Kinetic shock profiles for nonlinear hyperbolic conservation laws

C. M. Cuesta[*], S. Hittmeir[†], and C. Schmeiser[†]

Abstract

A unified framework for studying the existence and stability of kinetic shock profiles is presented. This includes small amplitude waves for the situation when the macroscopic model is a hyperbolic system of conservation laws with genuine nonlinearity. For the case of scalar conservation laws, also large amplitude waves can be understood. Applications range from BGK-models for general scalar conservation laws and for gas dynamics, to an equation for fermions in a scattering background under the action of an electric field and to the Boltzmann equation of gas dynamics.

Contents

[*]University of Nottingham, School of Mathematical Sciences, Division of Theoretical Mechanics, University Park, Nottingham, NG7 2RD, UK, E-mail: carlota.cuesta@nottingham.ac.uk

[†]Faculty for Mathematics, University of Vienna, Nordbergstraße 15, 1090 Wien, Austria, E-mail: sabine.hittmeir@univie.ac.at, christian.schmeiser@univie.ac.at

1 Introduction

This work contributes to the mathematical theory establishing the connection between kinetic transport equations and hyperbolic systems of conservation laws, occurring as their macroscopic limits. Shock waves are basic weak solutions of nonlinear hyperbolic conservation laws featuring a discontinuity. The main question considered in this work is the existence and dynamic stability of kinetic shock profiles, i.e. smooth travelling wave solutions of the kinetic equation, sharing the far-field states with the shock wave. Several recent results are reviewed and presented in a unified way.

For systems of nonlinear conservation laws only results for small amplitude shock waves are available. In this case, the Chapman-Enskog approximation, i.e. a diffusive regularization of the conservation laws, can be expected to provide a good approximation for solutions of the kinetic equation. Kinetic shock profiles can be constructed close to viscous shock profiles. The classical result on the existence of small amplitude kinetic shock profiles for the gas dynamics Boltzmann equation is due to Caflisch and Nicolaenko [11]. In Section 4, a modified and generalized version of their approach is presented, leading to more accurate approximation results. Stability of small amplitude kinetic shock profiles is the issue of Section 5. An approach based on energy (actually entropy) estimates in the spirit of the work of Liu and Yu [28] is presented. The main idea is to start from an approach for the system with diffusive regularization. This can actually be extended to proving convergence to rarefaction waves [16].

For scalar conservation laws, stronger results are possible. An approach for the construction of large amplitude kinetic shock profiles is presented in Section 6. The main ideas originate from the work of Golse [22] on the Perthame-Tadmor model [35]. Finally, dynamical stability is discussed, based on ideas from [5].

We consider a kinetic transport model for plane waves in the form

$$\partial_t f + v\partial_x f = Q(f)\,, \tag{1.1}$$

where $f(t,x,\mathbf{v})$ is a particle distribution function at time $t \in \mathbb{R}$ and position $x \in \mathbb{R}$. The components of the 'velocity' vector $\mathbf{v} = (v,w) \in V \subset \mathbb{R} \times [0,\infty)$ can be interpreted as the velocity component v in x-direction, and as an abbreviation w for $(v_2^2 + \cdots + v_d^2)^{1/2}$, where $(v, v_2, \dots, v_d) \in \mathbb{R}^d$ is the particle velocity. Thus, we assume x-axisymmetric velocity distributions. The set V of velocities is equipped with a measure $d\mu(\mathbf{v})$. Discrete sets V and, thus, hyperbolic relaxation models are permitted.

The so called *collision operator* Q is assumed to act on the velocity variable $\mathbf{v}$ only. Equations of the form (1.1) can be derived from fully d-dimensional kinetic transport equations, if the collision events are invariant under rotations (at least around the x-axis).

The collision operator is assumed to be nonlinear and to have the conservation property

$$\int_V \phi(\mathbf{v}) Q(f)(\mathbf{v}) d\mu(\mathbf{v}) = 0\,. \tag{1.2}$$

The (linearly independent) components of the vector $\phi(\mathbf{v}) \in \mathbb{R}^n$ are called *collision invariants*. As a consequence of (1.2), the macroscopic moments of f, collected in the vector

$$U_f(t,x) := \int_V \phi(\mathbf{v}) f(t,x,\mathbf{v}) d\mu(\mathbf{v})\,,$$

are the macroscopic densities of conserved quantities:

$$\partial_t U_f + \partial_x J_f = 0\,, \qquad \text{with } J_f := \int_V v\phi f\, d\mu\,. \tag{1.3}$$

As expected, the zero set of Q will be assumed to be n-dimensional and parametrizable by the macroscopic moments:

$$Q(f) = 0 \quad \Longleftrightarrow \quad f(\mathbf{v}) = \mathcal{M}(U_f, \mathbf{v})\,, \qquad \big(\text{implying } U_{\mathcal{M}(U)} = U\,.\big)$$

The *generalized Maxwellian* $\mathcal{M}(U)$ is the equilibrium distribution of the collision processes. It is plausible that the dynamics of close-to-equilibrium solutions of (1.1) is approximated by the system of conservation laws

$$\partial_t U_f + \partial_x J(U_f) = 0\,, \qquad J(U) := J_{\mathcal{M}(U)}\,, \tag{1.4}$$

obtained by replacing f by $\mathcal{M}(U_f)$ in the second term of (1.3). This approximation can only be expected to be physically relevant under a stability condition: We assume the existence of a *kinetic entropy density* $H(f, \mathbf{v})$, satisfying

$$\int_V \partial_f H(f) Q(f) d\mu \le 0\,. \tag{1.5}$$

where H is continuous in $\mathbf{v}$ and twice differentiable and convex in f. We also assume definiteness in the sense that equality in (1.5) only holds if $f = \mathcal{M}(U_f)$. This leads to the kinetic entropy inequality

$$\partial_t \int_V H(f) d\mu + \partial_x \int_V v H(f) d\mu \le 0\,. \tag{1.6}$$

The macroscopic system (1.4) will be assumed to be strictly hyperbolic meaning that for every U, the Jacobian $J'(U)$ of the *macroscopic flux* $J(U)$ has n distinct eigenvalues $\lambda_1(U) < \cdots < \lambda_n(U)$, and the corresponding left and right eigenvectors $l_k(U)$ and, respectively, $r_k(U)$, $k = 1, \ldots, n$, are assumed to be normalized such that $l_j(U) \cdot r_k(U) = \delta_{jk}$.

Piecewise constant weak solutions of (1.4) of the form

$$U(x,t) = \begin{cases} U_- & \text{for } x < st\,, \\ U_+ & \text{for } x > st\,, \end{cases}$$

are called *shock waves.* Here s is the shock speed and $U_\pm$ are the constant left and right states, which are related by the *Rankine-Hugoniot jump conditions*

$$s(U_+ - U_-) = J(U_+) - J(U_-)\,. \tag{1.7}$$

For a fixed left state U_- the *Hugoniot locus* is defined as the set of all U_+ such that (1.7) is satisfied for an appropriate s. In a neighbourhood of U_-, the Hugoniot locus consists of n curves intersecting in U_-. At U_-, the k-th curve is tangent to $r_k(U_-)$ and the shock speed s takes the value $\lambda_k(U_-)$ (see, e.g., [26]). If U_+ lies on the k-th curve of the Hugoniot locus, we refer to $\{U_\pm, s\}$ as a *k-shock.*

If the k-th field is *genuinely nonlinear,* i.e. $r_k \cdot \nabla\lambda_k \neq 0$, then the *Lax entropy condition*

$$\lambda_k(U_+) < s < \lambda_k(U_-) \tag{1.8}$$

is a stability condition for k-shocks (see, e.g., [26]) and we assume a normalization of r_k such that $r_k \cdot \nabla\lambda_k = 1$.

An alternative approach to entropy conditions starts from the kinetic entropy inequality (1.6) and uses the close-to-equilibrium approximation

$$\partial_t \eta(U_f) + \partial_x \Psi(U_f) \le 0\,, \tag{1.9}$$

as a side condition for weak solutions of (1.4). Here the macroscopic entropy density and entropy flux (satisfying $\nabla\Psi(U) = \nabla\eta(U) \cdot J'(U)$) are given by

$$\eta(U) = \int_V H(\mathcal{M}(U))d\mu\,, \qquad \Psi(U) = \int_V vH(\mathcal{M}(U))d\mu\,. \tag{1.10}$$

It can be shown that for small amplitude shock waves, i.e., small enough values of $|U_+ - U_-|$, the conditions (1.8) and (1.9) are equivalent.

Remark 1.1 *Further properties of the kinetic entropy density will be used below. The above implies the minimisation principle*

$$\eta(U) = \int_V H(\mathcal{M}(U))d\mu = \min_{\int_V \phi f d\mu = U} \int_V H(f)d\mu\,, \tag{1.11}$$

which has the further consequence that $\partial_f H(\mathcal{M}(U))$ is linear in the collision invariants, i.e. there exists a vector $b_U \in \mathbb{R}^n$ such that $\partial_f H(\mathcal{M}(U)) = b_U \cdot \phi$. If we take the gradient of the first relation in (1.10) it turns out that $b_U = \nabla\eta(U)$, such that $\partial_f H(\mathcal{M}(U)) = \nabla\eta(U) \cdot \phi$.

For later reference, we collect the assumptions on the collision operator made so far.

Assumption 1 *The collision operator Q has n linearly independent collision invariants $\phi_1(\mathbf{v}), \dots, \phi_n(\mathbf{v})$, and its zero set is given by $\{\mathcal{M}(U, \mathbf{v}) : \ U \in \mathbb{R}^n\}$. There exists a strictly convex kinetic entropy density $H(f, \mathbf{v})$ satisfying the inequality (1.5) (with equality iff $f = \mathcal{M}(U_f)$). The macroscopic system (1.4) is strictly hyperbolic.*

2 Examples

Not all collision operators can be interpreted as appropriate models for microscopic collision processes. In many cases they are just constructed as relaxation models towards a desired equilibrium. The so called *BGK-models* [7], [9] of the form $Q(f) = \mathcal{M}(U_f) - f$ belong to this class. All the examples presented below satisfy Assumption 1.

2.1 BGK-models for scalar conservation laws

A family of generalized Maxwellians for an arbitrary scalar hyperbolic conservation law with flux $J(U)$ is given by

$$\mathcal{M}(U, v) = \int_0^U m(v - J'(r))dr\,, \qquad v \in V = \mathbb{R}\,,$$

(see [18]), where $m(v) > 0$ can be any even function satisfying $\int_{-\infty}^{\infty} m(v)dv = 1$. The conservation law is conservation of mass with $\phi(v) = 1$ and $U_f = \int_{-\infty}^{\infty} f\, dv$. Noting that $\mathcal{M}$ is strictly increasing as a function of U, we define kinetic entropy densities by inverting it:

$$\zeta(f, v) = U \ :\Leftrightarrow\ \mathcal{M}(U, v) = f\,, \qquad H(f, v) := \int_0^f \eta'(\zeta(g, v))dg\,, \tag{2.1}$$

where η is an arbitrary convex function. Then the entropy inequality

$$\int_{-\infty}^{\infty} \partial_f H(f)[\mathcal{M}(U_f) - f]dv = \int_{-\infty}^{\infty} [\eta'(\zeta(f)) - \eta'(U_f)][\mathcal{M}(U_f, v) - f]dv \le 0$$

holds, since the equality follows from mass conservation and the inequality is a consequence of the monotonicities of $\zeta(f, v)$ with respect to f and of η'. The corresponding macroscopic entropy density is given by $\eta(U)$. So all the macroscopic entropies can be recovered from kinetic entropies.

There are of course also many other choices such as discrete velocity models, the simplest with two velocities: $d\mu(v) = (\delta(v + a) + \delta(v - a))dv$ and

$$\mathcal{M}(U, \pm a) = \frac{1}{2a}(aU \pm J(U))\,,$$

the corresponding BGK-model being equivalent to the standard relaxation model [24]

$$\partial_t U + \partial_x j = 0\,, \qquad \partial_t j + a^2 \partial_x U = J(U) - j\,,$$

with $U = f(-a) + f(a)$, $j = a(f(a) - f(-a))$. Kinetic entropy densities can be constructed as above, if the generalized Maxwellian is strictly monotone in U, i.e., if the subcharacteristic condition $|J'(U)| < a$ holds for all relevant values of U.

2.2 A BGK-model for isentropic and isothermal gas dynamics

The following class of generalized Maxwellians has been introcuced in [27]. Here $n = 2$, $U = (\rho, \rho u)$, $V = \mathbb{R}$, and

$$\mathcal{M}(\rho, u, v) = \alpha \left(\frac{2\gamma}{\gamma - 1} \rho^{\gamma - 1} - (v - u)^2 \right)_+^\beta\,,$$

with $1 < \gamma < 3$,

$$\beta = \frac{3 - \gamma}{2(\gamma - 1)}\,, \qquad \alpha = \frac{1}{J_\beta} \left(\frac{2\gamma}{\gamma - 1} \right)^{-1/(\gamma - 1)}\,, \qquad J_\beta = \int_{-1}^{1} (1 - z^2)^\beta dz\,.$$

The collision invariants $\phi(v) = (1, v)$ correspond to conservation of mass and momentum (in the x-direction), and the macroscopic flux vector is given by

$$J(\rho, m) = \int_{-\infty}^{\infty} v \begin{pmatrix} 1 \\ v \end{pmatrix} \mathcal{M}(\rho, m/\rho, v) dv = \begin{pmatrix} m \\ m^2/\rho + \rho^\gamma \end{pmatrix}\,. \tag{2.2}$$

Thus, the system (1.4) is the p-system of isentropic gas dynamics with adiabatic exponent γ. The eigenvalues of $J'(\rho, m)$ are given by

$$\lambda_{1/2} \;=\; u \mp c(\rho)\,, \qquad \text{where } c(\rho) = \sqrt{\gamma}\, \rho^{(\gamma - 1)/2}\,.$$

For $\rho > 0$ (away from vacuum) $\lambda_1 < \lambda_2$ holds everywhere and the system is strictly hyperbolic. With the corresponding right and left eigenvectors

$$r_1' = (1, \lambda_1)\,, \qquad r_2' = (1, \lambda_2)\,, \qquad l_1' = (\lambda_2, -1)\,, \qquad l_2' = (-\lambda_1, 1)\,,$$

one can see that the system is also genuinely nonlinear. The primes indicate that the eigenvectors are not scaled as assumed in Section 1. Moreover one can show that the Lax admissibility condition for a 1-shock reduces to

$$\rho_- < \rho_+ \tag{2.3}$$

and for a 2-shock to $\rho_- > \rho_+$.

A kinetic entropy density is given by

$$H(f, v) = \frac{v^2}{2} f + \frac{f^{1+1/\beta}}{2\alpha^{1/\beta}(1 + 1/\beta)}\,,$$

leading to the macroscopic entropy density

$$\eta(\rho, u) = \frac{\rho u^2}{2} + \frac{\rho^\gamma}{\gamma - 1}\,,$$

whose physical interpretation is, of course, energy.

From the Maxwellians

$$\mathcal{M}(\rho, u, v) = \frac{\rho}{\sqrt{2\pi}} e^{-\frac{(v-u)^2}{2}}, \qquad \text{where } v \in \mathbb{R},$$

we recover the isothermal gas dynamics, where the flux is given by (2.2) with $\gamma = 1$. Then the eigenvalues are as above with $c \equiv 1$. Here the kinetic entropy

$$H(f, v) = \frac{v^2}{2} f + f \ln f$$

leads to the macroscopic one

$$\eta(\rho, u) = \frac{\rho u^2}{2} + (\rho \ln \rho - \rho \ln \sqrt{2\pi}),$$

see e.g. also [9].

2.3 The gas dynamics BGK-model

In the general d-dimensional Maxwellian

$$\frac{\rho}{(2\pi T)^{d/2}} \exp\left(-\frac{(v-u)^2 + (v_2 - u_2)^2 + \cdots + (v_d - u_d)^2}{2T}\right)$$

with density ρ, mean velocity $(u, u_2, \ldots, u_d)$, and temperature T, the axisymmetry assumption is equivalent to vanishing transversal mean velocities, $u_2 = \cdots = u_d = 0$, and leads to

$$\mathcal{M}(\rho, u, T; v, w) = \frac{\rho}{(2\pi T)^{d/2}} \exp\left(-\frac{(v-u)^2 + w^2}{2T}\right).$$

The integration measure is defined as $d\mu(v, w) := w^{d-2}|S^{d-1}|\, dv\, dw = d(v, v_2, \ldots, v_d)$, where $|S^{d-1}|$ is the surface of the $(d-1)$-dimensional unit sphere. Conservation of mass, momentum in the x-direction, and energy is required, i.e., $\phi(v, w) = (1, v, (v^2 + w^2)/2)$ and

$$U_f = \begin{pmatrix} \rho_f \\ \rho_f u_f \\ \rho_f u_f^2/2 + \frac{d}{2}\rho_f T_f \end{pmatrix} = \int_V \begin{pmatrix} 1 \\ v \\ (v^2 + w^2)/2 \end{pmatrix} f\, d\mu\,.$$

The macroscopic flux is given by

$$J(\rho, u, T) = \begin{pmatrix} \rho u \\ \rho u^2 + \rho T \\ u(\rho u^2/2 + \frac{d}{2}\rho T + \rho T) \end{pmatrix}.$$

The macroscopic system (1.4) are the compressible Euler equations for a d-dimensional ideal gas, reduced to one dimension by assuming plane wave solutions with vanishing transversal velocity components.

The Jacobian

$$J'(U) = \begin{pmatrix} 0 & 1 & 0 \\ \frac{\gamma-3}{2}u^2 & (3-\gamma)u & \gamma-1 \\ (\gamma-1)u^3 - \gamma E\frac{u}{\rho} & \gamma\frac{E}{\rho} - \frac{3}{2}(\gamma-1)u^2 & \gamma u \end{pmatrix}, \qquad \gamma = \frac{d+2}{d},$$

has the eigenvalues

$$\lambda_1 = u - c, \qquad \lambda_2 = u, \qquad \lambda_3 = u + c,$$

with the sound speed $c = \sqrt{\gamma T}$. The corresponding right and left eigenvectors are given by

$$\begin{aligned} r_k &= \left(1, \lambda_k, \frac{1}{\gamma-1}\left(\frac{3-\gamma}{2}u^2 - (3-\gamma)u\lambda_k + \lambda_k^2\right)\right), \\ l_k &= \frac{1}{c^2 d}\left(\frac{1}{\lambda_k}\left(\frac{\gamma-3}{2(\gamma-1)}(\lambda_k - \gamma u)u^2 + (\gamma-1)u^3 - \gamma E\frac{u}{\rho}\right), \frac{1}{\gamma-1}(\lambda_k - \gamma u), 1\right), \end{aligned}$$

satisfying $(l_k \cdot r_k)_{k=1}^3 = (1, -2, 1)$. In view of further calculations we give r_1 explicitly: $r_1 = (1, u-c, \frac{1}{2}(u-c)^2 + \frac{3-\gamma}{2(\gamma-1)}c^2)$. The first and third field are genuinely nonlinear, whereas the second field is linearly degenerate, i.e, $r_2 \cdot \nabla\lambda_2 \equiv 0$.

The kinetic entropy density is the classical $H(f) = f \ln f$, and

$$\int_V \partial_f H(f)[\mathcal{M}(U_f) - f]d\mu = \int_V [\ln f - \ln \mathcal{M}(U_f)][\mathcal{M}(U_f) - f]d\mu \le 0,$$

the equality being a consequence of the fact that the logarithm of the Maxwellian is a linear combination of the collision invariants. The macroscopic entropy density is given by

$$\eta(\rho, T) = \frac{\rho d}{2}\ln\left(\frac{\rho}{(2\pi T)^{d/2}}\right).$$

Subtracting a multiple of the conserved quantity ρ and dividing by a constant factor gives the classical $\hat{\eta} = -\rho\ln(\rho T/\rho^\gamma)$ with $\gamma = (d+2)/d$.

2.4 Fermions in a background medium and a constant electric field

Semiclassical modelling of the scattering of fermions with an equilibrium background medium leads to collision operators of the form [30]

$$Q_s(f)(v) = \int_{\mathbb{R}} \sigma(v, v')[f(v')(1 - f(v))M(v) - f(v)(1 - f(v'))M(v')]dv',$$

where the collision cross section $\sigma(v,v') \geq \underline{\sigma} > 0$ is symmetric, $M(v) = (2\pi)^{-1/2}e^{-v^2/2}$ is a normalized Gaussian, and the occurrence of the factors $(1-f)$ is a consequence of the quantum mechanical Pauli exclusion principle. The zero set of Q_s is one-dimensional (corresponding to the conservation of mass) and consists of the Fermi-Dirac distributions $(1+c/M)^{-1}$, $c>0$. The action of a constant electric field with the x-component E is included in the total collision operator

$$Q(f) = Q_s(f) - E\partial_v f \,.$$

The only conserved quantity is mass ($\phi(v)=1$), and it has been proven in [3] that the zero set of Q can be parametrized by the density:

$$Q(f) = 0 \quad \Longleftrightarrow \quad f(v) = \mathcal{M}(\rho_f, v)\,,$$

where the generalized Maxwellian is a strictly increasing function of $\rho_f = \int_{\mathbb{R}} f\,dv$. Without the lower bound on the collision cross section the existence of nontrivial equilibrium distributions is not guaranteed (see [37]).

The somewhat surprising result that the definition (2.1) yields a kinetic entropy density for the operator Q (including an acceleration term) has been proven in [5].

