_0 + \int_0^\xi E(\xi,y)h_z(y)dy \qquad \text{with} \quad E(\xi,y) = \exp\left(\int_y^\xi r(\eta)d\eta\right) .$$

For $|\xi| < \bar\xi$, $E(\xi,y)$ is bounded and thus, obviously,

$$\|z\|_{L^2_\xi(-\bar\xi,\bar\xi)} \le C\|h_z\|_{L^2_\xi(-\bar\xi,\bar\xi)} \,.$$

For $\xi > \bar\xi$, by (3.19), we have

$$\begin{aligned} |z(\xi)| &\le E(\bar\xi,0)e^{\gamma(\bar\xi-\xi)}|z_0| + \int_0^{\bar\xi} E(\bar\xi,y)e^{\gamma(\bar\xi-\xi)}|h_z(y)|dy + \int_{\bar\xi}^\xi e^{\gamma(y-\xi)}|h_z(y)|dy \\ &\le Ce^{-\gamma\xi}(|z_0| + \|h_z\|_{L^2_\xi(0,\bar\xi)}) + z_1(\xi) \,, \end{aligned}$$

where z_1 solves $\partial_\xi z_1 = -\gamma z_1 + |h_z|$, with $z_1(\bar\xi) = 0$. Multiplying by z_1 and integrating over $(\bar\xi,\infty)$ gives $\|z_1\|_{L^2_\xi(\bar\xi,\infty)} \le \frac{1}{\gamma}\|h_z\|_{L^2_\xi(\bar\xi,\infty)}$, hence

$$\|z\|_{L^2_\xi(\bar\xi,\infty)} \le C(|z_0| + \|h_z\|_{L^2_\xi(0,\infty)}) \,.$$

The interval $(-\infty,-\bar\xi)$ is treated analogously, completing the estimation of $\|z\|_\xi$.

The estimate on $\|\partial_\xi z\|_\xi$ is an obvious consequence of the differential equation and the boundedness of r. □

Theorem 3.6 *There exists a unique solution $w \in H^1_\xi(L^2_v)$ of (3.23). Moreover w satisfies*

$$\|\partial_\xi^k w\|_{\xi,v} \le \frac{1}{\kappa}\|\partial_\xi^k h_w\|_{\xi,v}, \quad \textit{for } k = 0, 1 \,,$$

with κ as in Lemma 3.3.

Proof. We introduce an approximation by discrete velocity models. We choose an increasing sequence $\{\Omega^N\}$ of bounded measurable subsets of the support of the velocity measure exhausting it:

$$\Omega^N \subset \Omega^{N+1}\,, \qquad \bigcup_{N=1}^{\infty} \Omega^N = \Omega\,.$$

Each of the Ω^N is written as a finite disjoint union $\Omega^N = \bigcup_{j=1}^N \Omega_j^N$ of connected measurable subsets Ω_j^N, and the discrete velocities are chosen from these subsets: $v_j^N \in \Omega_j^N$ such that

$$\frac{1}{F(v_j^N)} = \frac{1}{\mu(\Omega_j^N)} \int_{\Omega_j^N} \frac{d\mu(v)}{F(v)}\,,$$

which is possible by the continuity of F. A quadrature formula for v-integrals is then defined by

$$\int f(v)\, d\mu(v) \approx \sum_{j=1}^{N} f(v_j^N)\mu(\Omega_j^N)\,.$$

These choices imply that for functions f and g, whose support is a subset of Ω^N and which are piecewise constant, i.e., constant on each Ω_j^N, the quadrature formula is exact both for the scalar product $\langle f, g\rangle_v$ and for the integrals $\int f\, d\mu(v)$ and $\int g\, d\mu(v)$. Finally, we make the decomposition of Ω^N fine enough such that

$$\lim_{N\to\infty} \sup_{1\le j\le N} \left| (v_j^N - s)^2 - \frac{1}{\mu(\Omega_j^N)} \int_{\Omega_j^N} (v-s)^2 d\mu(v) \right| = 0\,. \tag{3.26}$$

Now we approximate (3.23) by the discrete velocity model

$$\varepsilon(v_j^N - s)\partial_\xi w_j^N - (\mathcal{M}^N w^N)_j = h_{wj}^N := \frac{F(v_j^N)}{\mu(\Omega_j^N)} \int_{\Omega_j^N} \frac{h_w(v)}{F(v)} d\mu(v)\,, \quad j = 1,\ldots,N\,, \tag{3.27}$$

where w^N denotes both the vector $(w_1^N,\ldots,w_N^N)$ and the piecewise constant function in L_v^2 defined by

$$w^N(v) = \begin{cases} w_j^N & \text{for } v \in \Omega_j^N\,, \\ 0 & \text{for } v \in \Omega \setminus \Omega^N\,. \end{cases}$$

