High Field Approximations of the Spherical Harmonics Expansion Model for Semiconductors

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Abstract. We present an asymptotic analysis (with the scaled mean free path as small parameter) of the spherical-harmonics expansion (SHE-) model for semiconductors in the case of a large electric field. The Hilbert and Chapman-Enskog expansions are performed and the dependence of macroscopic parameter-functions such as the mobility and the diffusivity on the details of the considered elastic and inelastic scattering processes are investigated.

For example, we verify so called velocity-saturation mobility models, so far obtained by heuristic considerations, by means of an asymptotic analysis for certain scattering processes.

Key words: Semiconductors, spherical-harmonics expansion model, high field asymptotics, Chapman-Enskog expansion

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1 Introduction

The accurate modelling of high field effects is an important task in the simulation of charge carrier flow in semiconductors. A standard and commonly used approach is the inclusion of field dependent transport parameters in macroscopic models. For example, the use of field dependent mobilities in the standard drift-diffusion (DD-) model [14] is a classical approach. The detailed form of the field dependence is usually obtained by a combination of heuristic arguments combined with fitting to experimental values or to the results of Monte-Carlo simulations of the semiconductor Boltzmann equation [6].

A more rigorous approach has also been considered. With the classical one it shares the assumption that the semiconductor Boltzmann equation correctly represents the physics. By the Hilbert expansion method (assuming smallness of the scaled mean free path), macroscopic models can be derived from this kinetic equation. In particular, the low field DD-model with field independent transport parameters has been justified in this way [5]. The inclusion of high field effects makes the problem significantly more difficult. Only with unrealistic assumptions on the scattering mechanisms, a DD-model with field dependent mobility has been derived in [11]. There the DD-model is the result of a two-step procedure. First, by a limit process an equation without a diffusion term is derived. Then the diffusion term (with a field dependent diffusion constant) is obtained as a higher order correction by a variant of the Chapman-Enskog method.

In physically accurate versions of the semiconductor Boltzmann equation, scattering effects of different types and of different orders of magnitude have to be considered. Recently, the Hilbert expansion method has been applied to different (low field) situations resulting in a hierarchy of macroscopic models, lying in the gap between the full Boltzmann equation and the DD-model [1]. Examples are the energy transport model (see also [2]) and the spherical harmonics epansion (SHE-) model. The SHE-model can be derived either by expansion of the distribution function in terms of spherical harmonics [15], [16] or by a Hilbert expansion assuming that elastic collisions are the dominating physical effect [1], [4], [13]. The latter approach seems more attractive since it can be easily applied to non-rotationally-symmetric band structures. The unknown in the SHE-model is a distribution function depending on the wave vector only through the energy. In a second limit procedure with dominating inelastic scattering, the DD-model can be derived from the SHE-model.

In this work the latter limit is considered, however, for a high field situation where the effects of the driving field balance the dominating inelastic collision mechanisms. In section 2 the SHE-model is presented and a scaling is introduced. In section 3 the Chapman-Enskog expansion is carried out as far as possible without specifying the dominating inelastic collision operator. The resulting equation is structurally similar to that of [11]. The limiting equation is a (first order) convection equation for the macroscopic density with field dependent mobility. By the Chapman-Enskog procedure a (second order) correction is constructed. Sections 4, 5, and 6 deal with three specific choices of the inelastic collision term.

In section 4 a model for collisions with phonons of small energy is considered. The model has been derived in [13] by simultaneously letting the scaled mean free path of elastic collisions and the scaled phonon energy tend to zero. For this model the field dependent mobility can be computed explicitly. It coincides with heuristically derived models for velocity saturation. Also the Chapman-Enskog correction is computed and shown that the resulting equation for the density is parabolic.

A relaxation time model is considered in section 5. The existence of the field dependent mobility is proven and its asymptotic behaviour for large fields is examined. In this case the mean velocity does not saturate in general.

Finally, section 6 deals with a model for phonon scattering with finite phonon energy. Here, a rigorous existence result for the field dependent mobility is still missing. However, its asymptotic behaviour for large and small fields is analyzed formally. The large field behaviour is as in section 4. Because of the properties of the phonon scattering operator, the low field limit is nontrivial. In particular, the low field equilibrium distribution is not the Maxwellian as in sections 4 and 5.

2 The SHE-Model — Scaling

The SHE-model has the form [1]

$$N(\varepsilon)\frac{\partial F}{\partial t} - \left(\nabla_x - q\mathcal{E}\frac{\partial}{\partial \varepsilon}\right)\cdot\left[D(\varepsilon)\left(\nabla_x F - q\mathcal{E}\frac{\partial F}{\partial \varepsilon}\right)\right] = Q(F)\,, \qquad (2.1)$$

where the unknown $F(x,\varepsilon,t)$ is the electron distribution function depending on position $x \in \mathbb{R}^3$, particle kinetic energy $\varepsilon \in \mathbb{R}$, and time $t \in \mathbb{R}$. The elementary charge is denoted by q and $\mathcal{E}(x,t) \in \mathbb{R}^3$ is the electric field (here considered given).