3 Macroscopic and small wave approximations

3.1 The hydrodynamic limit

The macroscopic approximation (1.4) for the kinetic equation (1.1) can be formally derived by rescaling position and time by $x \to x/\varepsilon$, $t \to t/\varepsilon$, and passing to the limit $\varepsilon \to 0$ in

$$\varepsilon\partial_t f^\varepsilon + \varepsilon v\partial_x f^\varepsilon = Q(f^\varepsilon)\,. \tag{3.1}$$

Assuming a strong enough convergence $f^\varepsilon \to f$ as $\varepsilon \to 0$, passing to the limit yields $Q(f)=0$ and, thus, $f(t,x,\mathbf{v}) = \mathcal{M}(U(t,x),\mathbf{v})$ with $U = \lim_{\varepsilon\to 0} U_{f^\varepsilon}$. Passing to the limit in the conservation laws

$$\partial_t U_{f^\varepsilon} + \partial_x J_{f^\varepsilon} = 0 \tag{3.2}$$

leads to the macroscopic system (1.4). This limit can be justified for all examples of Section 2 in the following sense: If the initial data $f_I(x,\mathbf{v}) = f^\varepsilon(0,x,\mathbf{v})$ are smooth and possess smooth moments $U_{f_I}(x)$, then a unique smooth solution of (1.4) taking these initial data exists for a short enough open time interval. The unique solution of the initial value problem for (3.1) converges to $\mathcal{M}(U)$, where U is the solution of (1.4), on any compact subinterval of the existence interval of the latter (see, e.g., [10] for the case of the Boltzmann equation of gas dynamics).

The harder question of global convergence to weak entropy solutions has also been answered for all the examples except for the gas dynamics BGK-model (Section 2.3). Proofs working for all cases in Section 2.1 can be found in [33], [34]. For the isentropic gas dynamics model (Section 2.2) it has been carried out in [6], and for the fermion model (Section 2.4) in [5].

3.2 The linearised collision operator

The properties of linearizations of the collision operator at equilibrium distributions will be needed throughout the rest of this work and, in particular, in the following section for the construction of a more accurate macroscopic approximation. For a fixed vector $\hat{U}$, the linearisation of the collision operator around $\hat{\mathcal{M}} := \mathcal{M}(\hat{U})$ is denoted by

$$\mathcal{L}f := Q'(\hat{\mathcal{M}})f.$$

The motivation for introducing a suitable functional analytic framework for the operator $\mathcal{L}$ comes from the entropy inequality

$$\frac{1}{\varepsilon^2}\int_V Q(\hat{\mathcal{M}} + \varepsilon f)\partial_f H(\hat{\mathcal{M}} + \varepsilon f)d\mu \le 0$$

Since $\partial_f H(\hat{\mathcal{M}})$ is a linear combination of the collision invariants (see Remark 1.1), the limit as $\varepsilon \to 0$ of the left hand side is equal to $\langle \mathcal{L}f, f\rangle_{\mathbf{v}}$ with the weighted scalar product

$$\langle f, g\rangle_{\mathbf{v}} := \int_V f\, g\, \partial_f^2 H(\hat{\mathcal{M}})d\mu\,. \tag{3.3}$$

The induced Hilbert space and its norm are denoted by $(L^2_{\mathbf{v}}, \|.\|_{\mathbf{v}})$. The operator $\mathcal{L}$ is assumed to be bounded and symmetric and, by passing to the limit in the above entropy inequality, it is negative semidefinite in $L^2_{\mathbf{v}}$. By the symmetry assumption, the functions $\phi_j/\partial_f^2 H(\hat{\mathcal{M}})$, $1 \le j \le n$, (where ϕ_j is the j-th collision invariant) lie in the null space $\mathcal{N}$ of $\mathcal{L}$. We assume that they span $\mathcal{N}$, but use the alternative basis $\partial_{U_j}\mathcal{M}(\hat{U}, \mathbf{v})$, $j = 1, \dots, n$, having the useful property

$$(U_{\partial_{U_j}\mathcal{M}(\hat{U})})_i = \partial_{U_j}\int_V \phi_i \mathcal{M}(\hat{U})d\mu = \delta_{ij}\,, \qquad 1 \le i, j \le n\,.$$

Since

$$\langle f, \phi/\partial_f^2 H(\hat{\mathcal{M}})\rangle_{\mathbf{v}} = U_f\,,$$

$\mathcal{N}^\perp = \{f \in L^2_{\mathbf{v}} : U_f = 0\}$ holds, where $\mathcal{N}^\perp$ is the orthogonal complement of $\mathcal{N}$ in $L^2_{\mathbf{v}}$. We assume that $\mathcal{L} :\ \mathcal{N}^\perp \to \mathcal{N}^\perp$ is invertible. In other words, $U_g = 0$ is the solvability condition for the equation $\mathcal{L}f = g$, which has a unique solution $f \in \mathcal{N}^\perp$. Finally, the orthogonal projection from $L^2_{\mathbf{v}}$ to $\mathcal{N}$ is given by $f \mapsto U_f \cdot \nabla_U \mathcal{M}(\hat{U})$.

In this section, we have posed further assumptions on the collision operator:

Assumption 2 *The linearized collsion operator $Q'(\hat{\mathcal{M}})$ is symmetric with respect to the scalar product (3.3). Its kernel $\mathcal{N}$ is n-dimensional, and its restriction to $\mathcal{N}^\perp$ is invertible.*

For BGK-models the linearized collision operator is given by $\mathcal{L}f = U_f \cdot \nabla_U \hat{\mathcal{M}} - f$, immediately showing $\dim(\mathcal{N}) = n$, $\mathcal{L}|_{\mathcal{N}^\perp} = -\mathrm{id}$, and implying

$$\langle \mathcal{L}f, g\rangle_{\mathbf{v}} - \langle \mathcal{L}g, f\rangle_{\mathbf{v}} = \int_V (gU_f - fU_g)\cdot \nabla_U \hat{\mathcal{M}}\, \partial_f^2 H(\hat{\mathcal{M}})d\mu\,.$$

Symmetry of $\mathcal{L}$ is now a consequence of the identity

$$\nabla_U \mathcal{M}(U)\,\partial_f^2 H(\mathcal{M}(U)) = \nabla^2\eta(U)\phi\,, \tag{3.4}$$

derived by computing the gradient of $\partial_f H(\mathcal{M}(U)) = \nabla\eta(U)\cdot\phi$ (see Remark 1.1). The identity (3.4) has other useful consequences. Taking its tensor product with $v\nabla_U\mathcal{M}(U)$ and integrating it with respect to $\mathbf{v}$ shows that the matrix $\nabla^2\eta(U)J'(U)$ is symmetric. This in turn implies that $\nabla^2\eta(U)r_k(U)$ is a left eigenvector of $J'(U)$ corresponding to the eigenvalue λ_k. Thus,

$$\nabla^2\eta(U)r_k(U) = \kappa_k(U)l_k(U)\,, \qquad \text{with } \kappa_k(U) = \nabla^2\eta(U)(r_k(U), r_k(U)) > 0\,,$$

implying the following relation between elements of $\mathcal{N}$:

$$r_k(\hat{U})\cdot\nabla_U\hat{\mathcal{M}} = \kappa_k(\hat{U})l_k(\hat{U})\cdot\frac{\phi}{\partial_f^2 H(\hat{\mathcal{M}})}\,. \tag{3.5}$$

3.3 The Chapman-Enskog approximation

There are two basic strategies for improving the approximation quality of the macroscopic limit $\mathcal{M}(U)$ as an approximation for f^ε. The idea of the *Hilbert expansion* [23] is rather straightforward and amounts to constructing an asymptotic expansion for f^ε in terms of powers of ε:

$$f^\varepsilon(t,x,\mathbf{v}) = \mathcal{M}(U(t,x),\mathbf{v}) + \sum_{j=1}^{n}\varepsilon^n f_n(t,x,\mathbf{v}) + O(\varepsilon^{n+1})\,.$$

Substitution of this ansatz in (3.1), (3.2), and in the initial conditions, and comparing coefficients of ε leads to equations determining the sequence $\{f_n\}$ recursively.

The second approach does not concentrate on solving arbitrary initial value problems for (3.1), but to approximate a solution manifold parametrized by the macroscopic moments $U^\varepsilon = U_{f^\varepsilon}$. It starts with the *micro-macro decomposition*

$$f^\varepsilon = \mathcal{M}(U^\varepsilon) + \varepsilon f^\perp\,,$$

and tries to compute $f^\perp$ in terms of U^ε and the dynamics of U^ε, such that f^ε solves (3.1). When, in this program, $O(\varepsilon^2)$-errors are accepted in the equation for U_ε, an approximation up to $O(\varepsilon)$-errors is needed for $f^\perp$. From (3.1) we obtain

$$Q'(\mathcal{M}(U^\varepsilon))f^\perp = \nabla_U\mathcal{M}(U^\varepsilon)\cdot(\partial_t U^\varepsilon + v\partial_x U^\varepsilon) = \nabla_U\mathcal{M}(U^\varepsilon)\cdot(v - J'(U^\varepsilon))\partial_x U^\varepsilon\,,$$

where $O(\varepsilon)$-terms have been neglected. Computing the gradient with respect to U of the relation $J(U) = \int_V v\phi\mathcal{M}(U)d\mu$ shows that the right hand side satisfies the solvability condition mentioned at the end of the previous section such that $f^\perp$ can be computed uniquely in terms of the approximation U of U^ε:

$$f^\perp[U] = Q'(\mathcal{M}(U))^{-1}[\nabla_U\mathcal{M}(U)\cdot(v - J'(U))]\partial_x U\,. \tag{3.6}$$

Using this in (3.2) gives the *Chapman-Enskog approximation* [12]

$$\partial_t U + \partial_x J(U) = \varepsilon \partial_x (D(U)\partial_x U)\,, \tag{3.7}$$

with the diffusivity matrix

$$D(U) = -\int_V v\phi \otimes Q'(\mathcal{M}(U))^{-1}[\nabla_U \mathcal{M}(U)\cdot(v - J'(U))]d\mu\,,$$

where the symbol $\otimes$ denotes the tensor product. For BGK-models, $Q'|_{\mathcal{N}^\perp} = -\mathrm{id}$ and a more explicit representation can be found:

$$D(U) = \int_V v^2\phi \otimes \nabla_U \mathcal{M}(U)d\mu - J'(U)^2\,.$$

The diffusion dissipates the macroscopic entropy, which is reflected by the fact that

$$\nabla^2\eta(U)D(U) = -\langle Q'(\mathcal{M}(U))\chi \otimes \chi\rangle_{\mathbf{v}} \geq 0$$

holds, where the identity (3.4) and the notation $\chi = \nabla_U \mathcal{M}(U)\cdot(v - J'(U))$ have been used.

Remark 3.1 *For the BGK-models for scalar macroscopic equations ($\phi = 1$) in Section 2.1, $\partial_U \mathcal{M} > 0$ holds, implying*

$$J'(U)^2 = \left(\int_V v\partial_U \mathcal{M}(U)d\mu\right)^2 < \int_V v^2 \partial_U \mathcal{M}(U)d\mu\,,$$

since $\int_V \partial_U \mathcal{M}(U)d\mu = 1$. Thus, the diffusivity $D(U)$ is strictly positive.

For the BGK-model for isentropic gas dynamics in Section 2.2,

$$D(\rho,u) = (3-\gamma)\rho^{\gamma-1}\begin{pmatrix} 0 & 0 \\ -u & 1 \end{pmatrix}$$

holds, leading to the Navier-Stokes model

$$D(\rho,u)\partial_x \begin{pmatrix} \rho \\ \rho u \end{pmatrix} = \begin{pmatrix} 0 \\ \mu(\rho)\partial_x u \end{pmatrix}\,,$$

with the viscosity $\mu(\rho) = (3-\gamma)\rho^\gamma$. This example shows that the diffusivity is in general not regular, such that diffusion does not act on all components of U.

The gas dynamics BGK-model in Section 2.3 gives

$$D(\rho,u,T) = T\begin{pmatrix} 0 & 0 & 0 \\ -\frac{2(d-1)}{d}u & \frac{2(d-1)}{d} & 0 \\ -\frac{(d+2)}{2}T - \frac{3(d-2)}{2d}u^2 & \frac{(d-4)}{d}u & \frac{d+2}{d} \end{pmatrix}\,,$$

leading to

$$D(\rho,u,T)\partial_x \begin{pmatrix} \rho \\ \rho u \\ \rho(u^2+dT)/2 \end{pmatrix} = \begin{pmatrix} 0 \\ \mu(\rho,T)\partial_x u \\ u\mu(\rho,T)\partial_x u + \kappa(\rho,T)\partial_x T \end{pmatrix},$$

with viscosity $\mu(\rho,T) = \frac{2(d-1)}{d}\rho T$ *and heat conductivity* $\kappa(\rho,T) = \frac{d+2}{2}\rho T$*. Note that with* $\gamma = \frac{d+2}{d}$ *and with the isentropic relation* $\rho T = \rho^\gamma$*, the viscosity of the previous example is recovered.*

Remark 3.2 *Similar results as those described in Section 3.1 are also available for the vanishing diffusion limit* $\varepsilon \to 0$ *in (3.7) (see, e.g., [20], or [8] for a recent result).*

3.4 Weakly nonlinear approximation for small waves

In this section we consider the slow modulation of travelling wave solutions of the linearization of the hydrodynamic system (1.4) at a constant state U_-. Modulations are caused by nonlinearity and by the dissipative terms in the Chapman-Enskog system (3.7).

We choose k such that the k-th field is strictly nonlinear ($r_k \cdot \nabla\lambda_k = 1$) and introduce a moving reference frame and a long time scale by $x = \eta + \lambda_k(U_-)t$ and $t = \tau/\varepsilon$ in (3.7):

$$\varepsilon\partial_\tau U + (J'(U) - \lambda_k(U_-))\partial_\eta U = \varepsilon\partial_\eta(D(U)\partial_\eta U)\,. \tag{3.8}$$

This motivates the ansatz

$$U(\tau,\eta) = U_- + \varepsilon y(\tau,\eta) r_k(U_-) + \varepsilon^2 U_2(\tau,\eta) + O(\varepsilon^3)\,,$$

which annihilates the $O(1)$- and $O(\varepsilon)$-terms in (3.8). At $O(\varepsilon^2)$, we obtain

$$\partial_\tau y\, r_k + y\partial_\eta y J''(r_k,r_k) + (J' - \lambda_k)\partial_\eta U_2 = D r_k \partial_\eta^2 y\,,$$

where all functions of U are evaluated at U_-. A solvability condition for this equation for U_2 is obtained by taking the scalar product with the left eigenvector l_k of J' (satisfying $l_k \cdot r_k = 1$):

$$\partial_\tau y + y\partial_\eta y = D_k(U_-)\partial_\eta^2 y\,, \tag{3.9}$$

with $D_k(U) = l_k(U)\cdot D(U) r_k(U)$, where the relation

$$\begin{aligned} 1 &= r_k\cdot\nabla\lambda_k = r_k\cdot\nabla(l_k\cdot J' r_k) = l_k\cdot J''(r_k,r_k) + r_k\cdot(\lambda_k r_k\cdot\nabla l_k + \lambda_k l_k\cdot\nabla r_k) \\ &= l_k\cdot J''(r_k,r_k) + \lambda_k r_k\cdot\nabla(l_k\cdot r_k) = l_k\cdot J''(r_k,r_k) \end{aligned} \tag{3.10}$$

has been used.

Approximately, the modulation of travelling wave solutions is described by the viscous Burgers equation (3.9). Positivity of the scalar diffusivity D_k will be assumed. It has to be checked example by example.

Remark 3.3 *For scalar conservation laws we obviously have* $D_1(U) = D(U)$*. For the isentropic gas dynamics model we obtain* $D_k = (3-\gamma)\rho^{\gamma-1}/2$ *for both* $k=1$ *and* $k=2$*, and for the full gas dynamics BGK-model* $D_k = T$ *for the genuinely nonlinear fields* $k=1$ *and* $k=3$*.*

3.5 Viscous shock profiles for weak shocks

Let the k-th field of (1.4) be genuinely nonlinear and let $\{U_\pm, s\}$ denote a k-shock, where the shock speed can be written as

$$s = \lambda_- + \varepsilon\sigma\,,$$

with $\sigma < 0$ and a small perturbation parameter $0 < \varepsilon \ll 1$. The sign conditions are due to the Lax entropy condition (1.8). Then the difference of the far field states has an asymptotic expansion

$$U_+ - U_- = 2\varepsilon\sigma r_- + O(\varepsilon^2)\,,$$

where $r_- := r_k(U_-)$ and $\lambda_- := \lambda_k(U_-)$. A travelling wave solution $U = U_{vsp}(\xi)$, $\xi = x - st$ of the Chapman-Enskog equations (3.7), satisfying the far-field conditions $U_{vsp} \to U_\pm$ for $\xi \to \pm\infty$, will be called a *viscous shock profile.* It can be seen as an heteroclinic orbit of the ODE system

$$\varepsilon D(U)\partial_\xi U = J(U) - J(U_-) - s(U - U_-)\,. \tag{3.11}$$

General results on the existence of viscous shock profiles are not available (even for artificial viscosity of the form $D(U) = I$). For small shocks, i.e. ε small enough, U_{vsp} can be expected to stay close to the constant state U_-, and therefore the asymptotics of the previous section can be used for an approximation. This leads to a travelling wave problem for the viscous Burgers equation (3.9) with wave speed σ and with the far-field values $y_- = 0$ and $y_+ = 2\sigma$. A travelling wave solution $y_{vsp}(\eta - \sigma\tau) = y_{vsp}(\xi)$ can be computed explicitly.

To make this approximation rigorous is a nontrivial problem of the theory of singularly perturbed ODEs. The details of the justification depend on the properties of the diffusivity matrix $D(U)$. A general rigorous treatment is, thus, impossible and we state the result as an assumption.

Assumption 3 *Let the k-th field of the macroscopic flux $J(U)$ be genuinely nonlinear, and let $D_k(U_-) = l_k(U_-) \cdot D(U_-)r_k(U_-) > 0$. Let, for ε small enough, (3.11) have a solution $U_{vsp}(\xi) = U_- + \varepsilon y_{vsp}(\xi)r_k(U_-) + \varepsilon^2 U_2^\varepsilon(\xi)$, satisfying $\lim_{\xi\to\pm\infty} U_{vsp}(\xi) = U_\pm$, such that U_2^ε and all its derivatives are uniformly bounded with respect to ε.*

We verify the assumption for the BGK-models of Section 2. For the case of a scalar conservation law (Section 2.1), genuine nonlinearity, w.l.o.g. $J'' > 0$, has to be assumed. In this case, by $D(U) > 0$, viscous shock profiles obviously exist, iff the entropy condition $U_+ < U_-$ is satisfied. Smallness of the shock is not needed.

For the case of isentropic gas dynamics (Section 2.2), both fields are genuinely nonlinear. By the form of $D(\rho, m)$ the first equation in the system (3.11) is algebraic, and the system can be reduced to the scalar ODE

$$\frac{\varepsilon\mu(\rho)}{\rho}\partial_\xi\rho = (\rho - \rho_-)(u_- - s) + \frac{\rho^{\gamma+1} - \rho\rho_-^\gamma}{\rho_-(s - u_-)}\,.$$

It is easily seen that $s - u_-$ is negative for a 1-shock and positive for a 2-shock. Obviously, this sign determines the convexity of the right hand side, and a viscous shock profile exists, whenever the entropy condition ($\rho_+ > \rho_-$ for a 1-shock, $\rho_+ < \rho_-$ for a 2-shock) is satisfied. Again smallness of the shock is not needed.

For the full gas dynamics BGK-model (Section 2.3) again one equation in (3.11) is algebraic. However, after elimination of one unknown, a singularly perturbed second order system remains. Existence of viscous profiles for small shocks (in the genuinely nonlinear fields $k = 1, 3$) has been shown for various applications.

4 Existence of kinetic profiles for weak shocks

In this section we shall present an approach for the construction of small amplitude travelling wave solutions of the kinetic equation (1.1). The macroscopic moments of their far field limits are connected by genuinely nonlinear entropic shock waves of the hyperbolic system (1.4). The main ideas are generalizations of the work of Caflisch and Nicolaenko [11] on the gas dynamics Boltzmann equation. Our approach is slightly different in several details. In particular, starting from a formal asymptotic approximation, a perturbation equation for the correction term is considered. This leads to a sharper error bound in the final result. Also the problem is in general not linearized around the far-field state. This is necessary for treating problems with equilibrium velocity distributions with compact support (like the BGK-model for isentropic gas dynamics in Section 2.2), in order to guarantee that the support of the state we linearize around contains the support of the travelling wave.

In the following two subsections, the general procedure is outlined. Applications to several examples are contained in the last subsection.

A kinetic shock profile is a solution $f = f(\xi, \mathbf{v})$ of

$$\varepsilon(v - s)\partial_\xi f = Q(f)\,, \qquad \lim_{\xi\to\pm\infty} f(\xi, \mathbf{v}) = \mathcal{M}_\pm(\mathbf{v}) := \mathcal{M}(U_\pm, \mathbf{v})\,. \tag{4.1}$$

Considering (3.11) as an approximation for (4.1), an approximative kinetic profile for a small k-shock is given by

$$f_{as} := \mathcal{M}(U_{vsp}) + \varepsilon f^\perp[U_{vsp}]\,, \tag{4.2}$$

where the microscopic correction term is defined in (3.6).

4.1 The micro-macro decomposition of the correction term

We start by analyzing the formal approximation properties of (4.2). The residual is given by

$$\begin{aligned} \varepsilon^3 h &:= \varepsilon(v - s)\partial_\xi f_{as} - Q(f_{as}) \\ &= \varepsilon(v - s)\partial_\xi \mathcal{M}(U_{vsp}) + \varepsilon^2(v - s)\partial_\xi f^\perp[U_{vsp}] - Q(\mathcal{M}(U_{vsp}) + \varepsilon f^\perp[U_{vsp}])\,. \end{aligned} \tag{4.3}$$

Using the asymptotic expansion of U_{vsp} given in Assumption 3, it is straightforward to show that the scaling of the residual is justified in the sense that, as a function of ξ, h and its derivatives are bounded uniformly with respect to ε.

Also the system (3.11) implies that the macroscopic moments of the residual vanish: $U_h = 0$. Finally, f_{as} satisfies the far-field conditions in (4.1) exactly.