Note that by the construction of the quadrature we have

$$\langle f^N, g^N\rangle_N := \sum_{j=1}^{N} \frac{f_j^N g_j^N}{F(v_j^N)} \mu(\Omega_j^N) = \langle f^N, g^N\rangle_v\,.$$

The matrix $\mathcal{M}^N$ is defined by

$$(\mathcal{M}^N w^N)_j := F(v_j^N) \sum_{l=1}^{N} w_l^N \mu(\Omega_l^N) - w_j^N - (v_j^N - s)^2 F(v_j^N) \sum_{l=1}^{N} (v_l^N - s)^2 w_l^N \mu(\Omega_l^N)\,. \tag{3.28}$$

The equations (3.27) are a system of linear constant coefficient ordinary differential equations. A proof analogous to that of Lemma 3.3 shows that $\mathcal{M}^N$ is symmetric and negative definite with respect to $\langle\cdot,\cdot\rangle_N$ and, as a consequence, the generalized eigenvalue problem

$$(\mathcal{M}^N - \lambda(\mathcal{V}^N - s\mathrm{Id}^N))\phi = 0\,, \qquad \text{with } \mathcal{V}^N = \mathrm{diag}(v_1^N, \dots, v_N^N)\,,$$

corresponding to the left hand side of (3.27) has only real eigenvalues away from zero. Thus, a unique bounded solution exists which converges to zero as $\xi \to \pm\infty$. Computing the scalar product of the resulting equation with w^N and integration with respect to ξ gives

$$-\int_{-\infty}^{\infty} \langle \mathcal{M}^N w^N, w^N\rangle_N \, d\xi = \int_{-\infty}^{\infty} \langle h_w^N, w^N\rangle_N \, d\xi\,.$$

With the definiteness of $\mathcal{M}^N$ and the properties of the quadrature this implies

$$\|w^N\|_{\xi,v} \le \frac{1}{\kappa}\|h_w\|_{\xi,v}\,.$$

The uniform boundedness of w^N in $L^2_{\xi,v}$ implies its weak convergence (for a subsequence) to $w \in L^2_{\xi,v}$. We can also pass to the limit in (3.27). Here we use (3.26) for proving convergence of the last term in (3.28). The above estimate carries over to the limit w. Then the estimate for $\partial_\xi w$ is obtained by differentiating equation (3.23). Uniqueness is an obvious consequence. □

3.4 The nonlinear problem

In this section we prove existence and uniqueness of solutions of the nonlinear problem (3.16), (3.22), subject to $z(0) = z_0$ and to the requirement that the solution is square integrable with respect to ξ, replacing homogeneous far-field conditions.

After the preparations in the previous subsections, the proof is a straightforward contraction argument. We need, however, estimates for the right hand sides of (3.16) and (3.22). In the following, C denotes (possibly different) ε-independent constants.

Lemma 3.7 *(i) The coefficients Λ and Γ satisfy*

$$\|\Lambda\|_{C^1_\xi(L^2_v)} + \|\Gamma\|_{C^1_\xi(L^2_v)} \le C\,.$$

(ii) The nonlinear term $R(\rho)$ is quadratic in ρ: Let $\rho_1, \rho_2 \in H^1_\xi$ satisfy $|\rho_1|, |\rho_2| \le C\varepsilon^{-2}$. Then

$$\|R(\rho_1) - R(\rho_2)\|_{H^1_\xi(L^2_v)} \le C\left(\|\rho_1\|_{H^1_\xi} + \|\rho_2\|_{H^1_\xi}\right)\|\rho_1 - \rho_2\|_{H^1_\xi}\,.$$

Proof. The proofs of the statements are straightforward. All that is needed is the boundedness in L^2_v of the derivatives $\partial^k_\rho M(\rho_{as}+\varepsilon^2\tilde\rho)$ for $k\le 3$ with $\tilde\rho$ between values of ρ_1 and ρ_2, as well as the continuous embedding $L^\infty_\xi \to H^1_\xi$. □

Lemma 3.8 *The projection $P:\ L^2_v \to L^2_v$, defined by (3.20), is a bounded operator.*