The range of the energy variable is determined by the band structure of the material:

$$\varepsilon \in \mathcal{R}(\varepsilon_c) := \{\varepsilon_c(k) : \ k \in B\}, \tag{2.2}$$

where $B \subset \mathbb{R}^3$ denotes the Brillouin zone, $k \in B$ is the particle momentum and $\varepsilon_c(k)$ the conduction band diagram. The Brillouin zone is the elementary cell of the dual L^* of the crystal lattice, and ε_c is assumed in $C^1(\mathbb{R}^3)$, L^*-periodic and symmetric with respect to reflections: $\varepsilon_c(-k) = \varepsilon_c(k)$. The density of states is then given by

$$N(\varepsilon) = \frac{1}{4\pi^3} \int_B \delta(\varepsilon_c(k) - \varepsilon) dk \, . \tag{2.3}$$

We assume that $N(\varepsilon)$ is continuous on $\mathcal{R}(\varepsilon_c)$ and that it vanishes on $\partial\mathcal{R}(\varepsilon_c)$. (This is a further assumption on the band diagram.) With the help of the density of states, the macroscopic particle density is given by

$$n(x,t) = \int_{\mathcal{R}(\varepsilon_c)} F(x,\varepsilon,t) N(\varepsilon) d\varepsilon \, .$$

The diffusivity tensor $D(\varepsilon) \in \mathbb{R}^{3\times 3}$ depends on the details of the elastic collision mechanisms underlying the derivation of the SHE-model. If those are described by the collision operator

$$\left(\frac{\partial f}{\partial t}\right)_{el} = Q_{el}(f) \, ,$$

acting on distribution functions $f(k)$, $k \in B$, then a vector $\lambda(k)$ is defined as a solution of the equation

$$Q_{el}(\lambda) = -\nabla_k \varepsilon_c \, ,$$

and the diffusivity is given by [1]

$$D(\varepsilon) = \frac{1}{4\pi^3 \hbar^2} \int_B \delta(\varepsilon_c(k) - \varepsilon) \nabla_k \varepsilon_c \otimes \lambda \, dk \, ,$$

with the reduced Planck constant $\hbar$. It is an important property of the SHE-model that D vanishes on $\partial\mathcal{R}(\varepsilon_c)$ and that the parabolic modes degenerate there. As a consequence, no boundary conditions for F are needed on this part of the boundary.

We require that the solution is regular enough for the flux $J = -D(\nabla_x F - q\mathcal{E}\frac{\partial F}{\partial \varepsilon})$ to vanish on the energy boundary:

$$D(\nabla_x F - q\mathcal{E}\frac{\partial F}{\partial \varepsilon}) = 0 \qquad \text{on } \partial\mathcal{R}(\varepsilon_c)\,. \tag{2.4}$$

This is necessary for the validity of the macroscopic continuity equation

$$\frac{\partial n}{\partial t} + \nabla_x \cdot \int_{\mathcal{R}(\varepsilon_c)} J d\varepsilon = 0\,.$$

Finally, $Q(F)$ results from the inelastic collision mechanism

$$\left(\frac{\partial f}{\partial t}\right)_{inel} = Q_{inel}(f)\,,$$

and is given by

$$Q(F)(\varepsilon) = \frac{1}{4\pi^3}\int_B \delta(\varepsilon_c(k) - \varepsilon) Q_{inel}(F(\varepsilon_c))(k) dk\,. \tag{2.5}$$

An a-priori splitting of the collision mechanism into a dominating elastic one and into inelastic ones is fundamental for the validity of the SHE-model.

We assume particle conservation

$$\int_{\mathcal{R}(\varepsilon_c)} Q(F) d\varepsilon = 0\,,$$

but leave details of the collision operator unspecified until sections 4 and 5.

For a scaling of (2.1) we start out by choosing a reference energy ε_0. Then a reference value k_0 for the modulus of wave vectors is determined from the requirement that the scaled band diagram ε_{cs} defined by

$$\varepsilon_c(k) = \varepsilon_0 \varepsilon_{cs}(k/k_0)$$

takes moderate values on B/k_0. It is then reasonable to use the reference value

$$N_0 = \frac{k_0^3}{4\pi^3 \varepsilon_0}$$

for the density of states. Reference values for λ and the diffusivity are chosen as

$$\lambda_0 = \frac{\tau_{el}\varepsilon_0}{k_0}, \qquad D_0 = \frac{\tau_{el}\varepsilon_0 k_0}{4\pi^3\hbar^2},$$

where τ_{el} is a typical relaxation time for the elastic collisions. Reference values for the field and for the collision operator in the SHE-equation are given by

$$E_0 = \frac{\varepsilon_0}{qL_0}, \qquad Q_0 = \frac{k_0^3}{4\pi^3\tau_{inel}\varepsilon_0},$$

where the reference length L_0 is determined together with the reference time by balancing coefficients in (2.1):

$$L_0 = \frac{\varepsilon_0}{\hbar k_0}\sqrt{\tau_{el}\tau_{inel}}, \qquad t_0 = \tau_{inel}.$$

The accordingly scaled version of (2.1) reads

$$N\frac{\partial F}{\partial t} - \left(\nabla_x - \mathcal{E}\frac{\partial}{\partial \varepsilon}\right)\cdot\left[D\left(\nabla_x F - \mathcal{E}\frac{\partial F}{\partial \varepsilon}\right)\right] = Q(F), \tag{2.6}$$

with

$$N(\varepsilon) = \int_B \delta(\varepsilon_c(k) - \varepsilon)dk, \qquad D(\varepsilon) = \int_B \delta(\varepsilon_c(k) - \varepsilon)\nabla_k\varepsilon_c \otimes \lambda\, dk, \tag{2.7}$$

where the same symbols have been used for scaled and unscaled quantities, in particular ε_c in (2.7) actually stands for ε_{cs}. In sections 4 and 5 we shall investigate various specializations of the model.