The problem (4.1) rewritten in terms of the correction term $\varepsilon^2 g = f - f_{as}$ reads

$$\varepsilon(v-s)\partial_\xi g - \mathcal{L}_{as} g = \varepsilon^2 R(g) - \varepsilon h\,, \tag{4.4}$$

with $\mathcal{L}_{as} = Q'(f_{as})$ and $R(g) = \varepsilon^{-4}(Q(f_{as}+\varepsilon^2 g) - Q(f_{as}) - \varepsilon^2 \mathcal{L}_{as} g)$, subject to

$$g(\pm\infty, \mathbf{v}) = 0 \qquad \text{for all } \ \mathbf{v} \in V\,. \tag{4.5}$$

By computing the moments and integration with respect to ξ, we derive the property

$$\int_V (v-s)\phi\, g\, d\mu = 0\,. \tag{4.6}$$

The collision operator has been linearized around the approximation f_{as}. This has the inconvenience to depend on the spatial variable ξ. Therefore we shall also use the linearization $\mathcal{L} := Q'(\mathcal{M}(\hat{U}))$ around the constant-in-ξ state $\hat{\mathcal{M}}$, chosen such that $\hat{U} = U_- + \varepsilon\tilde{U}$ (with $\tilde{U}$ bounded uniformly in ε) and, consequently, $f_{as} = \hat{\mathcal{M}} + O(\varepsilon)$ and $\mathcal{L}_{as} = \mathcal{L} + O(\varepsilon)$. Here and in the following, we use the abbreviations

$$\hat{\mathcal{M}} = \mathcal{M}(\hat{U})\,, \quad \hat{\lambda} = \lambda_k(\hat{U})\,, \quad \hat{r} = r_k(\hat{U})\,, \quad \hat{l} = l_k(\hat{U})\,.$$

The correction term is split into a macroscopic and a microscopic part:

$$g(\xi, \mathbf{v}) = z(\xi)\Phi(\mathbf{v}) + \varepsilon w(\xi, \mathbf{v})\,, \tag{4.7}$$

where the macroscopic variable is scalar and corresponds only to contributions from the k-th field. The choice of the profile function Φ is motivated by the work of Caflisch and Nicolaenko for the Boltzmann equation [11]. It is chosen such that it approximately solves a generalized eigenvalue problem:

$$\mathcal{L}\Phi = \varepsilon\tau(v-s)\Phi + O(\varepsilon^2)\,,$$

for a constant τ and, additionally has the moment property of g:

$$\int_V (v-s)\phi\,\Phi\, d\mu = 0 \qquad \Longrightarrow \qquad \int_V (v-s)\phi\, w\, d\mu = 0\,. \tag{4.8}$$

Hence expanding $\Phi = \Phi_0 + \varepsilon\Phi_1$ and decomposing the wave speed as $s = \hat{\lambda} + \varepsilon\hat{\sigma}$ with $\hat{\sigma} = \sigma - \tilde{U}\cdot\nabla\hat{\lambda} + O(\varepsilon)$, we determine the components Φ_0 and Φ_1 and the eigenvalue τ such that

$$\mathcal{L}\Phi_0 = 0 \quad \text{and} \quad \int_V (v-\hat{\lambda})\phi\,\Phi_0\, d\mu = 0, \tag{4.9}$$

$$\mathcal{L}\Phi_1 = \tau(v-\hat{\lambda})\Phi_0 \quad \text{and} \quad \int_V (v-s)\phi\,\Phi_1\, d\mu = \hat{\sigma}\int_V \phi\,\Phi_0\, d\mu\,. \tag{4.10}$$

Considering that the null space of $\mathcal{L}$ is spanned by the components of $\nabla_U \hat{\mathcal{M}}$, the problem (4.9) is solved by $\Phi_0 = \hat{r} \cdot \nabla_U \hat{\mathcal{M}}$. Note that the second equation in (4.9) is the solvability condition for the first equation in (4.10). We choose a solution of the form

$$\Phi_1 = \tau \mathcal{L}^{-1}[(v - \hat{\lambda})\Phi_0] + \sum_{j \neq k} \beta_j r_j(\hat{U}) \cdot \nabla_U \hat{\mathcal{M}} .$$

The second equation in (4.10) then becomes

$$-\tau D(\hat{U})\hat{r} + \sum_{j \neq k} \beta_j (\lambda_j(\hat{U}) - s) r_j(\hat{U}) = \hat{\sigma} \hat{r} ,$$

which can be solved for τ and the β_j by

$$\tau = -\frac{\hat{\sigma}}{D_k(\hat{U})} , \qquad \beta_j = \frac{\tau}{\lambda_j(\hat{U}) - s} l_j(\hat{U}) \cdot D(\hat{U})\hat{r} , \quad j \neq k .$$

As a consequence of (3.5) and of (4.9)

$$\psi = \mathcal{L}^{-1} \left((v - \hat{\lambda}) \frac{\hat{l} \cdot \phi}{\partial_f^2 H(\hat{\mathcal{M}})} \right) \in \mathcal{N}^{\perp}$$

is well defined. In order to make the decomposition (4.7) unique, we pose the orthogonality condition

$$\langle (v - s)\psi, w \rangle_{\mathbf{v}} = 0 . \tag{4.11}$$

The computation

$$\begin{aligned} -\tilde{D} &= \langle (v - s)\psi, \Phi \rangle_{\mathbf{v}} = \langle v\psi, \Phi_0 \rangle_{\mathbf{v}} + O(\varepsilon) \\ &= \hat{l} \cdot \int_V v\phi \mathcal{L}^{-1}[(v - \hat{\lambda})\hat{r} \cdot \nabla_U \hat{\mathcal{M}}] d\mu + O(\varepsilon) = -D_k(\hat{U}) + O(\varepsilon) \end{aligned}$$

shows that, for ε small enough, $\tilde{D} > 0$ and the decomposition (4.7) is well defined.

We now write the perturbation equation (4.4) in terms of the decomposition (4.7) and divide by ε:

$$(v - s)\Phi \partial_\xi z - z \frac{1}{\varepsilon} \mathcal{L}_{as} \Phi + \varepsilon (v - s) \partial_\xi w - \mathcal{L}_{as} w = \varepsilon R(z\Phi + \varepsilon w) - h, \tag{4.12}$$

It is part of the method of Caflisch and Nicolaenko that for projecting the equation to its macroscopic and microscopic parts, the alternative decomposition

$$g = Pg - \frac{(v - s)\Phi}{\tilde{D}} \Pi g , \qquad \text{with } \Pi g = \langle \psi, g \rangle_{\mathbf{v}} ,$$

is used. This definition and the property (4.8) of Φ imply $U_{Pg} = U_g$. Application of Π to (4.12) gives the macroscopic equation

$$-\tilde{D} \partial_\xi z + \Psi(\xi) z = \varepsilon \Gamma w + \varepsilon \Pi R - \Pi h, \tag{4.13}$$

where

$$\Psi = -\frac{1}{\varepsilon}\Pi\mathcal{L}_{as}\Phi \qquad \text{and} \qquad \Gamma w = \frac{1}{\varepsilon}\Pi\mathcal{L}_{as} w\,.$$

These terms are (formally) $O(1)$. This might not be obvious for the last one: The construction of ψ, the symmetry of $\mathcal{L}$ and (4.8) imply

$$\Pi\mathcal{L}_{as}w = \langle\mathcal{L}\psi, w\rangle_{\mathbf{v}} + O(\varepsilon) = \hat{l}\cdot\int_V (v-s)\phi\, w\, d\mu + O(\varepsilon) = O(\varepsilon)\,.$$

In the limit $\varepsilon\to 0$, z satisfies a linear equation independent from the microscopic solution component w. The following result shows that it is a small perturbation of the linearization of the travelling wave version of the viscous Burgers equation (3.9).

Lemma 4.1 *Let Assumption 3 hold. Then, formally, $\Psi(\xi) = y_{vsp}(\xi) - \sigma + O(\varepsilon)$.*

Proof. We shall use the formula

$$Q''(\mathcal{M}(U))(\nabla_U\mathcal{M}(U), \nabla_U\mathcal{M}(U)) = -Q'(\mathcal{M}(U))\nabla_U^2\mathcal{M}(U)\,, \tag{4.14}$$

which can be derived by computing the Hessian with respect to U of $Q(\mathcal{M}(U)) = 0$. Since Φ_0 is in the null space of $\mathcal{L}$,

$$\frac{1}{\varepsilon}\mathcal{L}_{as}\Phi = \frac{\mathcal{L}_{as}-\mathcal{L}}{\varepsilon}\Phi_0 + \mathcal{L}_{as}\Phi_1 \tag{4.15}$$

holds. By the definition of Φ_1,

$$\Pi\mathcal{L}_{as}\Phi_1 = \langle\psi, \mathcal{L}\Phi_1\rangle_{\mathbf{v}} + O(\varepsilon) = \tau\langle\psi, (v-s)\Phi\rangle_{\mathbf{v}} + O(\varepsilon) = \hat{\sigma} + O(\varepsilon)\,. \tag{4.16}$$

The definition of f_{as} and the expansion of the viscous shock profile U_{vsp} imply $f_{as} = U_- + \varepsilon y_{vsp} r_k(U_-) + O(\varepsilon^2)$ and, thus,

$$\begin{aligned}
\frac{\mathcal{L}_{as}-\mathcal{L}}{\varepsilon} &= \frac{Q'(\mathcal{M}(U_- + \varepsilon y_{vsp} r_k(U_-))) - Q'(\hat{\mathcal{M}})}{\varepsilon} + O(\varepsilon)\\
&= Q''(\hat{\mathcal{M}})(y_{vsp}\hat{r} - \tilde{U})\cdot\nabla_U\hat{\mathcal{M}} + O(\varepsilon)\,.
\end{aligned}$$

With (4.14) we therefore obtain

$$\frac{\mathcal{L}_{as}-\mathcal{L}}{\varepsilon}\Phi_0 = -\mathcal{L}\nabla_U^2\hat{\mathcal{M}}(y_{vsp}\hat{r} - \tilde{U}, \hat{r}) + O(\varepsilon)\,,$$

implying, with the symmetry of $\mathcal{L}$,

$$\begin{aligned}
\Pi\frac{\mathcal{L}_{as}-\mathcal{L}}{\varepsilon}\Phi_0 &= -\hat{l}\cdot\int_V (v-\hat{\lambda})\phi\nabla_U^2\hat{\mathcal{M}}d\mu\,(y_{vsp}\hat{r} - \tilde{U}, \hat{r}) + O(\varepsilon)\\
&= -\hat{l}\cdot J''(\hat{U})(y_{vsp}\hat{r} - \tilde{U}, \hat{r}) + O(\varepsilon) = -y_{vsp} + \tilde{U}\cdot\nabla\hat{\lambda} + O(\varepsilon)\,,
\end{aligned}$$

where the last equality requires a computation analogous to (3.10). Combining this with (4.15) and (4.16) and recalling $\hat\sigma = \sigma - \tilde U \cdot \nabla\hat\lambda + O(\varepsilon)$ completes the proof. $\square$

The microscopic projection P is used to derive from (4.12) an equation for w. The linearized collison operator is now approximated by $\mathcal{L}$:

$$\varepsilon(v-s)\partial_\xi w - \mathcal{L}w = z\frac{1}{\varepsilon}P\mathcal{L}_{as}\Phi + \varepsilon\tilde\Gamma w + \varepsilon PR - Ph\,, \tag{4.17}$$

where

$$\tilde\Gamma w = -\frac{1}{\varepsilon}\,\frac{(v-s)\Phi}{\tilde D}\Pi\mathcal{L}w + \frac{1}{\varepsilon}\,P(\mathcal{L}_{as}-\mathcal{L})w\,.$$

The operator Γ is formally $O(1)$. Like g, its micro- and macro-components z and, respectively, w have to satisfy homogeneous far-field conditions

$$z(\pm\infty) = w(\pm\infty, \mathbf{v}) = 0\,. \tag{4.18}$$

The problem (4.4), (4.5) for g is equivalent to the (z,w)-problem (4.13), (4.17), (4.18).

The basic idea for the solution is to produce a fixed point problem by considering the right hand sides of (4.13) and of (4.17) as given. This is made difficult by the nondefiniteness of $\mathcal{L}$. The following procedure of *removal of the null space* is again based on ideas from Caflisch and Nicolaenko [11].

We introduce a negative definite perturbation of $\mathcal{L}$, which coincides with $\mathcal{L}$ on the set of functions w satisfying the moment conditions (4.8) and the orthogonality condition (4.11):

$$\mathcal{K}w = \mathcal{L}w - (v-s)\psi\langle(v-s)\psi, w\rangle_{\mathbf{v}} - (v-s)\frac{\phi}{\partial_f^2 H(\hat{\mathcal{M}})}\cdot\int_V (v-s)\phi w\,d\mu\,. \tag{4.19}$$

We already know that $\mathcal{L}$ is negative definite on $\mathcal{N}^\perp$. Writing the general element of the null space of $\mathcal{L}$ as $w = \sum_{j=1}^n \alpha_j r_j(\hat U)\cdot\nabla_U\hat{\mathcal{M}} \in \mathcal{N}$, we compute

$$-\langle\mathcal{K}w, w\rangle_{\mathbf{v}} = \left(\sum_{j=1}^n \alpha_j \hat l\cdot D(\hat U) r_j(\hat U)\right)^2 + \left|\sum_{j=1}^n \alpha_j(\lambda_j(\hat U)-s)r_j(\hat U)\right|^2\,.$$

The limits $(\lambda_j(U_-)-\lambda_k(U_-))r_j(U_-)$, $j\neq k$, as $\varepsilon\to 0$ of the vectors $(\lambda_j(\hat U)-s)r_j(\hat U)$, $j\neq k$, in the last term are linearly independent. Therefore this term controls the coefficients α_j, $j\neq k$. The coefficent of α_k in the first term on the right hand side is equal to $D_k(\hat U)$, whose limit $D_k(U_-)$ as $\varepsilon\to 0$ is positive. So this term controls α_k, showing that $\mathcal{K}$ is negative definite.

We now replace the operator $\mathcal{L}$ in (4.17) by $\mathcal{K}$:

$$\varepsilon(v-s)\partial_\xi w - \mathcal{K}w = z\frac{1}{\varepsilon}P\mathcal{L}_{as}\Phi + \varepsilon\tilde\Gamma w + \varepsilon PR - Ph\,, \tag{4.20}$$

and look for a solution of (4.13), (4.18), (4.20) in the following. The equivalence of the problems is not obvious:

Lemma 4.2 *For given $z(\xi)$, the problems (4.17), (4.18) and (4.20), (4.18) for w are equivalent.*

Proof. Since any solution of (4.17), (4.18) satisfies (4.8) and (4.11), it also solves (4.20), (4.18).

Let on the other hand w be a solution of (4.20), (4.18). Multiplication of (4.20) by the components of ϕ and integration with respect to velocity as well as taking the scalar product of (4.20) with ψ results in a system of $n+1$ linear homogeneous first order ODEs for the quantitites

$$\int_V (v-s)\phi\, w\, d\mu \quad \text{and} \quad \langle (v-s)\psi, w\rangle_{\mathbf{v}}\,.$$

Due to the homogeneous far-field conditions, these quantities vanish for all ξ, implying (4.8) and (4.11) and, thus, (4.17). ☐

4.2 The existence result

The solvability of the nonlinear problem (4.13), (4.18), (4.20) is deduced by using a fix-point argument. Hence we first consider the leading linear system, where we regard the right hand sides of (4.13) and (4.20) as given inhomogenities:

$$\varepsilon(v-s)\partial_\xi w - \mathcal{K}w = h_w\,, \qquad \text{subject to } \; w(\pm\infty, \mathbf{v}) = 0\,, \tag{4.21}$$

$$\tilde{D}\partial_\xi z - \Psi(\xi) z = h_z\,, \qquad \text{subject to } \; z(\pm\infty) = 0\,. \tag{4.22}$$

Taking the scalar product of (4.21) with w and integrating with respect to ξ gives

$$-\int_{-\infty}^{\infty} \langle \mathcal{K}w, w\rangle_{\mathbf{v}}\, d\xi = \int_{-\infty}^{\infty} \langle h_w, w\rangle_{\mathbf{v}}\, d\xi\,. \tag{4.23}$$

This shows that the definiteness of $\mathcal{K}$ implies uniqueness of the solution of (4.21), whereas equation (4.22) has a one parameter set of solutions, which reflects the translational invariance of the travelling wave problem. Therefore we pose the initial condition

$$z(0) = z_0\,, \tag{4.24}$$

with an arbitrary $z_0 \in \mathbb{R}$. Lemma 4.1 and the far-field behaviour of y_{vsp} imply $\Psi(\infty) < 0$ and $\Psi(-\infty) > 0$. Therefore the fundamental solution Z satisfying

$$\tilde{D}\partial_\xi Z - \Psi(\xi) Z = 0\,, \qquad Z(0) = 1\,,$$

decays exponentially for $\xi \to \pm\infty$, and the solution

$$z(\xi) = Z(\xi) z_0 + \frac{1}{\tilde{D}} \int_0^\xi \frac{Z(\xi)}{Z(\eta)} h_z(\eta) d\eta$$

of (4.22), (4.24) is bounded for bounded h_z.

At this point it is necessary to choose a functional analytic framework for the further development. Different choices are possible and have been made in different situations in the past. In this general treatment, we stay abstract and assume that two norms $\|\cdot\|_\xi^*$ and

$\|\cdot\|_\xi^{**}$ for functions of the spatial variable ξ have been chosen, where the first one is used for solutions of (4.22) and the second one for the right hand sides. Similarly, the norms $\|\cdot\|_{\xi,\mathbf{v}}^{*}$ and $\|\cdot\|_{\xi,\mathbf{v}}^{**}$ for functions of $(\xi,\mathbf{v})$ are used for solutions of (4.21) and, respectively, right hand sides. In the following, C denotes (possibly different) ε-independent constants.

Assumption 4 *A solution of (4.21) exists and the solutions of (4.22), (4.24) and of (4.21) satisfy estimates of the form*

$$\|z\|_\xi^* \le C(|z_0| + \|h_z\|_\xi^{**}), \qquad \|w\|_{\xi,\mathbf{v}}^* \le C\|h_w\|_{\xi,\mathbf{v}}^{**}.$$

Caflisch and Nicolaenko use weighted L^∞-norms for the Boltzmann equation [11], whereas for BGK-models, as we will see later, L^2-based norms turn out to be convenient. In view of (4.23), an L^2-approach seems natural. However, for the control of the nonlinearities regularity with respect to ξ is needed. Control of nonlinearities is straightforward in a L^∞-approach. Estimating the solution of (4.21) in terms of L^∞-norms on the other hand, requires much more sophistication than the derivation of L^2-estimates.

The approach for the existence proof of a solution of (4.21) is based on spectral theory in [11]. In [17] the proof relies on a discretisation of the velocity component.

The existence and uniqueness proof of solutions of the nonlinear problem (4.13), (4.20) and (4.24) is now a contraction argument. Therefore we need estimates for the right-hand sides of (4.13) and (4.20). Corresponding to the spaces of the solutions and inhomogenities of the linear problem we define the norms

$$\|(z,w)\|^* := \|z\|_\xi^* + \varepsilon\|w\|_{\xi,\mathbf{v}}^*, \qquad \|(h_z,h_w)\|^{**} := \|h_z\|_\xi^{**} + \varepsilon\|h_w\|_{\xi,\mathbf{v}}^{**}, \tag{4.25}$$

weighted according to the decomposition $g = \Phi z + \varepsilon w$. In the following we identify g with the pair (z,w), i.e., $\|g\|^* = \|(z,w)\|^*$.

The following assumption contains rigorous statements concerning the formal properties of the terms on the right hand sides of (4.13), (4.20):

Assumption 5 *(i) The linear terms appearing in the right hand sides of (4.13) and (4.20) can be bounded as follows:*

$$\frac{1}{\varepsilon}\|P\mathcal{L}_{as}\Phi z\|_{\xi,\mathbf{v}}^{**} \le C\|z\|_\xi^*, \qquad \|\Gamma w\|_\xi^{**} + \|\tilde{\Gamma} w\|_{\xi,\mathbf{v}}^{**} \le C\|w\|_{\xi,\mathbf{v}}^*. \tag{4.26}$$

(ii) The residual terms are uniformly bounded:

$$\|\Pi h\|_\xi^{**} + \|Ph\|_{\xi,\mathbf{v}}^{**} \le C. \tag{4.27}$$

(iii) The nonlinear term $R(g)$ is quadratic:

$$\begin{aligned} &\|\Pi R(g_1) - \Pi R(g_2)\|_\xi^{**} + \|PR(g_1) - PR(g_2)\|_{\xi,\mathbf{v}}^{**} \\ &\le C(\|g_1\|^* + \|g_2\|^*)\|g_1 - g_2\|^*, \qquad \textit{for } \|g_1\|^*,\ \|g_2\|^* \le \frac{C_0}{\varepsilon}. \end{aligned} \tag{4.28}$$

Before stating the existence and uniqueness result we note that in terms of the original unknown $f = f_{as} + \varepsilon^2 g$, the condition $z(0) = z_0$ reads

$$\langle (v-s)\psi, f - f_{as}\rangle_{\mathbf{v}}(\xi = 0) = -\varepsilon^2 z_0 \tilde{D}\,. \tag{4.29}$$

Theorem 4.3 *Let the Assumptions 1 – 5 hold. Then for every $z_0 \in \mathbb{R}$ and every small enough $\varepsilon > 0$ there exists a solution of (4.1), (4.29), which is unique in a ball $\{f : \|f - f_{as}\|^* \le \varepsilon\delta\}$ with δ independent of ε. It satisfies*

$$\|f - \mathcal{M}(U_{vsp})\|^* \le C\varepsilon^2\,,$$

or, more precisely,

$$f = \mathcal{M}(U_{vsp}) + \varepsilon f^\perp[U_{vsp}] + \varepsilon^2 \Phi z + \varepsilon^3 w,$$

where U_{vsp} is the solution of (3.11) and $\|z\|_\xi^*$ and $\|w\|_{\xi,\mathbf{v}}^*$ are uniformly bounded as $\varepsilon \to 0$.