Proof. The proof is a straightforward consequence of (2.3)–(2.5). □

Before stating the existence and uniqueness result for travelling waves we recall the micro-macro decomposition $g(\xi,v) = z(\xi)\Phi(v) + \varepsilon w(\xi,v)$ of functions $g\in H^1_\xi(L^2_v)$, made unique by the requirement $\langle (v-s)\Phi, w\rangle_v = 0$, since

$$\langle (v-s)\Phi,\Phi\rangle_v = \frac{\varepsilon\sigma\tilde D}{\hat D} \ne 0\,.$$

We define a norm on $H^1_\xi(L^2_v)$ by

$$\|g\| := \|z\|_{H^1_\xi} + \varepsilon\|w\|_{H^1_\xi(L^2_v)}\,. \tag{3.29}$$

We also note that in terms of the original unknown $f = f_{as}+\varepsilon^2 g$, the condition $z(0)=z_0$ reads

$$\langle (v-s)\Phi, f-f_{as}\rangle_v(\xi=0) = \frac{\varepsilon^3\sigma\tilde D}{\hat D} z_0\,. \tag{3.30}$$

Theorem 3.9 *Let the assumptions stated in Section 2 be satisfied. Then for every $z_0\in\mathbb{R}$ and for ε small enough, there exists a solution f of (1.12) satisfying (3.30), unique in a ball in $(H^1_\xi(L^2_v),\|\ \|)$ with center f_{as} and a $O(\varepsilon)$-radius. It satisfies*

$$\|f - M(\rho)\|_{H^1_\xi(L^2_v)} = O(\varepsilon^2)\,,$$

where ρ is the solution of (3.4)–(3.6), or, more precisely,

$$f = M(\rho) - \varepsilon(v-s)\partial_\xi M(\rho) + \varepsilon^2 z\Phi + \varepsilon^3 w\,,$$

where $\|z\|_{H^1_\xi}$ and $\|w\|_{H^1_\xi(L^2_v)}$ are uniformly bounded as $\varepsilon\to 0$.

Proof. As a consequence of Lemma 3.7 and Lemma 3.8 we have

$$\left\|\frac{a'(\rho_{as})-s}{\varepsilon}\rho_w\right\|_{H^1_\xi} \le C\|w\|_{H^1_\xi(L^2_v)}\,,$$

$$\|P\Lambda z\|_{H^1_\xi(L^2_v)} \le C\|z\|_{H^1_\xi}\,,\qquad \|\tilde\Gamma\rho_w\|_{H^1_\xi(L^2_v)} \le C\|w\|_{H^1_\xi(L^2_v)}\,.$$

This implies that for ε small enough, the results from Lemma 3.5 and Lemma 3.6 can be extended to the linear system

$$\tilde D\,\partial_\xi z - r(\xi)z = \varepsilon\frac{a'(\rho_{as})-s}{\varepsilon}\rho_w + h_z\,,$$

$$\varepsilon(v-s)\partial_\xi w - \mathcal{M}w - zP\Lambda = \varepsilon\tilde\Gamma\rho_w + h_w\,,$$

with inhomogeneities h_z, h_w and $z_0 = z(0)$. Applying the solution operator for this system to (3.16), (3.22), we obtain a fixed point problem of the form

$$z = \varepsilon R_z(z\rho_\Phi + \varepsilon\rho_w) + \tilde{h}_z\,, \tag{3.31}$$

$$w = \varepsilon R_w(z\rho_\Phi + \varepsilon\rho_w) + \tilde{h}_w\,, \tag{3.32}$$

with R_z and R_w sharing the properties of R given in Lemma 3.7 and $\tilde{h}_z$ and $\tilde{h}_w$ are given and bounded. Using $\|z\rho_\Phi + \varepsilon\rho_w\|_{H^1_\xi} \le \|(z,w)\|$ (see (3.29)), the estimate

$$\|(\varepsilon R_z(z\rho_\Phi + \varepsilon\rho_w) + \tilde{h}_z, \varepsilon R_w(z\rho_\Phi + \varepsilon\rho_w) + \tilde{h}_w)\| \le c(1 + \varepsilon\|(z,w)\|^2)$$

follows. Here we have identified $g = z\Phi + \varepsilon w$ with the pair (z,w). The estimate implies that for ε small enough both the ball with radius $2c$ and the ball with radius $1/(2\varepsilon c)$ are mapped into themselves by the right hand side of (3.31), (3.32). Also, with the property of the nonlinearity from Lemma 3.7, the fixed point operator is a contraction on a ball with $O(\varepsilon^{-1})$ radius. We conclude that for ε small enough, (3.31), (3.32) has a solution with $\|(z,w)\| \le 2c$ which is unique in a ball with $O(\varepsilon^{-1})$ radius. Knowing this and returning to (3.32), boundedness of $\|w\|_{H^1_\xi(L^2_v)}$ follows. □

By its construction the approximating density ρ is strictly monotone. It will be important for the stability proof below to extend this property to the exact density ρ_f.