We want to investigate the dynamics of (2.6) at macroscopic scales. Denoting by $\alpha \ll 1$ the parameter which sets the ratio of the microscopic to the macroscopic scale, we introduce the rescaling

$$x \to \frac{x}{\alpha}, \qquad t \to \frac{t}{\alpha}.$$

While doing this, we let $\mathcal{E}$ unchanged, which means that we assume the potential to have variations of order 1 over the microscopic scale. Therefore, we are looking for a macroscopic limit at high fields. The rescaled version of (2.6) reads

$$\alpha N\frac{\partial F}{\partial t} - \left(\alpha\nabla_x - \mathcal{E}\frac{\partial}{\partial \varepsilon}\right)\cdot\left[D\left(\alpha\nabla_x F - \mathcal{E}\frac{\partial F}{\partial \varepsilon}\right)\right] = Q(F). \tag{2.8}$$

3 Chapman-Enskog Expansion

In this section an asymptotic expansion of (2.8) is carried out corresponding to the limit $\alpha \to 0$. Computing the derivatives in (2.8), we obtain

$$\alpha N \frac{\partial F}{\partial t} - \alpha^2 \nabla_x \cdot (D \nabla_x F) \quad + \quad \alpha \nabla_x \cdot \left(D\mathcal{E} \frac{\partial F}{\partial \varepsilon} \right) + \alpha \mathcal{E} \cdot \frac{\partial}{\partial \varepsilon} (D \nabla_x F) = Q_{\mathcal{E}}(F) \,, \quad (3.1)$$

with

$$Q_{\mathcal{E}}(F) = \frac{\partial}{\partial \varepsilon} \left(\mathcal{E}^{tr} D \mathcal{E} \frac{\partial F}{\partial \varepsilon} \right) + Q(F) \,.$$

Here and in the sequel the superscript 'tr' denotes transposition.

Passing to the limit $\alpha \to 0$, we have to investigate the equation $Q_{\mathcal{E}}(F) = 0$.

Hypothesis 1: *The kernel of $Q_{\mathcal{E}}$ is one-dimensional and spanned by a function $M_{\mathcal{E}}(\varepsilon) \geq 0$ with*

$$\int_{\mathcal{R}(\varepsilon_c)} N(\varepsilon) M_{\mathcal{E}}(\varepsilon) d\varepsilon = 1 \,. \quad (3.2)$$

For carrying out the Chapman-Enskog expansion we shall also need to assume the unique solvability of inhomogeneous equations of the form

$$Q_{\mathcal{E}}(F) = g \,, \qquad \text{subject to } \int_{\mathcal{R}(\varepsilon_c)} NF \, d\varepsilon = 0 \,, \quad (3.3)$$

for certain inhomogeneities g, which have to satisfy the solvability condition $\int_{\mathcal{R}(\varepsilon_c)} g \, d\varepsilon = 0$.

Note that this condition is a straightforward consequence of the conservation property of Q and of the fact that D vanishes on $\partial \mathcal{R}(\varepsilon_c)$. In the following sections this hypothesis will be verified for two examples of $Q(F)$.

The distribution function is now decomposed into

$$F(x, \varepsilon, t) = n(x,t) M_{\mathcal{E}(x,t)}(\varepsilon) + \alpha F^\perp(x, \varepsilon, t) \quad (3.4)$$

with

$$n = \int_{\mathcal{R}(\varepsilon_c)} NF d\varepsilon \,, \qquad \int_{\mathcal{R}(\varepsilon_c)} NF^\perp d\varepsilon = 0 \,,$$

and, formally,

$$n = n_0 + O(\alpha), \quad F^\perp = F_0^\perp + O(\alpha) \quad \text{as} \quad \alpha \to 0.$$

Substitution of (3.4) into the SHE-equation (3.1) gives (after division by α):

$$\begin{aligned} &N\frac{\partial(nM_\mathcal{E})}{\partial t} + \alpha N\frac{\partial F^\perp}{\partial t} - \alpha\nabla_x \cdot (D\nabla_x(nM_\mathcal{E})) + \nabla_x \cdot (D\mathcal{E}n\frac{\partial M_\mathcal{E}}{\partial \varepsilon}) \\ &+\alpha\nabla_x \cdot (D\mathcal{E}\frac{\partial F^\perp}{\partial \varepsilon}) + \mathcal{E} \cdot \frac{\partial}{\partial \varepsilon}(D\nabla_x(nM_\mathcal{E})) + \alpha\mathcal{E}\frac{\partial}{\partial \varepsilon}(D\nabla_x F^\perp) + O(\alpha^2) \\ &= Q_\mathcal{E}(F^\perp)\,. \end{aligned} \tag{3.5}$$

We integrate with respect to ε and obtain the continuity equation

$$\frac{\partial n}{\partial t} - \nabla_x(n\mu^0(\mathcal{E})\mathcal{E}) - \alpha\nabla_x \cdot \int_{\mathcal{R}(\varepsilon_c)} D(\nabla_x(nM_\mathcal{E}) - \mathcal{E}\frac{\partial F^\perp}{\partial \varepsilon})d\varepsilon = O(\alpha^2) \tag{3.6}$$

with the mobility tensor in leading order

$$\mu^0(\mathcal{E}) = -\int_{\mathcal{R}(\varepsilon_c)} D\frac{\partial M_\mathcal{E}}{\partial \varepsilon}d\varepsilon\,. \tag{3.7}$$