Proof. It remains to prove the existence and uniqueness of the full nonlinear problem (4.13), (4.20), (4.24). As a consequence of assumption (4.26), the estimates from assumption 4 can be extended to the full linear problem

$$\begin{aligned} \tilde{D}\partial_\xi z - \Psi(\xi) z &= \varepsilon \Gamma w + h_z\,, \\ \epsilon(v-s)\partial_\xi w - \mathcal{K}w - z\frac{1}{\varepsilon} P\mathcal{L}_{as}\Phi &= \varepsilon\tilde{\Gamma} w + h_w\,, \end{aligned}$$

with given inhomogenities h_z, h_w and $z(0) = z_0$. In terms of the norms defined in (4.25) the estimate on the solution of the linear problem can be written as

$$\|(z,w)\|^* \le C(|z_0| + \|(h_z,h_w)\|^{**})\,.$$

Applying the solution operator for this system to (4.13), (4.20) implies a fixed point problem of the form

$$z = \varepsilon R_z(z,w) + \tilde{h}_z\,, \tag{4.30}$$

$$w = \varepsilon R_w(z,w) + \tilde{h}_w\,, \tag{4.31}$$

where R_z and R_w share the property in (4.28), and $\tilde{h}_z, \tilde{h}_w$ are the terms containing the residual, hence given and bounded due to (4.27). Using (4.28), the fix-point operator can be estimated by

$$\|(\varepsilon R_z(z,w) + \tilde{h}_z, \varepsilon R_w(z,w) + \tilde{h}_w)\|^* \le c(1 + \varepsilon(\|(z,w)\|^*)^2)\,,$$

for a constant $c > 0$.

This implies that for ε small enough both the ball with radius $2c$ and the ball with radius $\varepsilon^{-1}\min\{1/(2c), C_0\}$ are mapped into themselves by the right hand side of (4.30), (4.31). Due to the properties of the nonlinearity, the fix-point operator is a contraction on a ball with an $O(\varepsilon^{-1})$ radius. And we conclude that for ε small, (4.30), (4.31) has a solution with $\|(z,w)\|^* \le 2c$, which is unique in a ball with an $O(\varepsilon^{-1})$ radius. Knowing this and returning to (4.31), also the boundedness of $\|w\|_{\xi,\mathbf{v}}^*$ follows. $\square$

Lemma 4.4 *Let the assumptions of Theorem 4.3 hold and let the norm $\|\cdot\|^*_{\xi,\mathbf{v}}$ be such that $\|U_w\|_\infty \le C\|w\|^*_{\xi,\mathbf{v}}$. Then the macroscopic moments $U_{f,j}(\xi)$, $j = 1,\dots,n$, of the solution f of (4.1), (4.29) are strictly monotone. Due to the asymptotic expansion of the travelling wave solution f, $sgn(\partial_\xi U_{f,j}) = sgn(r_k(U_-)_j \partial_\xi y_{vsp})$ follows.*

Proof. We proceed as in [17]. One can easiliy extend the proof of Theorem 4.3 to show that the difference of two solutions (z,w) and $(\hat{z},\hat{w})$ is depending Lipschitz continuously on the initial data

$$\|z-\hat{z}\|^*_\xi \le C|z_0-\hat{z}_0|, \qquad \|w-\hat{w}\|^*_{\xi,\mathbf{v}} \le C|z_0-\hat{z}_0|\,.$$

For the corresponding solutions f and $\hat{f}$ of (4.1), (4.29) the relation

$$U_{f,k}(0) - U_{\hat{f},k}(0) = \varepsilon^2 U_{\Phi,k}(z_0-\hat{z}_0) + \varepsilon^3 (U_{w,k}(0) - U_{\hat{w},k}(0))$$

holds. The assumption $\|U_{w,k} - U_{\hat{w},k}\|_\infty \le C\|w-\hat{w}\|^*_{\xi,\mathbf{v}}$ now implies

$$|U_{w,k}(0) - U_{\hat{w},k}(0)| \le C|z_0-\hat{z}_0|\,.$$

Since $U_{\Phi,k} \ne 0$, the map $z_0 \mapsto U_{f,k}(0)$ is invertible for ε small, meaning that the travelling wave can also be made locally unique by prescribing the value of $U_{f,k}(0)$ instead of z_0. This argument can of course be repeated with $U_{f,k}(\xi_0)$ for every $\xi_0 \in \mathbb{R}$ instead of the origin.

Now assume $U_{f,k}(\xi)$ is not strictly monotone. Then there exist two ξ-values ξ_0 and $\xi_0+\delta$ with an arbitrarily small positive δ, such that $U_{f,k}(\xi_0) = U_{f,k}(\xi_0+\delta)$. Now also $\tilde{f}(\xi,v) = f(\xi+\delta,v)$ is a travelling wave with $U_{\tilde{f},k}(\xi_0) = U_{f,k}(\xi_0)$. By the uniqueness result we obtain $f \equiv \tilde{f}$. Consequently f must be periodic, which is a contradiction to the far-field conditions. □

4.3 Examples

For the BGK-models introduced in Section 2, it only remains to check Assumptions 4 and 5. The standard norms and spaces of functions of ξ we denote by $(L^2_\xi, \|.\|_\xi)$, $(H^m_\xi, \|.\|_{H^m_\xi})$, $(L^\infty_\xi, \|.\|_\infty)$ and recall the definition of the inner product in $\mathbf{v}$ in (3.3). Then the Hilbert space $L^2_{\xi,\mathbf{v}}$ is naturally defined by the scalar product

$$\langle f,g\rangle_{\xi,\mathbf{v}} \;=\; \int_{\mathbb{R}} \langle f,g\rangle_{\mathbf{v}}\, d\xi\,, \qquad \text{where } \; supp\ f, supp\ g \subset V\,,$$

with the induced norm $\|\cdot\|_{\xi,\mathbf{v}}$. Similarly the spaces $H^k_\xi(L^2_\mathbf{v})$ of functions, whose derivatives in ξ up to order k are in $L^2_\mathbf{v}$, are defined by

$$\|f\|_{H^k_\xi(L^2_\mathbf{v})} \;=\; \left(\|f\|^2_{\xi,\mathbf{v}} + \ldots + \|\partial^k_\xi f\|^2_{\xi,\mathbf{v}}\right)^{\frac{1}{2}}\,.$$

In terms of these norms we have to make some assumptions on the Maxwellians and kinetic entropies. We require that for a fixed $\mathbf{v} \in V$ the equilibrium distribution $\mathcal{M}(U,\mathbf{v})$ with support in V is five times continuously differentiable in U and

$$\int |\phi(\mathbf{v})^\alpha\, \nabla^\beta_U \mathcal{M}(U,\mathbf{v})|d\mu < C, \quad \int |\phi(\mathbf{v})^\alpha| \left(\nabla^\beta_U \mathcal{M}(U,\mathbf{v})\right)^2 \partial^2_f H(\hat{\mathcal{M}}) d\mu < C, \quad (4.32)$$

where α and β are multiindices with $|\alpha| \leq 4$ and $|\beta| \leq 5$ and U is $O(\varepsilon)$-close to $\hat{\mathcal{M}}$. Moreover we assume

$$\left| \int \phi(\mathbf{v})^\alpha f d\mu \right| \leq C\|f\|_{\mathbf{v}} , \qquad \text{for } |\alpha| \leq 3 , \tag{4.33}$$

implying

$$\|U_{f,k}\|_{H^m_\xi} \leq C\|f\|_{H^m_\xi(L^2_{\mathbf{v}})}, \qquad k = 1, \dots, n . \tag{4.34}$$

We are now prepared to investigate the existence of a solution to the linear problem corresponding to (4.21), (4.22). We shall mention that for the existence proof H^1_ξ-based norms are sufficient. In this case we would only need the moment conditions in (4.32) up to $|\beta| = 3$. But since showing the asymptotic stability requires L^∞-bounds on the macroscopic profiles of the travelling wave and also on their derivatives up to second order, we shall rather look for solutions in the spaces H^3_ξ, respectively $H^3_\xi(L^2_{\mathbf{v}})$:

$$\varepsilon(v - s)\partial_\xi w - \mathcal{K}w = h_w, \qquad \text{with } \ h_w \in H^3_\xi(L^2_{\mathbf{v}}) , \tag{4.35}$$

$$\tilde{D}\partial_\xi z - \Psi(\xi)z = h_z, \qquad \text{with } \ h_z \in H^2_\xi . \tag{4.36}$$

As we have already indicated before, there exist constants γ , $\bar{\xi} > 0$ such that

$$\Psi(\xi) \leq -\gamma \quad \text{for } \xi \geq \bar{\xi} , \quad \Psi(\xi) \geq \gamma \quad \text{for } \xi \leq \bar{\xi} . \tag{4.37}$$

Using this property of Ψ it was shown in [17] that the solution z of (4.36) with $z(0) = z_0$ satisfies the estimate

$$\|z\|_{H^3_\xi} \leq C(|z_0| + \|h_z\|_{H^2_\xi}) .$$

Additionally, based on a discretisation of the velocity component v, it was proven in [17] that there exists a unique solution $w \in H^3_\xi(L^2_{\mathbf{v}})$ of (4.21) satisfying

$$\|\partial^k_\xi w\|_{\xi,\mathbf{v}} \leq \frac{1}{\kappa}\|\partial^k_\xi h_w\|_{\xi,\mathbf{v}} , \qquad \text{for } k = 0, \dots, 3 .$$

Here the positive constant κ is the one from the coercivity estimate $-\langle \mathcal{K}w, w\rangle_{\mathbf{v}} \geq \kappa\|w\|^2_{\mathbf{v}}$. This coercivity estimate in particular holds for the negative definite operators $\mathcal{K}$ appearing in the examples under consideration.

If we can also verify the bounds on the linear and nonlinear terms in assumption 5 the contraction argument can be carried out. Using the boundedness of $\Pi : L^2_{\mathbf{v}} \to \mathbb{R}$ and $P : L^2_{\mathbf{v}} \to L^2_{\mathbf{v}}$, the moment conditions (4.32), (4.33) and the smoothness of U_{vsp}, we obtain the desired bounds on the linear terms

$$\frac{1}{\varepsilon}\|P\mathcal{L}_{as}\Phi z\|_{H^3_\xi(L^2_{\mathbf{v}})} \leq C\|z\|_{H^3_\xi}, \qquad \|\Gamma w\|_{H^3_\xi} + \|\tilde{\Gamma} w\|_{H^3_\xi(L^2_{\mathbf{v}})} \leq C\|w\|_{H^3_\xi(L^2_{\mathbf{v}})} . \tag{4.38}$$

We observe that for the particular examples under consideration the behaviour of U_{vsp} is exponential as $\xi \to \pm\infty$. This allows us to integrate the derivatives in ξ, and enables us to deduce the boundedness of the residual-terms

$$\|Ph\|_{H^3_\xi(L^2_{\mathbf{v}})} + \|\Pi h\|_{H^3_\xi} \leq C . \tag{4.39}$$

Here we have again additionally used the smoothness of U_{vsp}, the boundedness of P and Π and the moment conditions (4.32), (4.33).

Now it only remains to control the nonlinear term

$$\begin{aligned} R(g) &= \frac{1}{\varepsilon^4}[\mathcal{M}(U_{vsp}+\varepsilon^2 U_g)-\mathcal{M}(U_{vsp})-\varepsilon^2\nabla\mathcal{M}(U_{vsp})\cdot U_g] \\ &= U_g\mathcal{M}''(U_{vsp}+\varepsilon^2\vartheta U_g)U_g\,, \end{aligned}$$

for a $\vartheta\in(0,1)$. By differentiation, the moment conditions in (4.32) and the one-dimensional Sobolev imbedding, the estimate

$$\|R(g_1)-R(g_2)\|_{H^k_\xi(L^2_{\mathbf{v}})} \le C(\|g_1\|_{H^k_\xi(L^2_{\mathbf{v}})}+\|g_2\|_{H^k_\xi(L^2_{\mathbf{v}})})\|g_1-g_2\|_{H^k_\xi(L^2_{\mathbf{v}})} \tag{4.40}$$

can be deduced to hold for all g_1,g_2 with $\|g_1\|_{H^k_\xi(L^2_{\mathbf{v}})},\|g_2\|_{H^k_\xi(L^2_{\mathbf{v}})}\le C\varepsilon^{-2}$ in general. If V is compact, the ball of admissible functions g_1,g_2 has to be reduced to a ball with a radius $C_0\varepsilon^{-1}$. Due to the construction of V this guarantees that the supports of the Maxwellians resulting from Taylor expansions stay in V.

We shall give the norm of g according to (4.25) explicitly:

$$\|g\|^* = \|(z,w)\|^* = \|z\|_{H^3_\xi}+\varepsilon\|w\|_{H^3_\xi(L^2_{\mathbf{v}})}\,. \tag{4.41}$$

Hence obviously $\|g\|_{H^3_\xi(L^2_{\mathbf{v}})}\le C\|g\|^*$, and the existence and uniqueness result is an immediate consequence from Theorem 4.3.

For the oncoming examples it now only remains to give a concrete setting for the Maxwellians and the kinetic entropies, such that (4.32)-(4.33) hold.

BGK-models for scalar conservation laws

We have already mentioned that the monotonicity condition on the Maxwellian, $\partial_U\mathcal{M}>0$, provides a kinetic entropy. Considering $V=\mathbb{R}$, we linearize around the left states and hence the inner product in v can be written as

$$\langle f,g\rangle_v = \int\frac{fg}{\partial_U\mathcal{M}_-}d\mu\,.$$

As long as the Maxwellians satisfy the conditions corresponding to (4.32) and (4.33), the existence result is an immediate consequence.

The BGK-model for the isothermal system and the gas dynamics

In both cases we have smooth Maxwellians with $V=\mathbb{R}$. The conditions (4.32) and (4.33) can be checked by direct calculations.

The BGK-model for the isentropic system

In this example the Maxwellians under consideration have a compact support. Hence we have to construct a Maxwellian $\hat{\mathcal{M}} := \mathcal{M}(\hat\rho, \hat u)$ with a bigger support than all other functions appearing in our calculations. For simplicity we denote in the following $\gamma = 1 + 2\alpha$. Then the support of $\mathcal{M}(\rho_f, u_f)$ is bounded by

$$u_f - \frac{c_f}{\sqrt{\alpha}} \le v \le u_f + \frac{c_f}{\sqrt{\alpha}}\,. \tag{4.42}$$

As we have already seen, the macroscopic profiles of the travelling wave will be monotone, i.e.

$$\partial_\xi \rho_f > 0, \quad \partial_\xi u_f = \frac{s\rho_f - \rho_f u_f}{\rho_f^2}\partial_\xi\rho_f = \frac{s(\rho_- - \rho_- u_-)}{\rho_f^2}\partial_\xi\rho_f = -(c_- - \varepsilon\sigma)\frac{\rho_-}{\rho_f^2}\partial_\xi\rho_f < 0\,.$$

Now one can see that the left hand side of (4.42) is strictly decreasing. An expansion shows that also the right hand side of (4.42) is decreasing, and hence neither $\mathcal{M}_-$ nor $\mathcal{M}_+$ provide a large enough support. We choose

$$\hat u = u_-, \qquad \hat c = c_+(1 + \varepsilon/\rho_+)\,, \tag{4.43}$$

defining $\hat\rho$ and $\hat u$ uniquely. Then for ε small $\hat{\mathcal{M}}$ has the desired properties, i.e. the support of $\hat{\mathcal{M}}$ includes the supports of all $\mathcal{M}(\rho_f, u_f)$ plus an additional range of order ε. And thus we linearize from now on around the Maxwellian $\hat{\mathcal{M}}$ with the support $V := \left[\hat u - \frac{\hat c}{\sqrt{\alpha}}, \hat u + \frac{\hat c}{\sqrt{\alpha}}\right]$. The inner product (3.3) reads

$$\langle f, g\rangle_v := \frac{1}{2\beta d^{\frac{1}{\beta}}} \int f\, g\, \hat{\mathcal{M}}^{\frac{1}{\beta}-1} dv\,, \qquad \text{for } \ supp\, f, supp\, g \subset V\,.$$

Now it only remains to check (4.32), i.e. for $\mathcal{M}(\rho, u)$ with $supp\,\mathcal{M}(\rho,u) \subset V$:

$$\sup_\xi \left| \int \left(\partial_\rho^j \partial_{\rho u}^k \mathcal{M}(\rho, u)\right)^2 \hat{\mathcal{M}}^{\frac{1}{\beta}-1} dv \right| < C\,, \tag{4.44}$$

for $\ j + k = 0, \ldots, 5$. In order to guarantee that this holds, we have to make a technical assumption and restrict in the following α to the values

$$0 < \alpha < \frac{1}{17}\,, \quad \text{or equivalently} \quad 1 < \gamma < 1 + \frac{2}{17}\,.$$

It is sufficient to show the uniform boundedness of

$$\int_{supp\,\mathcal{M}(\rho,u)} \left(\frac{c^2}{\alpha} - (v-u)^2\right)^{2(\beta-n)} \left(\frac{\hat c^2}{\alpha} - (v - \hat u)^2\right)^{1-\beta} dv, \quad \text{for } n = 0, \ldots, 5\,. \tag{4.45}$$

The assumption $supp\,\mathcal{M}(\rho, u) \subset V$ implies

$$\left(\frac{c}{\sqrt{\alpha}} + u - v\right)\left(\frac{c}{\sqrt{\alpha}} - u + v\right) \le \left(\frac{\hat c}{\sqrt{\alpha}} + \hat u - v\right)\left(\frac{\hat c}{\sqrt{\alpha}} - \hat u + v\right)\,,$$

for all $v \in supp\,\mathcal{M}(\rho,u)$ and $\xi \in \mathbb{R}$, and hence, assuming for the moment $\beta > 1$, the integral in (4.45) is bounded by

$$\int \left(\frac{c^2}{\alpha} - (v-u)^2\right)_+^{\beta+1-2n} dv\,.$$

A transformation of variable leads to the Beta-function and hence (4.32) is valid only if $\beta + 1 - 2n > -1$, i.e. $\beta > 8$ or equivalently $0 < \alpha < 1/17$.

5 Stability of kinetic shock profiles for weak shocks

5.1 Stability of viscous shock profiles

Goodman [21] shows the asymptotic stability of viscous shock profiles for hyperbolic conservation laws with a positive definite visosity. Kawashima and Matsumura investigated the asymptotic stability of traveling wave solutions of some systems for one-dimensional gas motion [25]. In particular a decay rate for the scalar conservation law and a stability proof for the Navier-Stokes equations in Lagrangian coordinates are given. The stability for the isentropic gas dynamics in Lagrangian coordinates was derived by Matsumura and Nishihara in [31].

We consider a viscous regularization of the conservation law in terms of travelling wave coordinates and of a parabolic time scale:

$$\varepsilon\partial_t U + (J'(U) - s)\partial_\xi U = \varepsilon\hat{D}\partial_\xi^2 U\,,$$

where, for simplicity, the diffusivity matrix is considered constant. A viscous profile U_{vsp} satisfies the stationary version

$$(J'(U_{vsp}) - s)\partial_\xi U_{vsp} = \varepsilon\hat{D}\partial_\xi^2 U_{vsp}\,.$$

We introduce the perturbation by $\varepsilon U_G(t,\xi) := U(t,\xi) - U_{vsp}(\xi)$ and assume the 'well-preparedness' condition

$$\int_{-\infty}^{\infty} U_G(0,\xi)d\xi = 0\,, \tag{5.1}$$

for the initial data. This should fix the shift of the asymptotic travelling wave such that we expect convergence of U_G to zero. The equation for U_G can be written as

$$\partial_t U_G + \frac{1}{\varepsilon}\partial_\xi[(J'(U_{vsp}) - s)U_G] + \partial_\xi r(U_G) = \hat{D}\partial_\xi^2 U_G\,, \tag{5.2}$$

with the nonlinearity $r(U) = [J(U_{vsp} + \varepsilon U) - J(U_{vsp}) - \varepsilon J'(U_{vsp})U]/\varepsilon^2$. One of the basic assumptions of the analysis we present here, will be the existence of a symmetric, positive definite, U-dependent matrix $\Lambda(U)$, such that $\Lambda(U)J'(U)$ is symmetric and such that $\Lambda(U_{vsp})\hat{D} \geq \kappa > 0$ is positive definite. A possible candidate is the Hessian $\nabla^2\eta(U)$ of the entropy density, which satisfies the symmetrization property, and the matrix $\nabla^2\eta(U)D(U)$ with the Chapman-Enskog diffusivity is always symmetric and positive semidefinite (compare to [29]). Positive definiteness cannot be expected in general, as the examples in Section 3.3 show. For the case of non-definiteness, the details of the stability estimates will depend on the structure of $D(U)$. An example is carried out below.

Positive definiteness of $\nabla^2\eta(U_{vsp})D(U_{vsp})$ is of course preserved, when $D(U_{vsp})$ is replaced by a constant approximation, say $\hat{D} = D(\hat{U})$.

Assumption 6 *For every $U \in \mathbb{R}^n$ there exists a symmetric positive definite matrix $\Lambda(U)$, smoothly depending on U, such that $\Lambda(U)J'(U)$ is symmetric and $\Lambda(U_{vsp})\hat{D} \geq \kappa > 0$.*

Taking the scalar product in L^2_ξ of (5.2) with $\Lambda(U_{vsp})U_G$ gives

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|U_G\|^2_\Lambda - \frac{1}{2\varepsilon}\langle U_G, [\partial_\xi\Lambda(J'-s) - \Lambda\partial_\xi J']U_G\rangle_\xi - \langle\partial_\xi(\Lambda U_G), r(U_G)\rangle_\xi \\
&= -\langle\partial_\xi(\Lambda U_G), \hat{D}\partial_\xi U_G\rangle_\xi \,, \qquad (5.3)
\end{aligned}$$

where we used the weighted L^2-norm $\|U\|^2_\Lambda := \langle\Lambda U, U\rangle_\xi$. It is well known that stability cannot be proven based only on this equation. The main reason is that the bracket in the second term has the unfavourable definiteness in general. An example is the scalar case, where $\Lambda = 1$ and J' is a decreasing function of ξ along a shock profile.