Lemma 3.10 *Let the assumptions of Theorem 3.9 hold and let f be the solution of (1.12), (3.30). Then the macroscopic density $\rho_f(\xi)$ is strictly monotone.*

Proof. The previous proof is easily extended to show that the dependence of z and w on z_0 is Lipschitz continuous with ε-independent Lipschitz constant. Actually, the difference of two solutions (z,w) and $(\hat{z},\hat{w})$ with different z_0-values z_0 and $\hat{z}_0$, respectively, satisfies a system similar to (3.31), (3.31) with inhomogeneities proportional to $z_0 - \hat{z}_0$. With the properties of the nonlinearities from Lemma 3.7 it is straightforward that

$$\|z - \hat{z}\|_{H^1_\xi}\,,\ \|w - \hat{w}\|_{H^1_\xi(L^2_v)} \le C|z_0 - \hat{z}_0|\,. \tag{3.33}$$

For the corresponding solutions f and $\hat{f}$ of (1.12), (3.30)

$$\rho_f(0) - \rho_{\hat{f}}(0) = \varepsilon^2(z_0 - \hat{z}_0)\rho_\Phi + \varepsilon^3(\rho_w(0) - \rho_{\hat{w}}(0))$$

holds. The continuous embedding of $C(\mathbb{R})$ in H^1_ξ and (3.33) imply

$$|\rho_w(0) - \rho_{\hat{w}}(0)| \le C|z_0 - \hat{z}_0|\,,$$

and, thus, strict monotonicity (and therefore invertibility) of the map $z_0 \mapsto \rho_f(0)$ for ε small enough. This in turn implies that the travelling wave can also be made locally unique by prescribing the value of $\rho_f(0)$ instead of z_0. This argument can of course be repeated with $\rho_f(\xi_0)$ for every $\xi_0 \in \mathbb{R}$ instead of the origin.

Now assume tha ρ_f is not strictly monotone. Then there are two ξ-values ξ_0 and $\xi_0 + \delta$ with arbitrarily small positive δ such that $\rho_f(\xi_0) = \rho_f(\xi_0 + \delta)$. Now $\tilde{f}(\xi, v) = f(\xi + \delta, v)$ is a travelling wave with $\rho_{\tilde{f}}(\xi_0) = \rho_f(\xi_0)$ and $\tilde{f}$ arbitrarily close to f by making δ small. By the uniqueness result $\tilde{f} \equiv f$, and, consequently, f is periodic, which is a contradiction to the far-field boundary conditions. □

4 Local stability of small amplitude travelling waves

In this section we prove dynamic stability of the small amplitude travelling waves constructed above. As mentioned in the introduction, the techniques we employ are commonly used for conservation laws regularised with diffusion terms. This motivates the following scaling: we write equation (1.1) in the travelling wave variable $\xi = (x - st)\varepsilon$, and introduce the *parabolic* scaling $t \to t/\varepsilon^2$, where ε is the amplitude of the wave. Then equation (1.1) reads

$$\varepsilon^2 \partial_t f + \varepsilon(v - s)\partial_\xi f = M(\rho_f) - f\,, \tag{4.1}$$

and we pose the same far-field boundary conditions as for the travelling wave:

$$f(t, \xi = \pm\infty, v) = M(\rho_\pm, v)\,.$$

Let us denote by ϕ a travelling wave solution as constructed in Theorem 3.9. By Lemma 3.10 its macroscopic profile is monotone implying

$$\partial_\xi(a'(\rho_\phi)) \leq 0\,. \tag{4.2}$$

Observe that formally the integral of $\rho_f - \rho_\phi$ is constant in t. This allows to choose ϕ, by shifting in the ξ-direction if necessary, such that

$$\int_{\mathbb{R}} (\rho_f - \rho_\phi)\, d\xi = 0\,. \tag{4.3}$$

Condition (4.3) fixes the shift in ξ; we expect the solution f to approach this particular ϕ as $t \to \infty$.