In first order we thus derive the convection equation

$$\frac{\partial n_0}{\partial t} - \nabla_x \cdot (n_0\mu^0(\mathcal{E})\mathcal{E}) = 0 \tag{3.8}$$

with the particle velocity

$$v^0(\mathcal{E}) = -\mu^0(\mathcal{E})\mathcal{E}\,. \tag{3.9}$$

In order to obtain an $O(\alpha^2)$-approximation of the continuity equation (3.6) it is clearly sufficient to compute an $O(\alpha)$-approximation $F^\perp$ from (3.5):

$$Q_\mathcal{E}(F^\perp) = N\frac{\partial(nM_\mathcal{E})}{\partial t} + \nabla_x \cdot (D\mathcal{E}n\frac{\partial M_\mathcal{E}}{\partial \varepsilon}) + \mathcal{E} \cdot \frac{\partial}{\partial \varepsilon}(D\nabla_x(nM_\mathcal{E})) + O(\alpha).$$

Now we neglect the $O(\alpha)$-term, use (3.6) in the form

$$\frac{\partial n}{\partial t} - \nabla_x \cdot (n\mu^0(\mathcal{E})\mathcal{E}) = O(\alpha)$$

and compute the $O(\alpha)$-approximation $\overline{F_0}^\perp$ from

$$Q_{\mathcal{E}}(\overline{F_0}^\perp) = Nn\frac{\partial M_{\mathcal{E}}}{\partial t} + \nabla_x \cdot (nD\mathcal{E}\frac{\partial M_{\mathcal{E}}}{\partial \varepsilon}) + NM_{\mathcal{E}}\nabla_x \cdot (n\mu^0(\mathcal{E})\mathcal{E}) + \mathcal{E}\cdot\frac{\partial}{\partial \varepsilon}(D\nabla_x(nM_{\mathcal{E}}))\,. \quad (3.10)$$

The $O(\alpha^2)$-Chapman-Enskog expansion of the SHE-model is now obtained by solving (3.10) for $\overline{F_0}^\perp$ and replacing $F^\perp$ in (3.6) by $\overline{F_0}^\perp$.

We develop the vector expressions using Einstein's summation convention. After simple computations, we obtain:

$$Q_{\mathcal{E}}(\overline{F_0}^\perp) = u_{ij}\mathcal{E}_j\frac{\partial n}{\partial x_i} + n\left(y_{ijk}\mathcal{E}_j\frac{\partial \mathcal{E}_k}{\partial x_i} + w_k\frac{\partial \mathcal{E}_k}{\partial t} + z_{ij}\frac{\partial \mathcal{E}_j}{\partial x_i}\right),$$

with

$$\begin{aligned} z_{ij}(\varepsilon) &= \mu^0_{ij}NM_{\mathcal{E}} + D_{ij}\frac{\partial M_{\mathcal{E}}}{\partial \varepsilon}, \\ y_{ijk}(\varepsilon) &= \frac{\partial \mu^0_{ij}}{\partial \mathcal{E}_k}NM_{\mathcal{E}} + D_{ij}\frac{\partial^2 M_{\mathcal{E}}}{\partial\varepsilon\partial \mathcal{E}_k} + \frac{\partial}{\partial\varepsilon}\left(D_{ij}\frac{\partial M_{\mathcal{E}}}{\partial \mathcal{E}_k}\right), \\ w_k(\varepsilon) &= \frac{\partial M_{\mathcal{E}}}{\partial \mathcal{E}_k}N\,, \\ u_{ij}(\varepsilon) &= z_{ij}(\varepsilon) + \frac{\partial}{\partial\varepsilon}(D_{ij}M_{\mathcal{E}})\,. \end{aligned}$$

Each of the functions z_{ij}, y_{ijk}, w_k and u_{ij} satisfies the necessary condition $\int g d\varepsilon = 0$ for the solvability of the equation $Q(F) = g$ separately. Indeed, for z_{ij}, as well as for the first two terms of y_{ijk}, this follows from the definition of μ^0. For the last terms of u_{ij} and y_{ijk}, it is a consequence of the fact that D vanishes on $\partial\mathcal{R}(\varepsilon_c)$. Finally, the solvability condition for w_k is obtained by differentiation of the normalization condition (3.2) with respect to $\mathcal{E}_k$. At this point we assume the existence of unique solutions of (3.3) for the right hand sides $g = z_{ij}$, y_{ijk}, w_k, and u_{ij}; we denote these solutions by Z_{ij}, Y_{ijk}, W_k, and U_{ij}, respectively. We can write:

$$\overline{F_0}^\perp = U_{ij}\mathcal{E}_j\frac{\partial n}{\partial x_i} + n\left(Y_{ijk}\mathcal{E}_j\frac{\partial \mathcal{E}_k}{\partial x_i} + W_k\frac{\partial \mathcal{E}_k}{\partial t} + Z_{ij}\frac{\partial \mathcal{E}_j}{\partial x_i}\right).$$