We shall still extract some information from an estimate based on (5.3). Using the fact that $\partial_\xi U_{vsp} = O(\varepsilon)$, and that r is quadratic in the sense that $|r(U)| \leq C(|U|)U^2$ (with an increasing function C), standard estimation leads to

$$\frac{d}{dt}\|U_G\|^2_\Lambda + \kappa\|\partial_\xi U_G\|^2_\xi \leq C(\|U_G\|_\infty)\|U_G\|^2_\xi \,. \qquad (5.4)$$

It is by now a standard method to introduce the primitive $W(t,\xi) = \int_{-\infty}^{\xi} U_G(t,\xi')d\xi'$. The assumption (5.1) on the initial data and the conservation property imply the far field conditions

$$W(t,\pm\infty) = 0\,. \qquad (5.5)$$

Integration of (5.2) gives

$$\partial_t W + \frac{1}{\varepsilon}(J'(U_{vsp}) - s)\partial_\xi W + r(U_G) = \hat{D}\partial^2_\xi W\,. \qquad (5.6)$$

As above, we test with $\Lambda(U_{vsp})W$:

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|W\|^2_\Lambda - \frac{1}{2\varepsilon}\langle W, \partial_\xi[\Lambda(J'-s)]W\rangle_\xi + \langle\Lambda W, r(U_G)\rangle_\xi \\
&= -\langle\partial_\xi W, \Lambda\hat{D}\partial_\xi W\rangle_\xi - \langle W, \partial_\xi\Lambda\hat{D}\partial_\xi W\rangle_\xi \,. \qquad (5.7)
\end{aligned}$$

Now it is reasonable to assume that the second term has the favourable sign. The last term we estimate as

$$\left|\langle W, \partial_\xi\Lambda\hat{D}\partial_\xi W\rangle_\xi\right| \leq \frac{\kappa}{2}\|\partial_\xi W\|^2_\xi + c\|\partial_\xi\Lambda W\|^2_\xi \,.$$

A somewhat stronger version of the above assumption is that

$$-\frac{1}{2\varepsilon}\langle W, \partial_\xi[\Lambda(J'-s)]W\rangle_\xi - c\|\partial_\xi \Lambda W\|_\xi^2 \geq 0\,.$$

With the properties of the nonlinearity we obtain

$$\frac{d}{dt}\|W\|_\Lambda^2 + \Big(\kappa - C(\|U_G\|_\infty)\|W\|_\infty\Big)\|U_G\|_\xi^2 \leq 0\,. \tag{5.8}$$

For our estimates (5.4) and (5.8) to be useful we need pointwise-in-time control of $\|U_G\|_\infty$. This will be provided by an L^2-estimate on $V := \partial_\xi U_G$ and Sobolev imbedding. The derivative of (5.2) with respect to ξ can be written as

$$\partial_t V + \frac{1}{\varepsilon}\partial_\xi\Big((J'(U_{vsp}) - s)V + \partial_\xi J'\, U_G\Big) + \partial_\xi^2 r(U_G) = \hat{D}\partial_\xi^2 V\,, \tag{5.9}$$

We treat this equation similarly to (5.2) and (5.6), but omit the details. The result is the estimate

$$\frac{d}{dt}\|\partial_\xi U_G\|_\Lambda^2 + \kappa\|\partial_\xi^2 U_G\|_\xi^2 \leq C(\|U_G\|_\infty)(\|U_G\|_\xi^2 + \|\partial_\xi U_G\|_\xi^2)\,. \tag{5.10}$$

The stability proof is completed by a combination of (5.4), (5.8), and (5.10). For positive constants γ_1, γ_2, we define

$$I(t) := \|W\|_\Lambda^2 + \gamma_1\|U_G\|_\Lambda^2 + \gamma_2\|\partial_\xi U_G\|_\Lambda^2\,.$$

Then, by Sobolev imbedding,

$$\|W\|_\infty + \|U_G\|_\infty \leq cI\,.$$

With $M := cI(0)$, we assume that M is small enough, so γ_1 and γ_2 can be chosen such that

$$\kappa > C(M)(M + \gamma_1 + \gamma_2)\,, \qquad \kappa\gamma_1 > C(M)\gamma_2\,.$$

Then there is a positive constant λ such that

$$\frac{dI}{dt} \leq -\lambda\|U_G\|_{H_\xi^2}^2\,.$$

Thus, I is a Lyapunov functional. By integration with respect to time, U_G converges to zero as $t \to \infty$ in the sense that

$$\int_0^\infty \|U_G\|_{H_\xi^2}^2\, dt < \infty\,.$$

5.2 A Lyapunov functional for BGK-models

Now the ideas of the preceding section will be carried over to kinetic shock profiles. Here the L^2-energy methods for the macroscopic system will be extended to also control the microscopic part. Similar techniques have been used by Liu and Yu for the Boltzmann equation [28].

We start with the kinetic equation, written in travelling wave variables and a macroscopic diffusion scaling:

$$\varepsilon^2 \partial_t f + \varepsilon(v-s)\partial_\xi f = \mathcal{M}(U_f) - f\,.$$

Let φ denote a kinetic shock profile:

$$\varepsilon(v-s)\partial_\xi \varphi = \mathcal{M}(U_\varphi) - \varphi\,.$$

The perturbation $\varepsilon G = f - \varphi$ satisfies

$$\varepsilon^2 \partial_t G + \varepsilon(v-s)\partial_\xi G = \frac{1}{\varepsilon}[\mathcal{M}(U_\varphi + \varepsilon U_G) - \mathcal{M}(U_\varphi)] - G\,. \tag{5.11}$$

The micro-macro decomposition of the perturbation is defined by $G = U_G \cdot \nabla_U \hat{\mathcal{M}} + \varepsilon g$, where $\hat{U}$ in $\hat{\mathcal{M}} = \mathcal{M}(\hat{U})$ is a constant approximation of U_φ, and $\varepsilon g = -\mathcal{L}G$ is the microscopic projection with the linearization $\mathcal{L}$ of the collision operator around $\hat{\mathcal{M}}$. Computing the macroscopic moments of (5.11) gives

$$\partial_t U_G + \frac{1}{\varepsilon}(J'(\hat{U}) - s)\partial_\xi U_G + \partial_\xi J_g = 0\,. \tag{5.12}$$

Like in the previous section, we obtain an equation for $W(t,\xi) = \int_{-\infty}^{\xi} U_G(t,\xi')d\xi'$ by integration:

$$\partial_t W + \frac{1}{\varepsilon}(J'(\hat{U}) - s)\partial_\xi W + J_g = 0\,. \tag{5.13}$$

An equation for the microscopic part is derived by applying the microscopic projection to (5.11):

$$\begin{aligned} &\varepsilon^2 \partial_t g - \varepsilon \mathcal{L}((v-s)\partial_\xi g) + \nabla_U \hat{\mathcal{M}} \cdot (v - J'(\hat{U}))\partial_\xi U_G \\ &= -g + \frac{1}{\varepsilon} U_G \cdot [\nabla_U \mathcal{M}(U_\varphi) - \nabla_U \hat{\mathcal{M}}] + R(U_G)\,, \end{aligned} \tag{5.14}$$

with

$$R(U) = \frac{1}{\varepsilon^2}[\mathcal{M}(U_\varphi + \varepsilon U) - \mathcal{M}(U_\varphi) - \varepsilon U \cdot \nabla_U \mathcal{M}(U_\varphi)]\,.$$

The next step is to compute the last term in (5.13) in the spirit of the Chapman-Enskog approximation by computing g from (5.14):

$$\partial_t W + \frac{1}{\varepsilon}(J'(U_\varphi) - s)\partial_\xi W - \hat{D}\partial_\xi^2 W = \varepsilon^2 \partial_t J_g - \varepsilon \partial_\xi J_{\mathcal{L}((v-s)g)} - r(U_G)\,, \tag{5.15}$$

with $\hat{D} = D(\hat{U})$ and

$$r(U) = J_{R(U)} = \frac{1}{\varepsilon^2}[J(U_\varphi + \varepsilon U) - J(U_\varphi) - \varepsilon J'(U_\varphi)U]\,.$$

In the same way we derived (5.8) in the previous section, we obtain

$$\frac{d}{dt}\|W\|_\Lambda^2 + [\kappa - C(\|U_G\|_\infty)\|W\|_\infty]\,\|U_G\|_\xi^2 \le \varepsilon^2\langle \Lambda W, \partial_t J_g\rangle_\xi - \varepsilon\langle \Lambda W, \partial_\xi J_{\mathcal{L}((v-s)g)}\rangle_\xi\,.$$

The first term on the right hand side we rewrite using (5.13):

$$\langle \Lambda W, \partial_t J_g\rangle_\xi = \frac{d}{dt}\langle \Lambda W, J_g\rangle_\xi + \frac{1}{\varepsilon}\langle \Lambda(J'(\hat{U}) - s)U_G, J_g\rangle_\xi + \|J_g\|_\Lambda^2\,,$$

leading to

$$\begin{aligned}&\frac{d}{dt}\left(\|W\|_\Lambda^2 - \varepsilon^2\langle \Lambda W, J_g\rangle_\xi\right) + [\kappa - C(\|U_G\|_\infty)\|W\|_\infty]\,\|U_G\|_\xi^2\\ &\le \varepsilon\langle \Lambda(J'(\hat{U}) - s)U_G, J_g\rangle_\xi + \varepsilon^2\|J_g\|_\Lambda^2 + \varepsilon\langle \Lambda U_G, J_{\mathcal{L}((v-s)g)}\rangle_\xi\,,\end{aligned} \tag{5.16}$$

where an integration by parts has been carried out in the last term. An estimate for the microscopic part of the perturbation is derived by taking the $L^2_{\xi,\mathbf{v}}$-scalar product of the full perturbation equation (5.11) with G:

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left[\|U_G\|_{\hat{\Lambda}}^2 + \varepsilon^2\|g\|_{\xi,\mathbf{v}}^2\right] + \|g\|_{\xi,\mathbf{v}}^2\\ &= \left\langle U_G \cdot \frac{\nabla_U\mathcal{M}(U_\varphi) - \nabla_U\hat{\mathcal{M}}}{\varepsilon}\,, g\right\rangle_{\xi,\mathbf{v}} + \langle R(U_G), g\rangle_{\xi,\mathbf{v}}\,,\end{aligned} \tag{5.17}$$

with $\hat{\Lambda} = \nabla^2\eta(\hat{U})$. Now we assume that the factors in the scalar products on the right hand sides of (5.16) and (5.17) are bounded linear maps of U_G and of g with the exception of the quadratic term $R(U_G)$:

$$\begin{aligned}&\frac{d}{dt}\left[\|W\|_\Lambda^2 - \varepsilon^2\langle \Lambda W, J_g\rangle_\xi\right] + \left(\frac{\kappa}{2} - C(\|U_G\|_\infty)\|W\|_\infty\right)\|U_G\|_\xi^2 \le \varepsilon^2 c\|g\|_{\xi,\mathbf{v}}^2\,,\\ &\frac{d}{dt}\left[\|U_G\|_{\hat{\Lambda}}^2 + \varepsilon^2\|g\|_{\xi,\mathbf{v}}^2\right] + \|g\|_{\xi,\mathbf{v}}^2 \le C(\|U_G\|_\infty)\|U_G\|_\xi^2\,.\end{aligned}$$

Adding these inequalities after multiplying the second by a positive constant δ gives

$$\begin{aligned}&\frac{d}{dt}\left[\|W\|_\Lambda^2 - \varepsilon^2\langle \Lambda W, J_g\rangle_\xi + \varepsilon^2\delta\|g\|_{\xi,\mathbf{v}}^2 + \delta\|U_G\|_{\hat{\Lambda}}^2\right]\\ &+ \left(\frac{\kappa}{2} - (\delta + \|W\|_\infty)C(\|U_G\|_\infty)\right)\|U_G\|_\xi^2 + (\delta - \varepsilon^2 c)\|g\|_{\xi,\mathbf{v}}^2 \le 0\,.\end{aligned}$$

For fixed δ and ε small enough, the term under the time derivative can be bounded from below by

$$c\left[\|W\|_\xi^2 + \|U_G\|_\xi^2 + \varepsilon^2\|g\|_{\xi,\mathbf{v}}^2\right],$$

with a positive constant c. So it controls $\|W\|_\infty$, but not $\|U_G\|_\infty$.

By taking the derivatives of (5.11) and (5.15) with respect to ξ, we obtain equations for G and for $H := \partial_\xi G = \partial_\xi U_G \cdot \nabla_U \hat{\mathcal{M}} + \varepsilon h$:

$$\partial_t U_G + \frac{1}{\varepsilon}\partial_\xi\Big((J'(U_\varphi) - s)U_G\Big) - \hat{D}\partial_\xi^2 U_G = \varepsilon^2 \partial_t J_h - \varepsilon \partial_\xi J_{\mathcal{L}((v-s)h)} - \partial_\xi r(U_G)\,,$$
$$\varepsilon^2 \partial_t H + \varepsilon(v-s)\partial_\xi H = \frac{1}{\varepsilon}\partial_\xi\Big(\mathcal{M}(U_\varphi + \varepsilon U_G) - \mathcal{M}(U_\varphi)\Big) - H\,.$$

Treating the first equation like in the previous section and the second like (5.11), we obtain

$$\frac{d}{dt}\Big[\|U_G\|_\Lambda^2 - \varepsilon^2\langle \Lambda U_G, J_h\rangle_\xi\Big] + \kappa\|\partial_\xi U_G\|_\xi^2 \le C(\|U_G\|_\infty)\|U_G\|_\xi^2 + \varepsilon^2 c\|h\|_{\xi,\mathbf{v}}^2\,,$$
$$\frac{d}{dt}\Big[\|\partial_\xi U_G\|_{\hat\Lambda}^2 + \varepsilon^2\|h\|_{\xi,\mathbf{v}}^2\Big] + \|h\|_{\xi,\mathbf{v}}^2 \le C(\|U_G\|_\infty)(\|U_G\|_\xi^2 + \|\partial_\xi U_G\|_\xi^2)\,.$$

Now we take a linear combination of these inequalities like above:

$$\frac{d}{dt}\Big[\|U_G\|_\Lambda^2 - \varepsilon^2\langle \Lambda U_G, J_{\partial_\xi g}\rangle_\xi + \varepsilon^2\delta\|\partial_\xi g\|_{\xi,\mathbf{v}}^2 + \delta\|\partial_\xi U_G\|_{\hat\Lambda}^2\Big]$$
$$+ (\kappa - \delta C(\|U_G\|_\infty))\,\|\partial_\xi U_G\|_\xi^2 + (\delta - \varepsilon^2 c)\|\partial_\xi g\|_{\xi,\mathbf{v}}^2 \le C(\|U_G\|_\infty)\|U_G\|_\xi^2\,.$$

Again, the term under the time derivative is positive definite. Finally, with $\gamma > 0$ we define the Lyapunov functional by

$$\begin{aligned} I(t) \;:=\; & \|W\|_\Lambda^2 - \varepsilon^2\langle \Lambda W, J_g\rangle_\xi + \varepsilon^2\delta\|g\|_{\xi,\mathbf{v}}^2 + \delta\|U_G\|_{\hat\Lambda}^2 \\ & + \gamma\Big[\|U_G\|_\Lambda^2 - \varepsilon^2\langle \Lambda U_G, J_{\partial_\xi g}\rangle_\xi + \varepsilon^2\delta\|\partial_\xi g\|_{\xi,\mathbf{v}}^2 + \delta\|\partial_\xi U_G\|_{\hat\Lambda}^2\Big]\,, \end{aligned}$$

and obtain

$$\frac{dI}{dt} + \Big(\frac{\kappa}{2} - (\delta + \gamma + \|W\|_\infty)C(\|U_G\|_\infty)\Big)\,\|U_G\|_\xi^2 + (\delta - \varepsilon^2 c)\|g\|_{\xi,\mathbf{v}}^2$$
$$+\gamma\Big(\kappa - \delta C(\|U_G\|_\infty)\Big)\|\partial_\xi U_G\|_\xi^2 + \gamma(\delta - \varepsilon^2 c)\|\partial_\xi g\|_{\xi,\mathbf{v}}^2 \le 0\,.$$

The functional I controls $\|W\|_\xi^2 + \|U_G\|_\xi^2 + \|\partial_\xi U_G\|_\xi^2 + \varepsilon^2\|g\|_{\xi,\mathbf{v}}^2 + \varepsilon^2\|\partial_\xi g\|_{\xi,\mathbf{v}}^2$. So, by Sobolev imbedding,

$$\|W\|_\infty + \|U_G\|_\infty \le cI$$

holds. With $M := cI(0)$, I is indeed a Lyapunov functional, if

$$\frac{\kappa}{2} > (\delta + \gamma + M)C(M) \quad \text{and} \quad \delta > \varepsilon^2 c\,.$$

This can of course be achieved by choosing δ, γ, M, and ε small enough.

5.3 Stability of weak kinetic profiles for the isentropic gas dynamics BGK-model

The model of Section 2.2 satisfies the assumptions used in the previous section except the regularity of the Chapman-Enskog diffusivity. Therefore the main steps of the analysis will be recalled from [15].

To derive estimates for the macroscopic part we adapt ideas from [31], where the stability of travelling waves for the isentropic system for a compressible viscous gas in Lagrangian coordinates is proven by L^2-energy estimates. Control of the microscopic terms will be obtained like in the previous section.

As in the previous section, we start with the kinetic equation in diffusion scaling:

$$\varepsilon^2 \partial_t f + \varepsilon(v-s)\partial_\xi f = \mathcal{M}(\rho_f, m_f) - f\,, \tag{5.18}$$

with the far-field conditions $f(t, \xi = \pm\infty, v) = \mathcal{M}(\rho_\pm, m_\pm, v)$. As in Section 2.2 we shall switch between the momentum density and the mean velocity, connected by $m_f = \rho_f u_f$, as second macroscopic variable. Let φ be the travelling wave solution. The well-preparedness condition for the initial data now reads

$$\int_{\mathbb{R}} (\rho_{f_0} - \rho_\varphi) d\xi = 0, \qquad \int_{\mathbb{R}} (m_{f_0} - m_\varphi) d\xi = 0. \tag{5.19}$$

Introducing the perturbation G

$$\varepsilon G = f - \varphi, \quad \rho := \rho_G, \quad m := m_G\,,$$

we obtain

$$\varepsilon^2 \partial_t G + \varepsilon(v-s)\partial_\xi G = \frac{1}{\varepsilon}\left[\mathcal{M}(\rho_\varphi + \varepsilon\rho, m_\varphi + \varepsilon m) - \mathcal{M}(\rho_\varphi, m_\varphi)\right] - G\,. \tag{5.20}$$

As in [15] we apply a micro-macro decomposition to the deviation G

$$G = \nabla_U \mathcal{M}(\hat\rho, \hat m) \cdot \begin{pmatrix} \rho \\ m \end{pmatrix} + \varepsilon g, \tag{5.21}$$

where as before $U = (\rho, m)$. Observe that $-\mathcal{L}G = \varepsilon g$. Then the norm of G satisfies

$$\|G\|^2_{\xi,v} = \frac{1}{\hat\rho}\left[\hat c^2 \|\rho\|^2_\xi + \|(m - \rho\hat u))\|^2_\xi\right] + \varepsilon^2 \|g\|^2_{\xi,v}\,. \tag{5.22}$$

Macroscopic equations for ρ and m are obtained by computing the zeroth and first order moments of equation (5.20)

$$\varepsilon\partial_t\rho + \partial_\xi(m - \rho s) = 0\,, \tag{5.23}$$

$$\varepsilon\partial_t m + \partial_\xi\left(\nabla_U j(\hat\rho, \hat m) \cdot \begin{pmatrix} \rho \\ m \end{pmatrix} - sm\right) + \varepsilon\partial_\xi \int v^2 g\, dv = 0\,. \tag{5.24}$$

Next we apply $-\mathcal{L}$ to (5.20) to get an equation for g

$$\varepsilon^2\partial_t g-\partial_\xi\left(\nabla_U\hat{\mathcal{M}}\cdot\left[(J'(\hat\rho,\hat m)-v)\cdot\begin{pmatrix}\rho\\ m\end{pmatrix}\right]\right)-\varepsilon\partial_\xi\mathcal{L}((v-s)g)=R(\rho,m)-g\,,\tag{5.25}$$

with the nonlinearity

$$R(\rho,m)=\frac{1}{\varepsilon^2}\left[\mathcal{M}\left(\rho_\varphi+\varepsilon\rho,m_\varphi+\varepsilon m\right)-\mathcal{M}(\rho_\varphi,m_\varphi)-\varepsilon\nabla_U\hat{\mathcal{M}}\cdot\begin{pmatrix}\rho\\ m\end{pmatrix}\right].$$

Using equation (5.25) we calculate

$$\int_{\mathbb{R}} v^2 g\,dv \;=\; q(\rho,m)-\varepsilon S(g)-\hat D\partial_\xi(m-\rho\hat u)\,,\tag{5.26}$$

with the constant $\hat D:=(3-\gamma)\hat\rho^{\gamma-1}>0$, the nonlinearity $q(\rho,m):=\int_{\mathbb{R}}v^2R\,dv$ and

$$S(g)=\int_{\mathbb{R}}v^2(\varepsilon\partial_t g-\mathcal{L}((v-s)\partial_\xi g))dv\,.\tag{5.27}$$

The stability of the shock profiles will be investigated by introducing primitives of the macroscopic variables. According to (5.22) and the diffusion term in (5.26), it is convenient to use

$$W_\rho(t,\xi)=\int_{-\infty}^{\xi}\rho(t,\xi')\,d\xi'\,,\qquad W_u(t,\xi)\;=\;\int_{-\infty}^{\xi}(m(t,\xi')-\rho(t,\xi')\hat u)d\xi'\,.$$

Integrating (5.23),(5.24) with respect to ξ gives the macroscopic equations

$$\partial_t W_\rho+\frac{1}{\varepsilon}\left[\partial_\xi W_u+(\hat c-\varepsilon\hat\sigma)\partial_\xi W_\rho\right]=0,\tag{5.28}$$

$$\partial_t W_u+\frac{1}{\varepsilon}\left[(\hat c-\varepsilon\hat\sigma)\partial_\xi W_u+\hat c^2\partial_\xi W_\rho\right]+q-\hat D\partial_\xi^2 W_u=\varepsilon S(g).\tag{5.29}$$

Observe that the second equation is obtained by a linear combination of (5.23),(5.24). We expand q as follows

$$\begin{aligned}q(\rho,m)&=\frac{1}{\varepsilon}(\nabla_U j(\rho_\varphi,m_\varphi)-\nabla_U j(\hat\rho,\hat m))\cdot\begin{pmatrix}\rho\\ m\end{pmatrix}+\tilde q(\rho,m)\,,\\ \tilde q(\rho,m)&=\frac{1}{\varepsilon^2}\left(j\left(\rho_\varphi+\varepsilon\rho,m_\varphi+\varepsilon m\right)-j(\rho_\varphi,m_\varphi)-\varepsilon\nabla_U j(\rho_\varphi,m_\varphi)\cdot\begin{pmatrix}\rho\\ m\end{pmatrix}\right).\end{aligned}$$

and note that $\tilde q$ is purely quadratic in (ρ,m).