Let us denote by G the deviation of f from ϕ, namely

$$\varepsilon G = f - \phi\,. \tag{4.4}$$

Then G satisfies the equation

$$\varepsilon \partial_t G + (v - s)\partial_\xi G = \frac{1}{\varepsilon^2}\left[M(\rho_\phi + \varepsilon\rho_G) - M(\rho_\phi)\right] - \frac{1}{\varepsilon} G\,. \tag{4.5}$$

We recall that condition (4.3) allows to deal with the *macroscopic* unknown $W = \int_{-\infty}^{\xi} \rho_G\, d\xi$, see (1.20).

We decompose G into a macroscopic part and into a microscopic part, by simply using the natural *macroscopic* projection $f \to F\rho_f$, thus we write

$$G = \rho F + \varepsilon g, \quad \text{i.e.} \quad \rho := \rho_G\,.$$

Analogously, we split equation (4.5) into its microscopic and macroscopic part, i.e. we apply the macroscopic projection, and its complementary *microscopic* projection, which is in fact the operator $-\mathcal{L}$. Application of the macroscopic projection and division by F gives the equation

$$\partial_t \rho + \frac{1}{\varepsilon}(a'(\rho_-) - s)\partial_\xi \rho + \partial_\xi \int (v - s) g\, d\mu(v) = 0\,, \tag{4.6}$$

and application of $-\mathcal{L}$ gives

$$\varepsilon^2\partial_t g + (v - a'(\rho_-))F\partial_\xi\rho - \varepsilon\partial_\xi\mathcal{L}((v-s)g) = R_2[\rho] - g\,, \tag{4.7}$$

with

$$R_2[\rho](t,\xi,v) = \frac{1}{\varepsilon^2}\left[M(\rho_\phi(\xi)+\varepsilon\rho(t,\xi),v) - M(\rho_\phi(\xi),v) - \varepsilon\rho(t,\xi)F(v)\right]. \tag{4.8}$$

Integrating (4.6) with respect to ξ gives, in terms of $W = \int_{-\infty}^{\xi}\rho(t,y)\,dy$,

$$\partial_t W - \sigma\partial_\xi W + \int (v-s)g\,d\mu(v) = 0\,. \tag{4.9}$$

Using (4.7) we compute

$$\begin{aligned}\int (v-s)g\,d\mu(v) = \int (v-s)R_2[\rho]\,d\mu(v) - \varepsilon^2\int (v-s)\partial_t g\,d\mu(v)\\ + \varepsilon\,\partial_\xi\int (v-s)\mathcal{L}((v-s)g)d\mu(v) - D_0\partial_\xi\rho\,.\end{aligned} \tag{4.10}$$

Substituting (4.10) into (4.9) and setting

$$r_2[\rho](t,\xi) := \int (v-s)R_2[\rho](t,\xi,v)\,d\mu(v)\,, \tag{4.11}$$

we arrive at the integrated *macroscopic* equation

$$\partial_t W - \sigma\partial_\xi W + r_2[\rho] - D_0\partial_\xi^2 W = \varepsilon S[g]\,, \tag{4.12}$$

with

$$S[g] = \partial_\xi\int (v-s)\left(\varepsilon\partial_t g - \mathcal{L}((v-s)\partial_\xi g)\right)\,d\mu(v)\,. \tag{4.13}$$

Observe that (4.12) has the form of the perturbation equation for viscous shock profiles with a microscopic perturbation on the right hand side and the nonlinearity

$$r_2[\rho] = \frac{1}{\varepsilon^2}(a(\rho_\phi+\varepsilon\rho) - a(\rho_\phi) - \varepsilon\rho a'(\rho_-))\,. \tag{4.14}$$

For controlling the nonlinear terms, a uniform (in ε) bound on the L^∞-norm of the density ρ is needed. For the macroscopic equation without the microscopic perturbation on the right hand side, this is a consequence of the maximum principle. Here we shall employ bounds in H_ξ^1 for the same purpose.