Then, we define

$$\mathcal{U}^{\ell m}_{ij} = -\int_{\mathcal{R}(\varepsilon_c)} D_{\ell m}\frac{\partial U_{ij}}{\partial \varepsilon}d\varepsilon\,,$$

and $\mathcal{Y}_{ijk}^{\ell m}$, $\mathcal{W}_k^{\ell m}$ and $\mathcal{Z}_{ij}^{\ell m}$ analogously. The $O(\alpha^2)$-approximation of the continuity equation (3.6) now is:

$$\frac{\partial n}{\partial t} - \nabla_x \cdot (n\mu^\alpha[\mathcal{E}]\mathcal{E}) - \alpha\nabla_x \cdot (n\tilde{\mu}(\mathcal{E})\nabla_x\mathcal{E}) - \alpha\nabla_x \cdot (\overline{D}(\mathcal{E})\nabla_x n) = 0\,, \quad (3.11)$$

with the first order mobility μ^α, the diffusion matrix $\overline{D}(\mathcal{E})$, and the higher rank mobility (i.e. operating on the derivative of the field) $\tilde{\mu}(\mathcal{E})$ given by

$$\begin{aligned}
\mu^\alpha[\mathcal{E}] &= \mu^0(\mathcal{E}) + \alpha\mu^1[\mathcal{E}]\,, && (3.12)\\
\mu^1_{\ell m}[\mathcal{E}] &= \mathcal{Y}_{ijk}^{\ell m}\mathcal{E}_j\frac{\partial \mathcal{E}_k}{\partial x_i} + \mathcal{W}_k^{\ell m}\frac{\partial \mathcal{E}_k}{\partial t} + \mathcal{Z}_{ij}^{\ell m}\frac{\partial \mathcal{E}_j}{\partial x_i}\,, &&\\
\overline{D}_{\ell i}(\mathcal{E}) &= \mathcal{U}_{ij}^{\ell m}\mathcal{E}_j\mathcal{E}_m + \int_{\mathcal{R}(\varepsilon_c)} D_{\ell i}M_{\mathcal{E}}\,d\varepsilon\,, &&\\
(\tilde{\mu}(\mathcal{E})\nabla_x\mathcal{E})_\ell &= \tilde{\mu}_{\ell k m}\frac{\partial \mathcal{E}_k}{\partial x_m}\,, && (3.13)\\
\tilde{\mu}_{\ell k m}(\mathcal{E}) &= \int_{\mathcal{R}(\varepsilon_c)} D_{\ell m}\frac{\partial M_{\mathcal{E}}}{\partial \mathcal{E}_k}\,d\varepsilon\,. &&
\end{aligned}$$

The first-order drift velocity v^α,

$$v^\alpha = \mu^\alpha[\mathcal{E}]\mathcal{E} + \tilde{\mu}(\mathcal{E})\nabla_x\mathcal{E}\,,$$

has two contributions: the first one, $\mu^\alpha[\mathcal{E}]\mathcal{E}$, is the usual drift 'in the direction of the field', the second one, $\tilde{\mu}\nabla_x\mathcal{E}$, is a drift 'in the directions of the field gradients' (actually the last statement is rigorously true only for the rotationally symmetric case below). The first order mobility μ^α depends on the electric field and its derivatives (which is indicated by the brackets). The higher rank mobility $\tilde{\mu}$ maps the tensor $\nabla_x\mathcal{E}$ to vectors, which appears in formula (3.13). Note that, if the field gradients are $O(\alpha)$, the continuity equation (3.11) simplifies into a usual drift-diffusion equation:

$$\frac{\partial n}{\partial t} - \nabla_x \cdot (n\mu^0(\mathcal{E})\mathcal{E}) - \alpha\nabla_x \cdot (\overline{D}(\mathcal{E})\nabla_x n) = 0\,. \quad (3.14)$$

An open question in general is the positive-definiteness of the diffusion tensor $\overline{D}$. It will be proven in a particular case below.

The continuity equation (3.11) simplifies if the SHE diffusion matrix D is a scalar (which for instance happens for spherically symmetric band diagrams and rotationnally invariant collision operators). Then, the operator $Q_{\mathcal{E}}$, its equilibrium distribution $M_{\mathcal{E}}$, and the zero-th order mobility μ^0 only depend on the magnitude of the electric field $E = |\mathcal{E}|$. In this case, we denote $M_E = M_{\mathcal{E}}$. We compute

$$Q_{\mathcal{E}}(F_0^{\perp}) = u(\mathcal{E}\cdot\nabla_x)n + n\left(y(e\cdot\nabla_x)\frac{E^2}{2} + w\frac{1}{E}\frac{\partial}{\partial t}\frac{E^2}{2} + z\nabla_x\cdot\mathcal{E}\right), \quad (3.15)$$

with

$$\begin{aligned} z(\varepsilon) &= \mu^0 N M_E + D\frac{\partial M_E}{\partial\varepsilon}, & (3.16)\\ y(\varepsilon) &= \frac{\partial\mu^0}{\partial E}NM_E + D\frac{\partial^2 M_E}{\partial\varepsilon\partial E} + \frac{\partial}{\partial\varepsilon}\left(D\frac{\partial M_E}{\partial E}\right), \\ w(\varepsilon) &= \frac{\partial M_E}{\partial E}N, \\ u(\varepsilon) &= z + \frac{\partial}{\partial\varepsilon}(DM_E), & (3.17)\end{aligned}$$