Now the system (5.28),(5.29) can equivalently be stated as

$$\partial_t W_\rho+\frac{1}{\varepsilon}\left[\partial_\xi W_u+(\hat c-\varepsilon\hat\sigma)\partial_\xi W_\rho\right]=0,\tag{5.30}$$

$$\partial_t W_u+\frac{1}{\varepsilon}\left[K_2(\varphi)\partial_\xi W_u+K_1(\varphi)\partial_\xi W_\rho\right]+\tilde q-\hat D\partial_\xi^2 W_u=\varepsilon S(g),\tag{5.31}$$

where

$$K_1(\varphi) := c_\varphi^2 - (u_\varphi - \hat{u})^2\,, \qquad K_2(\varphi) := \hat{c} - \varepsilon\hat{\sigma} + 2\,(u_\varphi - \hat{u})\,. \tag{5.32}$$

We will need the signs of K_1, K_2 and of their derivatives. From Lemma 4.4 we know that the travelling wave is strictly increasing, which also implies $\partial_\xi u_\varphi < 0$. Then for ε small we get

$$\frac{\hat{c}^2}{2} < K_1(\varphi) < 2\hat{c}^2, \qquad \partial_\xi K_1(\varphi) > 0, \qquad \partial_\xi\left(K_1(\varphi)^{-1}\right) < 0, \tag{5.33}$$

$$\frac{\hat{c}}{2} < K_2(\varphi) < 2\hat{c}, \qquad \partial_\xi K_2(\varphi) < 0. \tag{5.34}$$

Recall from Theorem 4.3 that $\partial_\xi K_1(\varphi), \partial_\xi K_2(\varphi)$ are $O(\epsilon)$ uniformly in ξ.

We start with the derivation of estimates for the macroscopic parts. For controlling the nonlinear terms, L^∞_ξ-bounds of ρ, m are needed, which we shall control in H^1_ξ. This means we need to control the H^2_ξ-norm of W_ρ, W_u and therefore we give integral estimates for their derivatives up to second order in the following.

Expanding $(\rho_\varphi, m_\varphi)$ around $(\hat{\rho}, \hat{m})$ gives $\rho_\varphi = \rho_- + \varepsilon y = \hat{\rho} + \varepsilon\hat{y}_1$ and $m_\varphi = m_- + \varepsilon s y = \hat{m} + \varepsilon\hat{y}_2$ and we can write the nonlinearity as

$$R(\rho, m) \;=\; (\hat{y}_1, \hat{y}_2)\cdot\mathcal{H}(\mathcal{M}_1)\cdot\begin{pmatrix}\rho\\ m\end{pmatrix} + (\rho, m)\cdot\mathcal{H}(\mathcal{M}_2)\begin{pmatrix}\rho\\ m\end{pmatrix}, \tag{5.35}$$

where $\mathcal{M}_1 = \mathcal{M}(\hat{\rho}+\varepsilon\vartheta_1\hat{y}_1, \hat{m}+\varepsilon\vartheta_1\hat{y}_2), \mathcal{M}_2 = \mathcal{M}(\rho_\varphi+\varepsilon\vartheta_2\rho, m_\varphi+\varepsilon\vartheta_2 m)$ and $0 \le \vartheta_1, \vartheta_2 \le 1$. For $\|R\|_v$ to be well defined we have to guarantee that *supp* $\mathcal{M}_1$, *supp* $\mathcal{M}_2 \subset$ *supp* $\hat{\mathcal{M}}$. Due to the construction of $\hat{\mathcal{M}}$ this holds for $\mathcal{M}_1$. For $\mathcal{M}_2$ this is only true for sufficiently small $\|\rho\|_\infty, \|m\|_\infty$. We make this smallness assumption for the moment and prove it in the stability result at the end of this section. By differentiating (5.35) and using (4.32), we obtain

$$\|R\|^2_{H^k_\xi(L^2_v)} \le \tilde{C}\left[\|\partial_\xi W_\rho\|^2_{H^k_\xi} + \|\partial_\xi W_u\|^2_{H^k_\xi}\right] \quad \text{for } k = 0, 1, 2, \tag{5.36}$$

implying together with (4.33) the same bound for q

$$\|q\|^2_{H^k_\xi} \le \tilde{C}\left[\|\partial_\xi W_\rho\|^2_{H^k_\xi} + \|\partial_\xi W_u\|^2_{H^k_\xi}\right] \quad \text{for } k = 0, 1, 2. \tag{5.37}$$

Here and in the following $\tilde{C}$ depends on $\|\rho\|_\infty, \|m\|_\infty$.

Lemma 5.1 *Let W_ρ, W_u be the solution of the system (5.28), (5.29). Then there exists a constant C and $\tilde{C}(\|\rho\|_\infty, \|m\|_\infty)$ such that the following two estimates hold for any $\alpha_k > 0, k = 0, 1, 2$:*

$$\begin{aligned}
&\frac{d}{dt}J_0 + \int_{\mathbb{R}} \kappa(\varphi) W_u^2\, d\xi + \left(\alpha_0\left(\frac{\hat{c}^2}{2} - \varepsilon\tilde{C}\right) - \tilde{C}\|W_u\|_\infty\right)\|\partial_\xi W_\rho\|^2_\xi \\
&+\left(\frac{\hat{D}}{2\hat{c}^2} - \varepsilon C - \tilde{C}\|W_u\|_\infty - \alpha_0\left(1 + \varepsilon\tilde{C}\right)\right)\|\partial_\xi W_u\|^2_\xi \\
&\le\; \varepsilon\int_{\mathbb{R}}\left(K_1(\varphi)^{-1}W_u + \varepsilon\alpha_0\partial_\xi W_\rho\right) S(g)\, d\xi,
\end{aligned} \tag{5.38}$$

where

$$J_0 = \frac{1}{2}\int_{\mathbb{R}} \left[W_\rho^2 + K_1(\varphi)^{-1}W_u^2 + \varepsilon\alpha_0(\varepsilon\hat{D}(\partial_\xi W_\rho)^2 + 2(\partial_\xi W_\rho)W_u)\right] d\xi \tag{5.39}$$

and

$$\kappa(\phi) = \frac{1}{2\varepsilon}\left[-\partial_\xi(K_2(\varphi)\,K_1(\varphi)^{-1}) - 2\varepsilon\hat{D}|\partial_\xi(K_1(\varphi)^{-1})|\right] > 0\,,$$

and accordingly for the higher order derivatives $k = 1, 2$:

$$\begin{aligned}\frac{d}{dt}J_k + \alpha_k\left(\frac{\hat{c}^2}{2} - \varepsilon^2\tilde{C}\right)\|\partial_\xi^{k+1}W_\rho\|_\xi^2 + \left(\frac{\hat{D}}{2} - \alpha_k(1+\epsilon^2\tilde{C})\right)\|\partial_\xi^{k+1}W_u\|_\xi^2 \\ -\tilde{C}\left[\|\partial_\xi W_\rho\|^2_{H_\xi^{k-1}} + \|\partial_\xi W_u\|^2_{H_\xi^{k-1}}\right] \le \varepsilon\int_{\mathbb{R}}\left(\partial_\xi^k W_u + \varepsilon\alpha_k\partial_\xi^{k+1}W_\rho\right)\partial_\xi^k S(g)\,d\xi,\end{aligned} \tag{5.40}$$

where

$$J_k = \frac{1}{2}\int_{\mathbb{R}}\left[\hat{c}^2(\partial_\xi^k W_\rho)^2 + (\partial_\xi^k W_u)^2 + \varepsilon\alpha_k(\varepsilon\hat{D}(\partial_\xi^{k+1}W_\rho)^2 + 2\partial_\xi^{k+1}W_\rho\partial_\xi^k W_u)\right] d\xi. \tag{5.41}$$

Proof. We start with the proof of (5.38) and split it into two steps. First we derive estimate (5.38) with $\alpha_0 = 0$ and in the second step we prove the inequality for the remaining terms containing α_0.
Step 1: We test (5.30) with W_ρ and (5.31) with $K_1(\varphi)^{-1}W_u$ such that the integrals containing $W_\rho\partial_\xi W_u$ and $W_u\partial_\xi W_\rho$ cancel out. Here we also take advantage of the properties of K_1 and K_2, see (5.33), (5.34):

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int\left[W_\rho^2 + K_1(\varphi)^{-1}W_u^2\right]d\xi + \frac{1}{\varepsilon}\int\partial_\xi\left(-K_2(\varphi)\,K_1(\varphi)^{-1}\right)\frac{W_u^2}{2}d\xi + \int K_1(\varphi)^{-1}W_u\,\tilde{q}\,d\xi \\ +\hat{D}\int\left[K_1(\varphi)^{-1}(\partial_\xi W_u)^2 + \partial_\xi\left(K_1(\varphi)^{-1}\right)\partial_\xi\frac{W_u^2}{2}\right]d\xi = \varepsilon\int K_1(\varphi)^{-1}W_u\,S(g)\,d\xi.\end{aligned}$$

In the third term we estimate the quadratic term by

$$\left|\int\tilde{q}\,d\xi\right| \le \tilde{C}\left[\|\partial_\xi W_\rho\|_\xi^2 + \|\partial_\xi W_u\|_\xi^2\right]. \tag{5.42}$$

The triangle inequality is used for the fourth term

$$\left|\int\partial_\xi\left(K_1(\varphi)^{-1}\right)(\partial_\xi W_2)W_u\,d\xi\right| \ge -\int\left[|\partial_\xi\left(K_1(\varphi)^{-1}\right)|\,W_u^2 + \varepsilon C(\partial_\xi W_u)^2\right]d\xi.$$

From this estimate we cannot control $\|\partial_\xi W_\rho\|_\xi^2$. Therefore we will combine it with the next one.
Step 2: Testing the first derivative of (5.30) with $\partial_\xi W_\rho$ we obtain

$$\frac{1}{2}\frac{d}{dt}\int(\partial_\xi W_\rho)^2 d\xi + \frac{1}{\varepsilon}\int\partial_\xi^2 W_u\partial_\xi W_\rho d\xi = 0,$$

and we observe that

$$\frac{d}{dt}\int(\partial_\xi W_\rho)W_u\,d\xi \;=\; \int(\partial_t W_u\,\partial_\xi W_\rho - \partial_t W_\rho\,\partial_\xi W_u)\,d\xi.$$

By combining the equations in the corresponding way we get

$$\frac{1}{2}\frac{d}{dt}\int\left[\varepsilon^2\hat{D}(\partial_\xi W_\rho)^2 + 2\varepsilon(\partial_\xi W_\rho)W_u\right]d\xi + \int K_1(\varphi)(\partial_\xi W_\rho)^2 d\xi - \int(\partial_\xi W_u)^2 d\xi$$
$$+\int(K_2(\varphi)-(\hat{c}-\varepsilon\hat{\sigma}))\partial_\xi W_u\partial_\xi W_\rho d\xi + \varepsilon\int\partial_\xi W_\rho\,\tilde{q}\,d\xi = \varepsilon^2\int\partial_\xi W_\rho\,S(g)\,d\xi.$$

For the fourth term on the left hand side we use the triangle inequality together with $K_2(\varphi)-(\hat{c}-\varepsilon\hat{\sigma}) = 2(u_\varphi - \hat{u}) = O(\varepsilon)$. Finally applying (5.42) gives the estimate.

For the bounds on the higher order derivatives we proceed as above. First we differentiate the system (5.28), (5.29) k times and test it with $\hat{c}^2\partial_\xi^k W_\rho$, respectively $\partial_\xi^k W_u$. Applying

$$\int\partial_\xi^k q\,\partial_\xi^k W_u\,d\xi = -\int\partial_\xi^{k-1}q\,\partial_\xi^{k+1}W_u\,d\xi \geq -\frac{\hat{D}}{2}\left[\|\partial_\xi^{k+1}W_u\|_\xi^2 + \frac{1}{\hat{D}^2}\|q\|^2_{H_\xi^{k-1}}\right]$$

the inequality for $\alpha_k = 0$ is straightforward. The remaining part is analogous as Step 2 above. $\square$.

Now we concentrate on bounds for the small terms on the right hand sides in (5.38) and (5.40). Here the estimates from [17] are extended.

Lemma 5.2 *Let W_ρ, W_u, g and $S(g)$ satisfy (5.26)-(5.29). Then there exists a constant C such that*

$$\varepsilon\int K_1(\varphi)^{-1}W_u\,S(g)\,d\xi - \int\kappa(\varphi)W_u^2 d\xi \tag{5.43}$$
$$\leq \varepsilon^2\frac{d}{dt}\int K_1(\varphi)^{-1}W_u\int v^2\,g\,dv\,d\xi + \varepsilon C\left[\|g\|^2_{\xi,v} + \|\partial_\xi W_\rho\|_\xi^2 + \|\partial_\xi W_u\|_\xi^2\right],$$

and additionally for $k = 0, 1, 2$

$$\varepsilon\int\partial_\xi^k W_u\partial_\xi^k S(g)\,d\xi \tag{5.44}$$
$$\leq \varepsilon^2\frac{d}{dt}\int\partial_\xi^k W_u\int v^2\partial_\xi^k g\,dv\,d\xi + \varepsilon C\left[\|\partial_\xi^k g\|^2_{\xi,v} + \|\partial_\xi^{k+1}W_\rho\|_\xi^2 + \|\partial_\xi^{k+1}W_u\|_\xi^2\right].$$

Proof.

$$\int K_1(\varphi)^{-1}W_u\,S(g)\,d\xi$$
$$= \varepsilon\frac{d}{dt}\int K_1(\varphi)^{-1}W_u\int v^2 g\,dv\,d\xi + \varepsilon\int K_1(\varphi)^{-1}\left(\int v^2 g\,dv\right)^2 d\xi$$

$$
\begin{aligned}
&+\int K_1(\varphi)^{-1}(\hat{c}^2\partial_\xi W_\rho + (\hat{c}-\varepsilon\hat{\sigma})\partial_\xi W_u)\int v^2 g\,dv\,d\xi \\
&+\int \left[K_1(\varphi)^{-1}\partial_\xi W_u + \partial_\xi\left(K_1(\varphi)^{-1}\right)W_u\right]\int v^2\mathcal{L}((v-s)g)\,dv\,d\xi \\
\le\ &\varepsilon\frac{d}{dt}\int K_1(\varphi)^{-1}W_u\int v^2 g\,dv\,d\xi + C\left(\|g\|^2_{\xi,v} + \|\partial_\xi W_\rho\|^2_\xi + \|\partial_\xi W_u\|^2_\xi\right) \\
&+\int \partial_\xi\left(-\frac{1}{2K_1(\varphi)}\right)\left[\left(\int v^2\mathcal{L}((v-s)g)dv\right)^2 + W_u^2\right]d\xi.
\end{aligned}
$$

For the first equality we used (5.26) and (5.29), moreover (4.32) and (4.33), which also implies $\int v^2\mathcal{L}((v-s)\partial_\xi^k g)dv \le C\|\partial_\xi^k g\|_v$. Finally to control the last term the function $\kappa(\varphi)$ is needed. We multiply the above inequality with ε and use $\kappa(\varphi)+\varepsilon\partial_\xi\left(K_1(\varphi)^{-1}\right)/2 \ge 0$. The proof for the second estimate is similar. □

Lemma 5.3 *Let W_ρ, W_u, g and $S(g)$ satisfy (5.26)-(5.29). Then there exists a constant C and a $\tilde{C}$ such that for $k=0,1,2$*

$$
\varepsilon^2\int \partial_\xi^{k+1}W_\rho\partial_\xi^k S(g)\,d\xi \le \varepsilon^2\frac{\hat{D}}{2}\frac{d}{dt}\|\partial_\xi^{k+1}W_\rho\|^2_\xi + \varepsilon\tilde{C}\left[\|\partial_\xi W_\rho\|^2_{H^k_\xi} + \|\partial_\xi W_u\|^2_{H^k_\xi}\right] + \varepsilon C\|\partial_\xi^k g\|^2_{\xi,v}.
$$

Proof. These estimates cannot be derived in the same way as before, since the derivatives in the bounds would get too high and could not be controlled by the macroscopic estimates anymore. Here we take advantage of ε^2. We use equation (5.26) for $S(g)$

$$
\varepsilon^2\partial_\xi^k S(g) \;=\; \varepsilon\left(\partial_\xi^k q - \hat{D}\partial_\xi^{k+2}W_u - \int v^2\partial_\xi^k g dv\right).
$$

Now we substitute $\partial_\xi^{k+2}W_u$ according to (5.28) implying

$$
\begin{aligned}
\varepsilon^2\int \partial_\xi^{k+1}W_\rho\partial_\xi^k S(g)d\xi \;=\; &\varepsilon\int \partial_\xi^k q\partial_\xi^{k+1}W_\rho\,d\xi + \varepsilon^2\int \hat{D}\partial_t\frac{(\partial_\xi^{k+1}W_\rho)^2}{2}\,d\xi \\
&-\varepsilon\int\int v^2\partial_\xi^k g\partial_\xi^{k+1}W_\rho\,dv\,d\xi.
\end{aligned}
$$

□

For getting control of the microscopic terms we derive estimates from the full kinetic perturbation equation.

Lemma 5.4 *Let G, decomposed as in (5.21), be the solution of (5.20). Then there exists a $\tilde{C}$ such that for $k=0,1,2$*

$$
\frac{d}{dt}\left[\frac{1}{\hat{\rho}}\left(\hat{c}^2\|\partial_\xi^{k+1}W_\rho\|^2_\xi + \|\partial_\xi^{k+1}W_u\|^2_\xi\right) + \varepsilon^2\|\partial_\xi^k g\|^2_{\xi,v}\right] + \|\partial_\xi^k g\|^2_{\xi,v} \le \tilde{C}\left[\|\partial_\xi W_\rho\|^2_{H^k_\xi} + \|\partial_\xi W_u\|^2_{H^k_\xi}\right].
$$

Proof. The kth derivative of (5.20) is tested with $\partial_\xi^k G$. For more details see [17]. $\square$

Now we are able to prove the main result of this section.