Assuming such a bound, we write $R_2[\rho]$ as

$$R_2[\rho] = \rho\int_0^1 \frac{M'(\rho_\phi+\varepsilon\rho\eta) - M'(\rho_-)}{\varepsilon}\,d\eta\,.$$

Then by differentiation with respect to ξ and by assumption (2.4),

$$\|R_2[\rho]\|_{H^k_\xi(L^2_v)} \le C\|\rho\|_{H^k_\xi}\,, \quad \text{for } k=0,1,2\,, \tag{4.15}$$

and, consequently, by (2.5),

$$\|r_2[\rho]\|_{H^k_\xi} \le C\|\rho\|_{H^k_\xi}\,, \quad \text{for } k=0,1,2\,, \tag{4.16}$$

where here and in the following the symbols C as well as C_j with various j denote constants depending on $\|\rho\|_{L^\infty_\xi}$ but independent from ε.

As the next step we derive integral estimates as one would do for the purely macroscopic case.

Lemma 4.1 *Let W be a solution of (4.12) (for given g) and $\rho = \partial_\xi W$. Then the following estimates hold:*

$$\frac{1}{2}\frac{d}{dt}\|W\|_\xi^2 + (D_0 - C_0\|W\|_{L^\infty_\xi})\|\partial_\xi W\|_\xi^2 \le \varepsilon \int_{-\infty}^{\infty} W\, S[g]d\xi\,, \tag{4.17}$$

$$\frac{1}{2}\frac{d}{dt}\|\partial_\xi^k W\|_\xi^2 + \frac{D_0}{2}\|\partial_\xi^{k+1} W\|_\xi^2 - C_k\|\rho\|^2_{H^{k-1}_\xi} \le \varepsilon \int_{-\infty}^{\infty} \partial_\xi^k W\, S[\partial_\xi^k g]d\xi\,, \tag{4.18}$$

for $k=1,2$. The constants C_0, C_1, C_2 depend on $\|\rho\|_{L^\infty_\xi}$.

Proof. For proving (4.17), we test equation (4.12) with W. Let us look at the term containing $r_2[\rho]$ by writing

$$r_2[\rho] = \frac{1}{\varepsilon}\left(a'(\rho_\phi) - a'(\rho_-)\right)\rho + \frac{1}{2}a''(\tilde\rho)\rho^2\,,$$

with $\tilde\rho$ between ρ_ϕ and $\rho_\phi + \varepsilon\rho$. Then we get

$$\begin{aligned}\int_{\mathbb{R}} r_2[\rho]\, W d\xi &= \frac{1}{2\varepsilon}\int_{\mathbb{R}}(a'(\rho_\phi) - a'(\rho_-))\partial_\xi(W^2)\, d\xi + \frac{1}{2}\int_{\mathbb{R}} a''(\tilde\rho)(\partial_\xi W)^2 W\, d\xi \\ &\ge -\frac{1}{2\varepsilon}\int_{\mathbb{R}} \partial_\xi(a'(\rho_\phi))W^2 d\xi - C_0\|W\|_{L^\infty_\xi}\|\partial_\xi W\|_\xi^2\,. \end{aligned} \tag{4.19}$$

The first term on the right-hand side of (4.19) is positive by (4.2), completing the proof of (4.17).

For $k=1,2$, the corresponding ξ-derivatives of (4.12) are tested with $\partial_\xi^k W$. As already mentioned in the introduction, no positive term can be expected to arise from the nonlinearity. Therefore we just estimate the corresponding terms using (4.16):

$$\begin{aligned}&\left|\int_{\mathbb{R}} \partial_\xi^k\, r_2[\rho]\, \partial_\xi^k W d\xi\right| \\ &\le \int_{\mathbb{R}} |\partial_\xi^{k-1} r_2[\rho]\, \partial_\xi^{k+1} W| d\xi \le C_k\|\rho\|^2_{H^{k-1}_\xi} + \frac{D_0}{2}\|\partial_\xi^{k+1}W\|_\xi^2\,,\end{aligned} \tag{4.20}$$

completing the proof of (4.18). $\square$

Before deriving estimates for the microscopic contributions, we have to deal with the difficulty that the operator $S[g]$ describing the microscopic perturbation of the macroscopic equation, contains the time derivative $\partial_t g$.