and $e = \mathcal{E}/E$. Again, denote by U, Y, W and Z the solutions of problem (3.3) associated with the right-hand sides u, y, w, and z, and by $\mathcal{U}$, $\mathcal{Y}$,$\mathcal{W}$ and $\mathcal{Z}$, the integrals

$$\mathcal{U} = -\int_{\mathcal{R}(\varepsilon_c)} D\frac{\partial U}{\partial\varepsilon}d\varepsilon,$$

and so on. The continuity equation (3.11) is now

$$\begin{aligned}\frac{\partial n}{\partial t} - \nabla_x\cdot(n\mu^\alpha[\mathcal{E}]\mathcal{E}) \quad &- \quad \alpha\nabla_x\cdot\left(n\tilde{\mu}(\mathcal{E})\frac{1}{E}\nabla_x\frac{E^2}{2}\right) \\ &-\alpha\nabla_x\cdot(\overline{D}(\mathcal{E})\nabla_x n) = 0, \qquad (3.18)\end{aligned}$$

with μ^α, $\overline{D}$, and $\tilde{\mu}$ given by (3.12) and

$$\begin{aligned}\mu^1 &= \mathcal{Y}(e\cdot\nabla_x)\frac{E^2}{2} + \mathcal{W}\frac{1}{E}\frac{\partial}{\partial t}\frac{E^2}{2} + \mathcal{Z}\nabla_x\cdot\mathcal{E}, \\ \overline{D} &= \mathcal{U}\,\mathcal{E}\otimes\mathcal{E} + \int_{\mathcal{R}(\varepsilon_c)} DM_E d\varepsilon\,\mathrm{Id}, \\ \tilde{\mu}(\mathcal{E}) &= \int_{\mathcal{R}(\varepsilon_c)} D\frac{\partial M_E}{\partial E}d\varepsilon.\end{aligned}$$

In this case, the drift is a combination of two drifts, one parallel to the field and the other parallel to the gradient of its magnitude.

4 Interaction with phonons of small energy

We apply the above procedure to a collision operator derived in [13], [12] as an approximation for the interaction of electrons with phonons of small energy.

The scaled SHE collision operator Q deduced there is given by:

$$Q(F) = \frac{\partial}{\partial \varepsilon}\left[S(\varepsilon)\left(\frac{\partial F}{\partial \varepsilon} + F\right)\right], \tag{4.1}$$

with

$$S(\varepsilon) = \int_B \int_B \delta(\varepsilon_c(k) - \varepsilon)\delta(\varepsilon_c(k') - \varepsilon)\Phi_s(k, k')dk'dk$$

Here $\Phi_s(k, k')$ stands for the appropriately scaled transition matrix (cf. [12] and section 2).

In the following we shall deal with several examples sharing two basic assumptions. The first assumption is that Q_{el} is a *relaxation time operator* with

$$\Phi_s(k, k') = \Phi(\varepsilon_c(k)) .$$

Then,

$$S(\varepsilon) = \Phi(\varepsilon)N(\varepsilon)^2 ,$$

where N is the density-of-states (2.3) and the SHE-diffusion matrix D is given by

$$D(\varepsilon) = \frac{3}{4(2\pi)^{2/3}\Phi(\varepsilon)N(\varepsilon)} \int_B \delta(\varepsilon_c(k) - \varepsilon)\nabla_k\varepsilon_c \otimes \nabla_k\varepsilon_c dk .$$

The diffusion matrix still depends on the properties of the band diagram.

The second assumption is a *spherically symmetric band diagram*: We suppose that B is a ball, possibly with infinite radius and

$$\varepsilon_c(k) = \overline{\varepsilon}_c(|k|^2) ,$$

with a strictly monotone function $\overline{\varepsilon}_c$. We denote by γ the inverse function of $\overline{\varepsilon_c}$, defined on $\mathcal{R}(\varepsilon_c)$:

$$|k|^2 = \gamma(\varepsilon_c(k)) ,$$

We obtain

$$N = 2\pi\sqrt{\gamma}\gamma' , \quad S = 4\pi^2\gamma(\gamma')^2\Phi , \quad D = (2\pi)^{2/3}\frac{\gamma}{(\gamma')^2\Phi}Id .$$

The examples mentioned above are

(i) Parabolic band diagram: We assume that the band diagram is that of a free particle

$$\gamma = \varepsilon/(2\pi)^{2/3}$$

with $\varepsilon \in \mathcal{R}(\varepsilon_c) = [0,\infty)$. Then,

$$N = \sqrt{\varepsilon} , \quad S = \varepsilon\Phi(\varepsilon) , \quad D = \frac{\varepsilon}{\Phi(\varepsilon)}Id .$$

Two subcases are particularly interesting:

(i-a) Lyumkis case [9]: $\Phi \equiv 1$. Then,

$$N = \sqrt{\varepsilon} , \quad S = \varepsilon , \quad D = \varepsilon Id , \tag{4.2}$$

(i-b) Chen case [3]: $\Phi = \sqrt{\varepsilon}$. Then,

$$N = \sqrt{\varepsilon} , \quad S = \varepsilon^{3/2} , \quad D = \sqrt{\varepsilon} Id , \tag{4.3}$$