Theorem 5.5 *Let the assumptions of Theorem 4.3 hold and let φ be the travelling wave solution. Let $f_0(\xi, v)$ be the initial datum for (5.18) and let*

$$
\begin{aligned}
W_{\rho,0}(\xi) &= \frac{1}{\varepsilon}\int_{-\infty}^{\xi} [\rho_{f_0}(\xi') - \rho_\varphi(\xi')]\, d\xi' , \\
W_{u,0}(\xi) &= \frac{1}{\varepsilon}\int_{-\infty}^{\xi} [(m_{f_0}(\xi') - m_\varphi(\xi')) - \hat{u}(\rho_{f_0}(\xi') - \rho_\varphi(\xi'))]\, d\xi' .
\end{aligned}
$$

Moreover we assume $f_0 - \varphi \in H_\xi^2(L_v^2)$ (implying $f_0(\pm\infty, v) = \varphi(\pm\infty, v)$) and $W_{\rho,0}, W_{u,0} \in L_\xi^2$, which ensures assumption (5.19). Let

$$
\|W_{\rho,0}\|_{L_\xi^2} + \|W_{u,0}\|_{L_\xi^2} + \frac{1}{\varepsilon}\|f_0 - \varphi\|_{H_\xi^2(L_v^2)} \leq \delta \tag{5.45}
$$

for a δ small enough, which is independent from ε. Then for ε small enough equation (5.18) with initial data f_0 has a unique global solution. In particular, small amplitude travelling waves are locally stable in the sense that

$$
\lim_{t\to\infty} \int_t^\infty \|f(s,.) - \varphi(.)\|_{H_\xi^2(L_v^2)}^2 ds = 0.
$$

Proof. The main idea is to construct a Lyapunow functional, which is decaying in time. Recall (5.39), (5.41) and define

$$
I := I_0 + \gamma_1 I_1 + \gamma_2 I_2,
$$

where

$$
\begin{aligned}
I_0 &:= J_0 + \varepsilon C_0 \left[\frac{1}{\hat{\rho}}\left(\hat{c}^2\|\partial_\xi W_\rho\|_\xi^2 + \|\partial_\xi W_u\|_\xi^2\right) + \varepsilon^2\|g\|_{\xi,v}^2\right] \\
&\quad -\varepsilon^2\left[\int\int K_1(\varphi)^{-1} v^2 g\, W_u\, dv d\xi + \alpha_0\frac{\hat{D}}{2}\|\partial_\xi W_\rho\|_\xi^2\right],
\end{aligned}
$$

and for $k = 1, 2$

$$
\begin{aligned}
I_k &:= J_k + \varepsilon C_k \left[\frac{1}{\hat{\rho}}\left(\hat{c}^2\|\partial_\xi^{k+1} W_\rho\|_\xi^2 + \|\partial_\xi^{k+1} W_u\|_\xi^2\right) + \varepsilon^2\|\partial_\xi^k g\|_{\xi,v}^2\right] \\
&\quad -\varepsilon^2\left[\int\int v^2 \partial_\xi^k g\, \partial_\xi^k W_u\, dv d\xi + \alpha_k\frac{\hat{D}}{2}\|\partial_\xi^{k+1} W_\rho\|_\xi^2\right].
\end{aligned}
$$

Here the constants $C_j, j = 0, 1, 2$, are positive and independent from ε. Then for any $\gamma_1, \gamma_2 > 0$ the functional $H(t)$ is bounded from above and below by

$$
\|W_\rho\|_{H_\xi^2}^2 + \|W_u\|_{H_\xi^2}^2 + \varepsilon\left[\|\partial_\xi^3 W_\rho\|_\xi^2 + \|\partial_\xi^3 W_2\|_\xi^2\right] + \varepsilon^3\|g\|_{H_\xi^2(L_v^2)}^2,
$$

respectively by

$$\|W_\rho\|^2_{H^2_\xi} + \|W_u\|^2_{H^2_\xi} + \varepsilon\|G\|^2_{H^2_\xi(L^2_v)}.$$

We combine the estimates from Lemmata 5.1-5.4 to get a final one for the time derivative of I and write all terms on the left hand side. For an initial data and ε small enough, one can show that there exist constants $\gamma_1, \gamma_2 > 0$ and $\alpha_j > 0$, $j = 0, 1, 2$, such that the coefficients of $\|\partial_\xi^{k+1} W_\rho\|^2_\xi$, $\|\partial_\xi^{k+1} W_u\|^2_\xi$, $k = 0, 1, 2$, are positive initially. Since by the Sobolev-Imbedding I controls the L^∞_ξ-norms of W_ρ, W_u and their derivatives, these coefficients stay positive, if $I(0)$ is small enough, which is guaranteed by assumption (5.45). Hence

$$\frac{d}{dt} I(t) \le 0, \qquad \text{for all } t \ge 0,$$

and the proof is completed by integrating with respect to t. □

Remark 5.6 (Isothermal Case) *There is one difference in the isothermal case important to be mentioned. Since the sound speed c is constant, the derivative of $K_1(\varphi)$ corresponding to (5.32) is now of $O(\varepsilon^2)$ and has a different sign*

$$\partial_\xi K_1(\varphi) = -2(u_\varphi - u_-)\partial_\xi u_\varphi < 0.$$

Therefore the macroscopic estimate corresponding to (5.38) with $\alpha_0 = 0$ has to be derived differently. We test equation (5.30) with $K_1(\varphi)W_\rho$ and (5.31) with W_u to obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int (K_1(\varphi)W_\rho^2 + W_u^2)d\xi + \varepsilon C\|W_\rho\|^2_\xi + \frac{\hat{D}}{2}\|W_u\|^2_\xi \\ &-\tilde{C}\|W_u\|_\infty(\|\partial_\xi W_\rho\|^2_\xi + \|\partial_\xi W_u\|^2_\xi) + 2\|\partial_\xi W_u\|^2_\xi = \varepsilon\int S(g)\,\partial_\xi W_u\,d\xi \end{aligned}$$

Observe that here we do not need estimate (5.43).

6 Shock profiles for strong shocks of scalar conservation laws

The first existence proof of large kinetic shock profiles is due to Golse [22] for the Perthame-Tadmor model. The proof for other kinetic models for scalar conservation laws follows similar steps. It consists of obtaining the shock profile as the limit of profiles for ξ on a finite interval $[-L, L]$ as $L \to \infty$. Since the shock profile problem is translation invariant in the ξ direction, care has to be taken with fixing the profiles before taking this limit.

In this section we consider (1.1) and assume that its macroscopic limit is a scalar conservation law. Furthermore we assume throughout this section that $V = \mathbb{R}$ and $d\mu = dv$.

We start by giving some ingredients that are common to the examples that follow and that allow a way of proving existence and stability of strong shock profiles. A description of the proofs is then done in Section 6.1 and 6.2. This section is completed in Section 6.3 by presenting the program for the examples in sections 2.1 and 2.4.

We assume that the equilibrium distribution $\mathcal{M}(U,v)$ is continuous in v and has continuous derivatives with respect to U. The macroscopic flux is assumed to be genuinely nonlinear and, without loss of generality, strictly concave:

$$J''(U) < 0\,. \tag{6.1}$$

A key property of $\mathcal{M}$ is its invertibility with respect to U: we assume that there exists a $\zeta(f,v)$ such that

$$U = \zeta(f,v) \Longleftrightarrow f = \mathcal{M}(U,v)\,. \tag{6.2}$$

To be more precise, we shall assume that the function $\mathcal{M}(v) : U \to \mathcal{M}(U,v)$ is $C^2(\mathbb{R})$ and that there exist U_+, $U_- \in \mathbb{R} \cup \{-\infty, +\infty\}$ such that

$$\partial_U \mathcal{M}(U,v) > 0 \quad \text{for all} \quad U \in [U_-, U_+]\,. \tag{6.3}$$

We continue by briefly describing some of the additional features of the equations.

Existence and uniqueness and the maximum principle:
In general, local (in time) existence and uniqueness of the initial value problem

$$\partial_t f + v\partial_x f = Q(f)\,, \quad \text{on} \quad \mathbb{R}^+ \times \mathbb{R} \times V \tag{6.4}$$

$$f(0,x,v) = f_{init}(x,v) \quad \text{for} \quad (x,v) \in \mathbb{R} \times V \tag{6.5}$$

follows by considering the mild formulation of (1.1)

$$f(t,\cdot,\cdot) = T(t)f_{init}(\cdot,\cdot) + \int_0^t T(t-s)Q(f(s,\cdot,\cdot))\,ds\,, \tag{6.6}$$

where $T(t)$ denotes the continuous group generated by the linear transport operator $v\partial_x$. Thus, well-posedness follows if $Q(f)$ is Lipschitz continuous in the domain of $T(t)$ by a fixed point argument.

We shall assume that $Q(f)$ allows a form of comparison principle, which relates the solution to the distribution $\mathcal{M}$ at constant values of U: Let U_-, $U_+ \in \mathbb{R}$ be given, then if the initial condition satisfies

$$\mathcal{M}(U_-,v) \le f_{init}(x,v) \le \mathcal{M}(U_+,v) \tag{6.7}$$

then the solution of (6.4)-(6.5) satisfies

$$\mathcal{M}(U_-,v) \le f(t,x,v) \le \mathcal{M}(U_+,v) \quad \text{for all} \quad t > 0 \tag{6.8}$$

and thus also

$$U_- \le U_f(t,x) \le U_+ \quad \text{for all} \quad t > 0\,. \tag{6.9}$$

The general result for smooth initial data is the following:

Proposition 6.1 *Let $f_{init} \in C_0^1(\mathbb{R} \times V)$, such that there exists U_-, $U_+ \in \mathbb{R}$ with (6.7), then there exists a unique solution $f \in C^1(\mathbb{R}^+ \times \mathbb{R} \times V)$ satisfying (6.8) and (6.9).*

Kinetic entropy inequality For any increasing function χ the following holds

$$\int_V \chi(\zeta(f,v))Q(f)\,dv \le 0\,. \tag{6.10}$$

I.e. (1.5) holds with $H(f,v) = \int_{\bar{f}}^{f} \chi(\zeta(g,v))\,dg$. All convex entropies η are recovered from a kinetic entropy density H by taking $\chi = \eta'$, and so (1.10) holds.

Additionally, we assume that for $\eta(U) = U^2/2$ the entropy dissipation can be quantified. Namely, that there exists a constant $C > 0$ independent of ε such that

$$\int_V \varphi(f,v)Q(f)\,dv \le -C\int_V (f - \mathcal{M}(U_f))^2 w(v)\,dv \tag{6.11}$$

where the function w only depends on v and is positive and uniformly bounded.

L^1-contraction

Another property that is satisfied by scalar conservation laws is the L^1-contraction. For the kinetic equation we shall assume that for two given solutions of (6.4)-(6.5) such that $f - g \in L^1(\mathbb{R} \times V)$ for all $t > 0$

$$\int_V (Q(f) - Q(g))\mathrm{sign}(f-g)\,dv \le 0\,. \tag{6.12}$$

We also assume that the equality holds if and only if $\mathrm{sign}(f-g)$ is constant (independent of v). The L^1-contraction property now follows from (6.12). Subtracting the equations for f and g and multiplying by $\mathrm{sign}(f-g)$ implies that

$$\partial_t \int_V |f-g|\,dv + \partial_x \int_V v|f-g|dv \le 0\,. \tag{6.13}$$

Or integrating with respect to x:

$$\frac{d}{dt}\int_{\mathbb{R}}\int_V |f-g|\,dv\,dx \le 0\,. \tag{6.14}$$

Also a comparison principle (that generalizes Proposition 6.1) follows easily from (6.13) (by the Crandall-Tartar-Lemma [13]). Two solutions f and g with $f - g \in L^1(\mathbb{R}_x \times V)$ for all $t > 0$ clearly satisfy

$$\int_{\mathbb{R}}\int_V (f-g)\,dv\,d\xi = \int_{\mathbb{R}}\int_V (f_{init} - g_{init})\,dv\,d\xi \quad \text{for all } t > 0\,.$$

Then if $f_{init} \ge g_{init}$ then also $f \ge g$ for all $t > 0$.

6.1 Existence of kinetic profiles for strong shocks

We now look for traveling wave solutions with speed s connecting different equilibrium states. The traveling wave variable is defined by

$$\xi = x - st\,. \tag{6.15}$$

and we look for functions $f(\xi, v)$ that satisfy

$$(v-s)\partial_\xi f = Q(f) \tag{6.16}$$

$$\lim_{\xi\to-\infty} f(\xi,v) = \mathcal{M}(U_-,v) \qquad \lim_{\xi\to+\infty} f(\xi,v) = \mathcal{M}(U_+,v) \tag{6.17}$$

with $U_\pm \in \mathbb{R}$, where the far field conditions hold almost everywhere in v.

Indeed, integrating (6.16) with respect to v over V gives

$$\partial_\xi \int_V (v-s) f\, dv = 0$$

and integration with respect to ξ, using (6.17), implies that

$$\int_V (v-s) f\, dv = J(U_-) - s\,U_- = J(U_+) - s\,U_+ \tag{6.18}$$

thus we recover the Rankine-Hugoniot condition.

Observe that if a solution of (6.16)-(6.17) exists then (6.1) implies that $U_- < U_+$, and no solution exists if $U_- > U_+$. This is a consequence of (1.6): multiplying (6.16) by χ and integration with respect to v gives

$$\partial_\xi \int_V (v-s) H(f,v)\, dv \le 0$$

(this is the traveling wave version of (1.6)). Integration with respect to ξ and (6.17) implies

$$(\psi(U_+) - \psi(U_-)) - \frac{J(U_+) - J(U_-)}{U_+ - U_-}\,(\eta(U_+) - \eta(U_-)) \le 0\,. \tag{6.19}$$

It is now a standard exercise of scalar conservation laws to prove that $U_- < U_+$: since this inequality holds for all convex entropies, we choose to write if for $\eta(U) = U^2/2$, and ψ now satisfies $\psi'(U) = U\,J'(U)$. Defining

$$L(U) := (\psi(U) - \psi(U_-)) - \frac{1}{2}(J(U) - J(U_-))\,(U + U_-)$$

the inequality (6.19) becomes $L(U_+) \le 0$. We now compute $L(U_-) = 0$ and observe

$$L'(U) = \frac{1}{2}[J'(U)(U - U_-) - (J(U) - J(U_-))] < 0\,,$$

which holds by the concavity of J, thus $U_- < U_+$.

The general result is the following

Theorem 6.2 *There exists a traveling wave solution (unique up to translations in ξ) such that*

$$\lim_{\xi\to\pm\infty} f(\xi, v) \to \mathcal{M}(U_\pm, v) \quad \textit{weakly in } L^1_v(\omega)$$

Its macroscopic density is continuous and monotonically increasing.

We describe the steps of the proof in some detail.

Step 1: The slab problem First one constructs profiles on the intervals $[-L, +L]$ for all $L > 0$, by solving the equation

$$(v - s)\partial_\xi f^L = Q(f^L), \quad \xi \in (-L, +L)\,, \; v \in V \tag{6.20}$$

subject to the inflow boundary conditions

$$f^L(-L, v) = \mathcal{M}(U_-, v)\,, \quad \text{for } v > s \tag{6.21}$$

$$f^L(+L, v) = \mathcal{M}(U_+, v)\,, \quad \text{for } v < s\,. \tag{6.22}$$

The definition of a fixed point map will depend in each case on $Q(f)$, and is similar to the fixed point map defined to prove existence of the evolution equation. In particular, this fixed point map iteration will preserve the maximum and the comparison principles. In general, regularity of the macroscopic slab profiles might need to be proved additionally by means of averaging lemmas, for instance.

A general result can be formulated as follows

Proposition 6.3 *With the assumptions (6.3) and that the collision operator admits a maximum principle and that (6.10) hold, the slab problem (6.20)-(6.22) has a solution $f^L \in L^1_{x,v}((-L, L) \times V)$ with continuous macroscopic density U^L, and satisfying*

$$\mathcal{M}(U_-, v) \le f^L(\xi, v) \le \mathcal{M}(U_+, v) \quad \textit{for all} \quad \xi \in \mathbb{R}\,, \; v \in V\,, \tag{6.23}$$

then also $U_+ \le U^L \le U_-$.

The analogous for the Boltzmann equation is still open. Some results on a slab appear in Arkeryd, Cercignani, and Illner [1], Arkeryd and Nouri [2] and Ukai [38]. A maximum principle is not available here.

Step 2: Centering the profile The limiting problem for $L = \infty$ is translation invariant with respect to ξ. For this reason, before taking the limit $L \to \infty$, we normalize the shift of the profiles f^L. First we observe that by (6.23) and the inflow boundary conditions

$$U^L(-L) = \int_{-\infty}^{s} f^L(-L, v)\, dv + \int_{s}^{+\infty} f^L(-L, v)\, dv$$

$$\le \int_{-\infty}^{s} \mathcal{M}(U_+, v)\, dv + \int_{s}^{+\infty} \mathcal{M}(U_-, v)\, dv \le U^L(L)\,.$$

Then, for all $L > 0$, by continuity of U^L we can take $\xi^L \in [-L, L]$ such that

$$U^L(\xi^L) = U^* := \int_{-\infty}^{s} \mathcal{M}(U_+, v)\, dv + \int_{s}^{+\infty} \mathcal{M}(U_-, v)\, dv\,.$$

Next we shift the point ξ^L to the origin. Before that we first need to extend the ξ domain of f^L to $\mathbb{R}$:

$$f_1^L(\xi, v) := \begin{cases} f^L(-L, v) & \xi < -L \\ f^L(\xi, v) & -L \le \xi \le L \\ f^L(L, v) & L > \xi\,. \end{cases}$$

For a sequence $L_n \to \infty$ as $n \to \infty$ we let $\xi_n := \xi^{L_n}$ and $f_n(\xi, v) := f_1^{L_n}(\xi - \xi_n, v)$, and $U_n := \int_V f_n\, dv$. Clearly now $U_n(0) = U^*$ for all n.

Passing to the limit $L \to \infty$ in the equation will differ in each case. But, the bound (6.23) extends trivially and holds for f_n. Then f_n converges weakly* in $L^\infty_{\xi,v}(\mathbb{R} \times V)$ to some f satisfying (6.23). Applying velocity averaging one obtains that $U_n \to U_f$ uniformly on compact sets and $U_f(0) = U^*$. The weak limit f of f_1^L solves the limit equation in the distributional sense (the limit of the non-linear terms is treated in a similar way as for the existence proof; it depends on the specific form of $Q(f)$). It is yet necessary to prove that the shifted intervals $[-L_n + \xi_n, L_n + \xi_n]$ tend to $\mathbb{R}$.

Proposition 6.4 *$L + \xi^L$, $L - \xi^L \to \infty$ as $L \to \infty$. And there exist sequences $\xi_n \to \infty$ and $\eta_n \to -\infty$ such that the solution of the limit problem satisfies*

$$f(\xi_n, v) \to \mathcal{M}(U_+, v)\,, \quad f(\eta_n, v) \to \mathcal{M}(U_-, v) \quad v - a.e.\,.$$

In the proof one argues by contradiction, assuming that for a sequence $L_n \to \infty$, $\xi^n - L_n \to \xi^* > -\infty$ as $n \to \infty$. Then, by passing to the limit in the equation in the distributional sense, the limit f of f_n satisfies a half-space problem for $\xi \ge \xi^*$ with equilibrium inflow data:

$$\begin{aligned} &(v - s)\partial_\xi f = Q(f)\,, \quad \text{for} \quad \xi \ge \xi^* \\ &f(\xi^*, v) = \mathcal{M}(U_-, v)\,, \quad \text{for} \quad v > s\,. \end{aligned}$$

One then proves that $f(\xi^*, v) = \mathcal{M}(U_-, v)$ also holds for $v \le s$ v-a.e. and actually that $f(\xi, v) = \mathcal{M}(U_-, v)$ for $\xi \ge \xi^*$ v-a.e..

With the aid of (6.11) and the continuity of U_f one proves that $U_f \to U_{+\infty}$ as $\xi \to \infty$, and that, restricted to a subsequence ξ_n, $f(\xi_n, v) \to \mathcal{M}(U_{+\infty})$ v-a.e. Using the maximum principle and the inflow boundary condition it can be shown that $U_- \le U_{+\infty} \le U_+$ and that

$$\int_{v<s} (v - s)(f(\xi^*, v) - \mathcal{M}(U_-, v))\, dv = 0\,, \quad J(U_{+\infty}) - sU_{+\infty} = J(U_-) - sU_-$$

then, by (6.23),

$$f(\xi^*, v) = \mathcal{M}(U_-, v) \quad v - \text{a.e.}\,, \tag{6.24}$$

as anticipated. It is next shown that

$$\int_V (Q(f) - Q(\mathcal{M}(U_-)))\,\mathrm{sign}(f - \mathcal{M}(U_-))\,dv = 0$$

and (6.12) gives that $\mathrm{sign}(f - \mathcal{M}(U_-))$ is constant, actually zero by (6.24), thus $f(\xi, v) = \mathcal{M}(U_-, v)$ for all $\xi \geq \xi^*$. This is in contradiction with $U_f(0) = U^*$, then $-L + \xi^L \to -\infty$.

A similar argument shows that $L + \xi^L \to \infty$ and that there is a sequence $\xi_k \to -\infty$ as $k \to \infty$ such that there exists $U_{-\infty}$ with $f(\xi_k, v) \to \mathcal{M}(U_{-\infty}, v)$ as $k \to \infty$ v-a.e. The entropy inequality again implies that $U_{-\infty} \leq U_{+\infty}$. Finally, $J(U_{-\infty}) - sU_{-\infty} = J(U_+) - sU_+$, as before, hence $U_{-\infty} = U_-$ and $U_{+\infty} = U_+$.

Step 3: monotonicity with respect to ξ

Monotonicity of the macroscopic profiles now follows as a consequence of (6.12). And in particular implies that the far-field conditions hold in the stronger sense of Theorem 6.2.

First we observe that the following holds

Lemma 6.5 *Let f and g be two solutions of (6.4) such that there exists sequences $\xi_n \to \infty$ and $\eta_n \to -\infty$ with*

$$\lim_{n\to\infty} (f(\xi_n, v) - g(\xi_n, v)) = 0 \qquad \lim_{n\to\infty} (f(\eta_n, v) - g(\eta_n, v)) = 0$$

then sign$(f - g)$ is independent of v.

This follows by subtracting the equations of f and g, multiplying by $\mathrm{sign}(f - g)$ and integrating with respect to v and ξ (as in (6.13)) gives

$$0 = \partial_\xi \int_{\mathbb{R}} \int_V (v - s)|f - g| = \int_{\mathbb{R}} \int_V (Q(f) - Q(g))\mathrm{sign}(f - g)\,dv\,dx\,,$$

thus $\mathrm{sign}(f - g)$ is independent of v.

This means that if we consider a traveling wave solution f, any translation of it $\hat{f}(\xi, v) = f(\xi + a, v)$ with $a > 0$, $\hat{f}$ is clearly a traveling wave solution as well, and the above lemma applies, giving that $\mathrm{sign}(f - \hat{f})$ is independent of v. In particular, that f is monotone with respect to ξ holds if an expression of the form $\int w(v) f\,dv$ for any positive w, is monotone. This step can be performed in the examples and will depend on the form of $Q(f)$.

We can now state the general result:

Theorem 6.6 *If $J''(U) < 0$ and $U_- < U_+$, there exists a traveling wave solution f (unique up to translation in ξ) such that*

$$\lim_{\xi\to\pm\infty} f(\xi, v) \rightharpoonup \mathcal{M}(U_\pm, v) \quad \text{weakly in} \quad L^1_v(V).$$

Moreover, its macroscopic density is continuous and monotonically increasing.

6.2 Stability of kinetic profiles for strong shocks

We now study dynamic stability of the traveling wave just constructed. We prove that solutions of the Cauchy problem (6.4)-(6.5) approach a traveling wave solution as $t \to \infty$ if the initial condition has the same far field behavior of a shock profile.