Lemma 4.2 *Let W and g satisfy (4.9), and let the operator S be defined by (4.13). Then for $k = 0, 1, 2$ the following holds:*

$$\int_{-\infty}^{\infty} \partial_\xi^k W \, S[\partial_\xi^k g] d\xi \le \varepsilon \frac{d}{dt} \int_{-\infty}^{\infty} \langle F \partial_\xi^k W, (v-s)\partial_\xi^k g \rangle_{\xi,v} + C(\|\partial_\xi^{k+1} W\|_\xi^2 + \|\partial_\xi^k g\|_{\xi,v}^2) \,. \tag{4.21}$$

Proof. A straightforward computation, using the kth order derivative of (4.9), gives

$$\int_{-\infty}^{\infty} \partial_\xi^k W \, S[\partial_\xi^k g] d\xi = \varepsilon \frac{d}{dt} \int_{-\infty}^{\infty} \partial_\xi^k W \int (v-s) \partial_\xi^k g d\mu(v) \, d\xi$$
$$-\varepsilon\sigma \int_{-\infty}^{\infty} \partial_\xi^{k+1} W \int (v-s) \partial_\xi^k g \, d\mu(v) \, d\xi + \int_{-\infty}^{\infty} \left(\int (v-s) \partial_\xi^k g \, d\mu(v) \right)^2 d\xi$$
$$+ \int_{-\infty}^{\infty} \partial_\xi^{k+1} W \int (v-s) \mathcal{L}((v-s)\partial_\xi^k g) d\mu(v) \, d\xi \,.$$

For estimating the last three terms we use the Cauchy-Schwartz inequality and (2.5). $\square$

For getting control of the microscopic part, we derive entropy estimates from the full kinetic perturbation equation (4.5).

Lemma 4.3 *Let $G = \rho F + \varepsilon g$ be a solution of (4.5). Then, for $k = 0, 1, 2$*

$$\frac{d}{dt} \left(\|\partial_\xi^k \rho\|_\xi^2 + \varepsilon^2 \|\partial_\xi^k g\|_{\xi,v}^2 \right) + \|\partial_\xi^k g\|_{\xi,v}^2 \le C \|\rho\|_{H_\xi^k}^2 \,. \tag{4.22}$$

Proof. Writing the right-hand side of (4.5) as $R_2[\rho] - g$ and taking the scalar product of its kth derivative with $\partial_\xi^k G = \partial_\xi^k \rho F + \partial_\xi^k g$, we get

$$\frac{1}{2} \frac{d}{dt} \|\partial_\xi^k G\|_{\xi,v}^2 + \|\partial_\xi^k g\|_{\xi,v}^2 = \langle \partial_\xi^k R_2[\rho], \partial_\xi^k g \rangle_{\xi,v} \,.$$

The result is a consequence of using $\|\partial_\xi^k G\|_{\xi,v}^2 = \|\partial_\xi^k \rho\|_\xi^2 + \varepsilon^2 \|\partial_\xi^k g\|_{\xi,v}^2$, and then applying the Cauchy-Schwarz inequality, the Young inequality, and (4.15) to the right hand side. $\square$

Now we are prepared for proving our stability result for small kinetic shock profiles.

Theorem 4.4 *Let the assumptions of Theorem 3.9 hold and let ϕ be a travelling wave solution. Let $f_0(\xi, v)$ be an initial datum for (4.1) and let $W_0(\xi) = \frac{1}{\varepsilon} \int_{-\infty}^{\xi} (\rho_{f_0}(\eta) - \rho_\phi(\eta)) d\eta$. Let $f_0 - \phi \in H_\xi^2(L_v^2)$ (implying that f_0 satisfies the same far-field conditions as ϕ) and $W_0 \in L_\xi^2$ (implying $W_0(\infty) = \int_{-\infty}^{\infty} (\rho_{f_0}(\xi) - \rho_\phi(\xi)) d\xi = 0$). Let*

$$\|W_0\|_{L_\xi^2} + \frac{1}{\varepsilon} \|f_0 - \phi\|_{H_\xi^2(L_v^2)} \le \delta \,, \tag{4.23}$$

for δ small enough, but independent from ε. Then the equation (4.1) subject to the initial condition $f(t = 0) = f_0$ has a unique global solution and

$$\lim_{t\to\infty} \int_t^{\infty} \|f(s, \cdot, \cdot) - \phi\|_{H_\xi^2(L_v^2)}^2 \, ds = 0 \,.$$

Remark 4.5 *The shortcomings of Theorem 4.4 are that it only gives local stability (i.e., the perturbations have to be small enough) of small amplitude travelling waves. Also convergence as $t \to \infty$ only holds in a very weak sense. The smallness of the wave is the basic assumption of this work allowing a perturbative treatment close to macroscopic equations. The other shortcomings are already present in the underlying results for viscous shock profiles.*

Proof. The proof is based on the construction of a Lyapunov functional H, such that both H and $-dH/dt$ measure the size of the perturbation. However, it will be impossible to estimate H in terms of $-dH/dt$, which is the reason why we do not have a result on the time decay rate.