(ii) Kane's band diagram [7]: This is a correction to the parabolic band approximation which accounts for non-parabolicity effects close to the bottom of the conduction band. It is given by

$$\gamma = \frac{1}{(2\pi)^{2/3}}\varepsilon(1+\alpha^*\varepsilon) , \quad \varepsilon \in [0,\infty) , \tag{4.4}$$

where α^* is the so-called 'non-parabolicity' coefficient. In this case,

$$\begin{aligned} N &= \sqrt{\varepsilon\,(1+\alpha^*\varepsilon)}\,(1+2\alpha^*\varepsilon) , \\ S &= \varepsilon\,(1+\alpha^*\varepsilon)\,(1+2\alpha^*\varepsilon)^2\Phi(\varepsilon) , \\ D &= \frac{\varepsilon(1+\alpha^*\varepsilon)}{(1+2\alpha^*\varepsilon)^2\Phi(\varepsilon)}Id . \end{aligned}$$

The coefficients have the same behaviour near the origin as in the parabolic case but their behaviour for large ε is very different.

(ii-a) Lyumkis case [9]: $\Phi \equiv 1$. Then, as $\varepsilon \to \infty$, we have

$$N \sim \varepsilon^2 , \quad S \sim 4(\alpha^*)^3\varepsilon^4 , \quad D \sim \frac{1}{4\alpha^*}Id .$$

(ii-b) Chen case [3]: $\Phi = \sqrt{\varepsilon}$, Then, as $\varepsilon \to \infty$, we have

$$N \sim \varepsilon^2\,, \quad S \sim 4(\alpha^*)^3\varepsilon^{9/2}\,, \quad D \sim \frac{1}{4\alpha^*}\frac{1}{\sqrt{\varepsilon}}Id\,.$$

Going back to the general case, we write

$$Q_{\mathcal{E}}(F) = (\mathcal{E}^{tr}D\mathcal{E}F')' + (S(F'+F))' = (D_{\mathcal{E}}F' + SF)'\,,$$

where primes denote derivatives with respect to ε and

$$D_{\mathcal{E}} = \mathcal{E}^{tr}D\mathcal{E} + S\,.$$

Hypothesis 2: *(i) The energy range $\mathcal{R}(\varepsilon_c)$ is an interval $[0,\varepsilon^*]$ with ε^* being finite or infinite.*
(ii) N, D, S and the entries of D are C^2 functions. N and S are positive, D is positive definite on $(0,\varepsilon^*)$, and N, S, and D vanish on $\partial\mathcal{R}(\varepsilon_c)$, i.e. at 0 if $\varepsilon^* = \infty$ and at 0 and ε^* if ε^* is finite.
(iii) If $\varepsilon^* = \infty$, the integral $I(\varepsilon) = \int_0^\varepsilon \frac{S}{D_{\mathcal{E}}}(u)du$ diverges as $\varepsilon \to \infty$. Furthermore the integral

$$J = \int_0^\infty \exp(-I(\varepsilon))N(\varepsilon)d\varepsilon\,, \tag{4.5}$$

converges.

These hypotheses match the physical requirements. The coefficients vanish on $\partial\mathcal{R}(\varepsilon_c)$ because this set is a set of critical values of the energy band diagram ε_c. The divergence of $I(\varepsilon)$ in the case $\varepsilon^* = \infty$ is a non-runaway hypothesis, which guarantees that the field will not drag too many particles to too large energies. This hypothesis is similar to the non-integrability of the scattering frequency required for the high-field diffusion approximation of the Boltzmann equation [11]. It is easy to check that these hypotheses are satisfied in the Chen and Lyumkis examples above, for both the parabolic and the Kane dispersion relations.

Assuming hypothesis 2, we can define

$$U(\varepsilon,\varepsilon') = \exp\left(-\int_\varepsilon^{\varepsilon'} \frac{S}{D_{\mathcal{E}}}(u)du\right)\,, \quad \varepsilon,\varepsilon' \in [0,\varepsilon^*]\,. \tag{4.6}$$

Theorem 4.1 *Suppose Hypothesis 2 is satisfied. Then the solutions F of*

$$Q_{\mathcal{E}}(F) = 0\,, \tag{4.7}$$

which are bounded on $[0,\varepsilon^*]$ *and such that* $D_{\mathcal{E}}F' \to 0$ *as* $\varepsilon \to 0$ *are given by* $F = nM_{\mathcal{E}}$ *with* $n \in \mathbb{R}$ *and*

$$M_{\mathcal{E}}(\varepsilon) = AU(0,\varepsilon)\,, \tag{4.8}$$

where A *is such that*

$$\int_0^{\varepsilon^*} M_{\mathcal{E}}(\varepsilon)N(\varepsilon)d\varepsilon = 1\,.$$

The requirement $D_{\mathcal{E}}F' \to 0$ as $\varepsilon \to 0$ is the energy boundary condition (2.4) at $\varepsilon = 0 \in \partial\mathcal{R}(\varepsilon_c)$. It is easily checked that the equilibrium distribution also satisfies $D_{\mathcal{E}}M'_{\mathcal{E}} \to 0$ as $\varepsilon \to \varepsilon^*$.