We let $\varphi(\xi, v)$ be a kinetic shock profile, i.e. a solution to (6.16) and (6.17). We also let it, by a shift in ξ if necessary, be chosen such that for the initial datum $f_{init}(\xi, v)$

$$\int_{\mathbb{R}}\int_V (f_{init} - \varphi)\, dv\, d\xi = 0\,. \tag{6.25}$$

We denote the difference between the solution of the initial value problem and the shock profile by $h(t, \xi, v) := f(t, \xi, v) - \varphi(\xi, v)$, so h satisfies

$$\partial_t h + (v - s)\partial_\xi h = Q(f) - Q(\varphi)\,, \tag{6.26}$$

$$h(t = 0) = f_{init} - \phi\,, \quad \int_{\mathbb{R}}\int_V h(t, \xi, v)\, dv\, d\xi = 0\,. \tag{6.27}$$

Multiplying (6.26) by $\mathrm{sign}(h)$ and integrating with respect to v and ξ we get

$$\frac{d}{dt}\int_{\mathbb{R}}\int_V |h|\, dv\, d\xi \le \int_{\mathbb{R}}\int_V (Q(f) - Q(\varphi))\,\mathrm{sign}(h)\, dv\, d\xi \le 0\,.$$

And so $\lim_{t\to\infty} \|h\|_{L^1_{x,v}} < \infty$, and also

$$\int_{\mathbb{R}}\int_V |h|\, dv\, d\xi \le \int_{\mathbb{R}}\int_V |h_{init}|\, dv\, d\xi \quad \text{for all } t > 0\,.$$

For each $t_n \to \infty$ we define $h_n(t, \xi, v) := h(t_n + t, \xi, v) \to h_\infty(t, \xi, v)$. The sequences $\{h_n\}_n$ are bounded in $L^\infty(0, \infty; L^1_{\xi,v} \cap L^\infty_{\xi,v}(\mathbb{R} \times V))$. Then (restricted to a subsequence) $h_n \to h_\infty$ as $n \to \infty$ in $L^\infty_{x,v}(\mathbb{R} \times V)$ weak*. Because of the translation invariance in ξ we get equicontinuity in the ξ-direction; by applying the L^1 contractivity to the difference $h(t, \xi + h, v) - h(t, \xi, v)$. Thus we can conclude that there is a subsequence of t_n such that

$$h_n \to h_\infty \quad \text{as } n \to \infty \quad \text{in} \quad L^\infty_{\xi,v}(\mathbb{R} \times V) \text{ weak}^*\,.$$

Also since $\int_{\mathbb{R}} |\partial_\xi(U_f - U_g)|\, d\xi \le \int_{\mathbb{R}}\int_V |\partial_\xi(f - g)|\, dv\, d\xi$, there exists a U_∞ such that

$$\int_V h_n\, dv = U_{h_n} \to U_\infty \quad \text{strongly in} \quad L^1_\xi(\mathbb{R})\,. \tag{6.28}$$

We now observe that

$$\int_0^\infty \int_{\mathbb{R}\times V} (Q(f_n) - Q(\varphi))\,\mathrm{sign}(h_n)\, dv\, d\xi\, dt$$

$$= \int_{t_n}^\infty \int_{\mathbb{R}\times V} (Q(f) - Q(\varphi))\,\mathrm{sign}(h)\, dv\, d\xi\, dt \to 0 \quad \text{as } n \to \infty\,.$$

The term $Q(f_n) - Q(\varphi)$ can be rearranged in the examples and one can take the limit in the weak formulation of the equation satisfied by h_n, using (6.28) in the nonlinear term. In addition the above implies that the $\mathrm{sign}(h_\infty)$ does not depend on v, and it is easy to check that $\int\int h_\infty \, dv \, d\xi = 0$, since this property is preserved in t. The limit equation is

$$\partial_t h_\infty + (v-s)\partial_\xi h_\infty = \tilde{Q} h_\infty$$
$$\mathrm{sign}(h_\infty) = \mathrm{sign}(U_{h_\infty}) \,, \int_{\mathbb{R}}\int_V h_\infty \, dv \, d\xi = 0$$

and holds in the weak sense. Here $\tilde{Q}$ results from the linearisation of $Q(f_n) - Q(\varphi)$ and taking the limit $n \to \infty$.

Let us see that $h_\infty \equiv 0$. We argue by contradiction, first we assume that there is a (t_0, ξ_0, v_0) such that $h_\infty(t_0, \xi_0, v_0) > 0$. There must be a (ξ_1, v_1) such that $h_\infty(t_0, \xi_1, v_1) < 0$. This implies $h_\infty(t_0, \xi_0, v) > 0$ and $h_\infty(t_0, \xi_1, v) < 0$ for all v, because $\mathrm{sign}(h_\infty)$ does not depend on v. In fact $\mathrm{sign}(h_\infty)$ does not change along characteristics. So taking $\xi_2 = \xi_0 + (v_0 - s)(t_1 - t_0)$ and $t_1 \neq t_0$, we get $h(t_1, \xi_2, v) > 0$. Now we can choose v_2 such that $\xi_1 = \xi_2 + (v_2 - s)(t_0 - t_1)$, implying $h_\infty(t_0, \xi_1, v) > 0$, a contradiction.

The stability result can now be stated

Theorem 6.7 *Let f be a solution of (6.4)-(6.5), with $\mathcal{M}(U_-, v) \leq f_{init} \leq \mathcal{M}(U_+, v)$, such that $\lim_{x\to\pm\infty} f_{init}(x, v) = \mathcal{M}(U_\pm, v)$. Let φ be a traveling wave solution such that (6.25) holds, then for every sequence $t_n \to \infty$, $f(t_n + t, \xi, v) \to \varphi(\xi, v)$ in $L^\infty(0, T \times \mathbb{R} \times V)$ weak*.*

6.3 Examples

BGK-model for scalar conservation laws

The above program has been carried out in [14], [18] for the BGK-model of scalar conservation laws described in Section 2.1.

Kinetic entropy inequalities are obtained from (6.3) by letting ζ be the inverse of $\mathcal{M}(U, v)$ as we already described in Section 2.1. For this model (6.11) readily holds with $w(v) = (\sup_{U_- \leq U \leq U_+} \partial_U \mathcal{M}(U, v))^{-1}$.

In several of the arguments that follow a subsequence of distribution functions will converge in $L^\infty_{x,v}(\mathbb{R} \times V)$ weak* as a consequence of the maximum principle. Then either by an averaging lemma or a uniform estimate derived from the equation, the corresponding sequence of macroscopic densities U_f converges strongly in some $L^p_x(\mathbb{R})$ with $1 \leq p < \infty$ or uniformly in $C_x(\mathbb{R})$. Passing to the limit in the equation can be done easily in the weak formulation because the only nonlinear term $\mathcal{M}(U_f, v)$ involves f through U_f.

Let us briefly see why Proposition 6.1 holds. The mild formulation (6.6) which now reads

$$f(t, x, v) = e^{-t} f_0(x - vt, v) + \int_0^t e^{(s-t)} \mathcal{M}(U_f(s, x - v(t-s)), v) \, ds \,,$$

and a standard fixed point argument gives local existence in time. A comparison principle follows easily from the mild formulation too. Let f^1_{init} and f^2_{init} be two initial conditions such that $f^1_{init} \leq f^2_{init}$, and let f^1 and f^2 denote the corresponding solutions, let also $g_{init} := f^2_{init} - f^1_{init}$ and $g := f^2 - f^1$, then g satisfies

$$\partial_t g + \partial_x g + g = \partial_U \mathcal{M}(\bar{U}, v) U_g \,,$$

for some $\bar{U}$. Thus by (6.3) the positivity of the initial condition is preserved in time. This can be applied to steady solutions $\mathcal{M}(U^*, v)$ where U^* is constant, and so the maximum principle follows.

We observe that in this case the mild formulation gives a self-consistent formulation in terms of U_f:

$$U_f(t,x) = e^{-t} \int_V f_0(x - vt, v)\, dv + \int_0^t e^{(s-t)} \int_V \mathcal{M}(U_f(s, x - v(t-s)), v)\, dv\, ds \,.$$

A similar formulation is used to solve the slab problem associated to the traveling wave equation, as we shall see.

Let us now check that the L^1-contraction property holds. We just need to show that (6.12) holds for any two solutions of the initial value problem, with different initial conditions, f and g and such that $f - g \in L^1_{x,v}(\mathbb{R} \times V)$. We compute

$$\int_V \{\mathcal{M}(U_f), v) - \mathcal{M}(U_g, v) - (f - g)\} \mathrm{sign}(f - g)\, dv =$$
$$\int_V \{|\mathcal{M}(U_f), v) - \mathcal{M}(U_g, v)| - |f - g|\}\, dv = |U_f - U_g| - \int_V |f - g|\, dv \leq 0 \,.$$

where we have used (6.3).

We now sketch the proofs of existence and stability of traveling waves. We recall that with the traveling wave variable (6.15) we look for solutions of the problem (6.16) subject to (6.17), where the traveling wave speed s is given by (6.18). That $U_- < U_+$ under the assumption (6.1) follows from (6.19), as before.

Let us turn now to the existence of traveling waves. To solve the slab problem (6.20) subject to (6.21) and (6.22), the following fixed point map can be used

$$\mathcal{T} : U^L \mapsto \int_V f^L\, dv \,,$$

where f^L solves (6.20), for a given U^L, subject to the boundary conditions (6.21) and (6.22). So the proof is an application of the Schauder's fixed point theorem. The special form of the collision operator allows to define a fixed point map that is in fact an operator of macroscopic densities and that maps a subspace of the locally continuous functions into itself. The compactness of the operator holds by applying an averaging lemma. The definition of the operator implies that there is a unique f^L that solves the equation and whose macroscopic density is the fixed point, giving the existence.

The passage to the limit $L \to \infty$ after the profile has been centered can be carried out in the same way as above and the proof of Proposition 6.4, in particular, follows similarly.

The only thing left to check in order to conclude that Theorem 6.6 holds is the monotonicity of the profiles. By Lemma 6.5 we only need to prove that if f is a traveling wave solution which satisfies the far field conditions in the sense of Proposition 6.4, then $\int_V w(v) f \, dv$ is monotone increasing for a positive function w. In this case we can take $w(v) \equiv (v-s)^2$; we let $g = f - \mathcal{M}(U_-, v)$ and multiply the equation satisfied by g by $(v-s)$. Integrating with respect to v yields

$$\partial_\xi \int_V (v-s)^2 g \, dv = J(U_g + U_-) - J(U_-) - sU_g - \int_V (v-s) g \, dv \,. \tag{6.29}$$

But integration with respect to v of the equation for g implies that $\int_V (v-s) g \, dv$ is constant, and further, after taking the limit along the sequences ξ_n and η_n, that $\int_V (v-s) g \, dv = 0$. Finally, by (6.1) and $U_- \leq U_f \leq U_+$,

$$\partial_\xi \int_V (v-s)^2 g \, dv = J(U_f) - J(U_-) - s(U_f - U_-) > 0 \,.$$

The proof of stability of the shock profiles can be readily adapted and we shall not comment further on it.

Fermions in a background medium

We now consider the kinetic model for fermion-phonon interaction in the presence of a large electric field E described in Section 2.4 and review the results from [5]. In one space dimension, numerical computations of $J(U)$ suggest that for $E \neq 0$ then $\operatorname{sign}(J''(U)) = \operatorname{sign}(E)$, see [5] and [19]. Let us assume without loss of generality that $E > 0$ and that $J''(U) > 0$.

Existence and the maximum principle are proved by noticing that $Q_s(f)$ can be split into a linear and a nonlinear part

$$Q_s(f) = \lambda_1(f) + \lambda_2(f) \, f \tag{6.30}$$

where the operators $\lambda_1(f)$ and $\lambda_2(f)$ are linear integral operators. The fixed point iteration is defined by solving at each step the linear equation

$$\varepsilon \partial_t f^{n+1} + \varepsilon v \partial_x f^{n+1} + E \, \partial_v f^{n+1} - \lambda_2(f^n) \, f^{n+1} = \lambda_1(f^n) \,, \tag{6.31}$$

with

$$f^0 = f_{init} \in L^1_v(\mathbb{R}; W^{1,1}_x(\mathbb{R}))$$

where (6.31) can be solved by the method of characteristics or by semigroup theory. Positivity of solutions f^n follows by observing that if $f \geq 0$ then $-\lambda_2(f) \geq 0$ and $\lambda_1(f) \geq 0$. That $f^n \leq 1$ for all n also follows, by writing the equation in terms of $g^n = 1 - f^n$ and using that $Q_s(1-g^n) = -Q_s(g^n)$ gives the equation (6.31) with f^{n+1} and f^n replaced by g^{n+1} and by g^n, respectively. So $g^n \geq 0$ for all n if $g^0 = 1 - f^0 \geq 0$.

Thus, the sequence f^n is uniformly bounded in L^∞ and there is a subsequence that converges weakly* to some $f \in L^\infty$. It has been shown, Mustieles [32], that the sequence

f^n converges strongly in $L^\infty([0,T]; L^1_{x,v}(\mathbb{R}^2))$, by deriving L^1 estimates of the form $\|f^{n+1} - f^n\|_{L^1} \le C\|f^n - f^{n-1}\|_{L^1}$, and the existence holds.

That (6.10) and (6.11) hold has been shown by Ben Abdallah, Chaker and Schmeiser [5]. We do not go into the proof here, we just remark that (6.11) holds for any convex entropy η, not only for $\eta = U^2/2$, namely

$$\int_{\mathbb{R}} (Q_s(f) - E\partial_v f)\chi(\zeta(f,v))\, dv \le -C \int_{\mathbb{R}} (f - F(\bar{U}_f))^2\, M(v)\, dv$$

holds for any strictly increasing function $\chi \in C^1$, with ζ, as before, defined by (6.2) and with $\bar{U}_f := \int \zeta(f,v)\, M(v)\, dv$. The inequality (6.12), and hence L^1-contraction, was proved by Poupaud [36].

To prove the existence of the slab problem (6.20)-(6.22) one can proceed as in [4] for the Milne problem. Using the iteration, analogous to (6.31),

$$(v-s)\partial_\xi f^{n+1} + E\partial_v f^{n+1} - \lambda_2(f^n)f^{n+1} = \lambda_1(f^n)\,,$$

for $E > 0$ the iteration procedure is started with $f^0 = \mathcal{M}(U_+, v)$, thus clearly $f^1 \le f^0$, the comparison principle implies that $f^{n+1} \le f^n$ for all n, thus the sequence defined by the iteration is decreasing, and $f^n \in L^1([-L,L]\times\Omega) \cap L^\infty([-L,L]\times\Omega)$. The existence of fixed points is achieved by passing to the limit $n \to \infty$; the convergence is strong in L^1 by the monotone convergence theorem, and weak* in L^∞, this allows passing to the limit in Q. An additional argument that uses the inequality (6.12) is needed to prove the uniqueness of the fixed point. For $E < 0$ the iteration procedure is started at $f^0 = \mathcal{M}(U_-, v)$ instead.

The rest of the existence proof now follows as in Section 6.1. We only remark that the monotonicity of f with respect to ξ is directly proved for U_f (here $w(v) = 1$), it requires a technical lemma that follows by analyzing the collision term, we refer to [5] for details.

The proof of stability follows the same lines. We only remark that the splitting of the operator (6.30) and averaging lemmas are used here to pass to the limit in the h equation (6.26).

Acknowledgments: The authors acknowledge support by the Austrian Science Fund under grant numbers W8 and P18367. The first author has also been supported by a EPSRC Postdoctoral Research Fellowship.

References

[1] L. Arkeryd, C. Cercignani, and R. Illner. Measure solutions of the steady Boltzmann equation in a slab. *Comm. Math. Phys.*, 142:285–296, 1991.

[2] L. Arkeryd and A. Nouri. A compactness result related to the stationary Boltzmann equation in a slab, with applications to the existence theory. *Indiana Univ. Math. J.*, 44:815–839, 1995.

[3] N. Ben Abdallah and H. Chaker. The high field asymptotics for degenerate semiconductors. *Math. Models Methods Appl. Sci.*, 11:1253–1272, 2001.

[4] N. Ben Abdallah and H. Chaker. The high field asymptotics for degenerate semiconductors: initial and boundary layer analysis. *Asymptot. Anal.*, 37:143–174, 2004.

[5] N. Ben Abdallah, H. Chaker, and C. Schmeiser. The high field asymptotics for a fermionic Boltzmann equation: entropy solutions and kinetic shock profiles. *J. Hyperbolic Differ. Equ.*, 4:679–704, 2007.

[6] F. Berthelin and F. Bouchut. Relaxation to isentropic gas dynamics for a BGK system with single kinetic entropy. *Methods Appl. Anal.*, 9:313–327, 2002.

[7] P. L. Bhatnagar, E. P. Gross, and M. Krook. A model for collision processes in gases. i. small amplitude processes in charged and neutral one-component systems. *Phys. Rev.*, 94:511–525, 1954.

[8] S. Bianchini and A. Bressan. Vanishing viscosity solutions of nonlinear hyperbolic systems. *Ann. of Math. (2)*, 161:223–342, 2005.

[9] F. Bouchut. Construction of BGK models with a family of kinetic entropies for a given system of conservation laws. *J. Statist. Phys.*, 95:113–170, 1999.

[10] R. E. Caflisch. The fluid dynamic limit of the nonlinear Boltzmann equation. *CPAM*, 33:651–666, 1980.

[11] R. E. Caflisch and B. Nicolaenko. Shock profile solutions of the Boltzmann equation. *Comm. Math. Phys.*, 86:161–194, 1982.

[12] S. Chapman and T.G. Cowling. *The mathematical theory of nonuniform gases.* Cambridge University Press, Cambridge, 1952.

[13] M. G. Crandall and L. Tartar. Some relations between nonexpansive and order preserving mappings. *Proc. Amer. Math. Soc.*, 78:385–390, 1980.

[14] C. M. Cuesta. A note on L^1 stability of travelling waves for a one-dimensional BGK model. In *Hyperbolic Problems: Theory, Numerics, Applications, proceedings of the 11th international conference on hyperbolic problems*, pages 431–438. Springer, 2008.

[15] C. M. Cuesta, S. Hittmeir, and C. Schmeiser. Weak shocks of a BGK kinetic model relaxing to the isentropic system of gas dynamics. In preparation.

[16] C. M. Cuesta and C. Schmeiser. Long-time behaviour of a one-dimensional bgk model: convergence to macroscopic rarefaction waves. Submitted.

[17] C. M. Cuesta and C. Schmeiser. Weak shocks for a one-dimensional BGK kinetic model for conservation laws. *SIAM Journal on Mathematical Analisys.*, 38:637–656, 2006.

[18] C. M. Cuesta and C. Schmeiser. Kinetic profiles for shock waves of scalar conservation laws. *Bull. Inst. Math. Acad. Sin. (N.S.)*, 2:391–408, 2007.

[19] L. Derbel. Résolution numérique de la limite champ fort de l'équation de Boltzmann pour les semiconducteurs dégénérés. Master's thesis, ENIT, Tunis, 2000.

[20] R. J. DiPerna. Convergence of approximate solutions to conservation laws. *Arch. Rational Mech. Anal.*, 82:27–70, 1983.

[21] J. Goodman. Nonlinear Asymptotic Stability of Viscous Shock Profiles for Conservation Laws. *Arch. Rational Mech. Anal.*, 95:325–344, 1986.

[22] F. Golse. Shock profiles for the Perthame-Tadmor kinetic model. *Comm. Partial Differential Equations*, 23:1857–1874, 1998.

[23] D. Hilbert. *Grundzüge einer allgemeinen Theorie der linearen Integralgleichungen.* Teubner, Leipzig, Berlin, 1912.

[24] S. Jin and Z. P. Xin. The relaxation schemes for systems of conservation laws in arbitrary space dimensions. *Comm. Pure Appl. Math.*, 48(3):235–276, 1995.

[25] S. Kawashima and A. Matsumura. Asymptotic stability of traveling wave solutions of systems of one-dimensional gas motion, *Commun. Math. Phys.*, 101:97–127, 1985.

[26] P. D. Lax. *Hyperbolic systems of conservation laws and the mathematical theory of shock waves.* Society for Industrial and Applied Mathematics, Philadelphia, Pa., 1973. Conference Board of the Mathematical Sciences Regional Conference Series in Applied Mathematics, No. 11.

[27] P.-L. Lions, B. Perthame, and E. Tadmor. Kinetic formulation of the isentropic gas dynamics and p-systems. *Comm. Math. Phys.*, 163:415–431, 1994.

[28] T.-P. Liu and S.-H. Yu. Boltzmann equation: micro-macro decompositions and positivity of shock profiles. *Comm. Math. Phys.*, 246:133–179, 2004.

[29] A. Majda and R. L. Pego. Stable viscosity matrices for systems of conservation laws. *J. Differential Equations*, 56(2):229–262, 1985.

[30] P. A. Markowich, C. A. Ringhofer, and C. Schmeiser. *Semiconductor equations.* Springer-Verlag, Vienna, 1990.

[31] C. Mascia and R. Natalini. L^1 nonlinear stability of traveling waves for a hyperbolic system with relaxation. *J. Differential Equations*, 132:275–292, 1996.

[32] F.-J. Mustieles. Global existence of solutions of the nonlinear Boltzmann equation of semiconductor physics. *Rev. Mat. Iberoamericana*, 6(1-2):43–59, 1990.

[33] R. Natalini. A discrete kinetic approximation of entropy solutions to multidimensional scalar conservation laws. *J. Differential Equations*, 148:292–317, 1998.

[34] B. Perthame. *Kinetic formulation of conservation laws*, volume 21 of *Oxford Lecture Series in Mathematics and its Applications*. Oxford University Press, Oxford, 2002.

[35] B. Perthame and E. Tadmor. A kinetic equation with kinetic entropy functions for scalar conservation laws. *Comm. Math. Phys.*, 136:501–517, 1991.

[36] F. Poupaud. A half-space problem for a nonlinear Boltzmann equation arising in semiconductor statistics. *Math. Methods Appl. Sci.*, 14:121–137, 1991.

[37] F. Poupaud. Runaway phenomena and fluid approximation under high fields in semiconductor kinetic theory. *Z. Angew. Math. Mech.*, 72:359–372, 1992.

[38] S. Ukai. Stationary solutions of the BGK model equation on a finite interval with large boundary data. In *Proceedings of the Fourth International Workshop on Mathematical Aspects of Fluid and Plasma Dynamics (Kyoto, 1991)*, volume 21, pages 487–500, 1992.