We start by defining a partial functional for each differentiation order k appearing in Lemma 4.1 and Lemma 4.3. By adding the corresponding inequality from Lemma 4.1 and the product of εA_k with the corresponding inequality from Lemma 4.3, we produce an inequality for the time derivative of

$$
\begin{aligned}
H_k &= \frac{1}{2}\|\partial_\xi^k W\|_{L_\xi^2}^2 - \varepsilon^2\langle F\partial_\xi^k W, (v-s)\partial_\xi^k g\rangle_{\xi,v} + \varepsilon^3 A_k\|\partial_\xi^k g\|_{L_\xi^2(L_v^2)}^2 + \varepsilon A_k\|\partial_\xi^{k+1}W\|_{L_\xi^2}^2 \\
&\geq \kappa_k(\|\partial_\xi^k W\|_{L_\xi^2}^2 + \varepsilon\|\partial_\xi^{k+1}W\|_{L_\xi^2}^2 + \varepsilon^3\|\partial_\xi^k g\|_{L_\xi^2(L_v^2)}^2)\,,
\end{aligned}
$$

where the last inequality holds with a ε-independent $\kappa_k > 0$, if $A_k > 0$ is chosen independently from ε (but otherwise arbitrarily). With two more positive constants γ_1 and γ_2, we define the Lyapunov functional as $H = H_0 + \gamma_1 H_1 + \gamma_2 H_2$ and observe that it can be bounded from above and below by

$$
\|W\|_{H_\xi^2}^2 + \varepsilon\|\partial_\xi^3 W\|_\xi^2 + \varepsilon^3\|g\|_{H_\xi^2(L_v^2)}^2\,.
$$

With the aid of the Lemmata 4.1, 4.2, and 4.3, it is now straightforward to obtain an inequality of the form

$$
\frac{dH}{dt} \leq -C(\|\rho\|_{L_\xi^\infty})(1-\|W\|_{L_\xi^\infty})(\|\rho\|_{H_\xi^2}^2 + \varepsilon\|g\|_{H_\xi^2(L_v^2)}^2)\,, \tag{4.24}
$$

with an ε-independent positive $C(\|\rho\|_{L_\xi^\infty})$. Since (by 1D-Sobolev embedding) H controls the L_ξ^∞-norms of W and ρ, the right hand side is negative at $t = 0$ and remains so, if $H(0)$ is small enough. This in turn is guaranteed by the assumption (4.23). Note that the L_ξ^2-norm of W (and, thus, H) is not controlled by the dissipation term in (4.24), which is the reason that we cannot obtain a decay rate. The proof is completed by integrating (4.24) with respect to t. □

References

[1] F. Bouchut. Construction of BGK models with a family of kinetic entropies for a given system of conservation laws. *J. Statist. Phys.*, 95:113–170, 1999.

[2] R. E. Caflisch and B. Nicolaenko. Shock profile solutions of the Boltzmann equation. *Comm. Math. Phys.*, 86:161–194, 1982.

[3] C. M. Cuesta and C. Schmeiser. Kinetic profiles for shock waves of scalar conservation laws. Submitted.

[4] F. Golse. Shock profiles for the Perthame-Tadmor kinetic model. *Comm. Partial Differential Equations*, 23:1857–1874, 1998.

[5] M. A. Katsoulakis and A. E. Tzavaras. Contractive relaxation systems and the scalar multidimensional conservation law. *Comm. Partial Differential Equations*, 22:195–233, 1997.

[6] S. Kawashima and A. Matsumura. Asymptotic stability of traveling wave solutions of systems for one-dimensional gas motion. *Comm. Math. Phys.*, 101:97–127, 1985.

[7] T.-P. Liu and S.-H. Yu. Boltzmann equation: micro-macro decompositions and positivity of shock profiles. *Comm. Math. Phys.*, 246:133–179, 2004.

[8] R. L. Pego. Remarks on the stability of shock profiles for conservation laws with dissipation. *Trans. Amer. Math. Soc.*, 291:353–361, 1985.

[9] B. Perthame and E. Tadmor. A kinetic equation with kinetic entropy functions for scalar conservation laws. *Comm. Math. Phys.*, 136:501–517, 1991.

[10] J. Smoller. *Shock waves and reaction-diffusion equations.* Springer, New York, 1983.