Proof: We solve $(D_{\mathcal{E}}F' + SF)' = 0$, or $D_{\mathcal{E}}F' + SF = C$, where C is a constant. At $\varepsilon = 0$, we require that F is bounded and that $D_{\mathcal{E}}F' \to 0$ as $\varepsilon \to 0$. This imposes the choice $C = 0$. Then F is given by (4.8) and it is an easy matter to check that it satifies all the requirements of the theorem. ∎

Example 4.1 *Parabolic band diagram, Lyumkis case (4.2):* In this case straightforward computations show that

$$M_{\mathcal{E}}(\varepsilon) = \frac{2}{\sqrt{\pi}}\frac{1}{T_E^{3/2}}e^{-\varepsilon/T_E}\,, \quad T_E = 1 + E^2\,. \tag{4.9}$$

The zeroth-order moblity μ^0 is given by

$$\mu^0(\mathcal{E}) = \frac{2}{\sqrt{\pi}\,T_E}\,. \tag{4.10}$$

The zeroth-order drift velocity $v^0(\mathcal{E})$ saturates at high field, a standard feature of mobility models in the literature [14]:

$$\lim_{E\to\infty} |v^0(\mathcal{E})| = \frac{2}{\sqrt{\pi}}\,.$$

Example 4.2 *Parabolic band diagram, Chen case (4.3):* In this case,

$$M_{\mathcal{E}}(\varepsilon) = Ae^{-\varepsilon}\left(1 + \frac{\varepsilon}{E^2}\right)^{E^2}\,.$$

and

$$A^{-1} = \sqrt{\pi}\, a(E^2)\,, \quad a(\kappa) = \int_0^\infty e^{-u}\left(1+\frac{u}{\kappa}\right)^\kappa \sqrt{u}\, du\,.$$

The mobility is given by

$$\mu^0(\mathcal{E}) = \frac{1}{\sqrt{\pi}}\frac{b(E^2)}{a(E^2)}\,,$$

with

$$b(\kappa) = \int_0^\infty e^{-u}\left(1+\frac{u}{\kappa}\right)^\kappa \frac{1}{2\sqrt{u}}\, du\,.$$

As E (and, thus, κ) tends to infinity and ε is kept fixed, the equilibrium distribution can be approximated by the Druyvenstein distribution

$$M_{\mathcal{E}}(\varepsilon) \sim \frac{2^{1/4}}{\Gamma(3/4)E^{3/2}} \exp\left(-\frac{\varepsilon^2}{2E^2}\right)\,,$$

and we have

$$a(\kappa) \sim \kappa^{3/4}\frac{\Gamma(3/4)}{2^{1/4}}\,, \qquad b(\kappa) \sim \kappa^{1/4}\frac{\Gamma(1/4)}{2^{7/4}}\,,$$

and therefore,

$$\mu^0(\mathcal{E}) \sim \frac{\Gamma(1/4)}{2^{3/2}\Gamma(3/4)\sqrt{\pi}}\frac{1}{E}\,.$$

Again, the velocity v^0 saturates as $E \to \infty$.

Example 4.3 *Kane's band diagram (4.4), Lyumkis case (4.2):* In this case,

$$\begin{aligned} M_{\mathcal{E}}(\varepsilon) &= Ae^{-\varepsilon}\left(\frac{u^2+\sqrt{2}u+1}{u^2-\sqrt{2}u+1}\right)^\sigma \\ &\quad \times \exp\left(2\sigma \arctan(\sqrt{2}\,u-1) + 2\sigma\arctan(\sqrt{2}\,u+1)\right)\,, \end{aligned}$$

with

$$u = \frac{1+2\alpha^*\varepsilon}{\sqrt{E}}\,, \qquad \sigma = \frac{\sqrt{E}}{8\sqrt{2}\,\alpha^*}\,.$$

A straightforward but lengthy asymptotic analysis for large fields gives

$$\begin{aligned} M_{\mathcal{E}}(\varepsilon) &\sim cE^{-6/5}\exp\left[-\frac{(2\alpha^*\varepsilon+1)^5}{10\alpha^*E^2}\right], \qquad \text{as } E\to\infty,\ \varepsilon \text{ fixed,} \\ \mu_0(\mathcal{E}) &\sim cE^{-6/5}, \end{aligned}$$

where c denotes (possibly different) constants depending only on α^*. Thus, the velocity v^0 decays as $E^{-1/5}$ after going through a maximum.

Example 4.4 *Kane's band diagram (4.4), Chen case (4.3):* The large field behaviour is now given by

$$\begin{aligned} M_{\mathcal{E}}(\varepsilon) &\sim cE^{-1}\exp\left[-\frac{(2\alpha^*\varepsilon+1)^5}{12\alpha^{*2}E^2}\left(\alpha^*\varepsilon-\frac{1}{10}\right)\right], \text{ as } E\to\infty,\ \varepsilon \text{ fixed,} \\ \mu_0(\mathcal{E}) &\sim cE^{-7/6}. \end{aligned}$$

The qualitative behaviour is as above with the velocity decay given by $E^{-1/6}$.

We note that for parabolic band diagrams, equilibria $M_{\mathcal{E}}$ have faster decay at large energy in the Chen case than in the Lyumkis case. In the Lyumkis case, D and S are comparable. Also the equilibrium $M_{\mathcal{E}}$ differs from the Maxwellian at lattice temperature over the whole energy range. Actually, the equilibrium is a Maxwellian with a field-modified temperature T_E. In the Chen case, D is negligable compared to S at large energy and the equilibrium is just the lattice Maxwellian with a polynomial weight, the degree of which increases as the field increases. The same comments hold for the Kane band diagram in both the Lyumkis and Chen cases.

We now investigate the solvability of inhomogeneous equations with the operator $Q_{\mathcal{E}}$. As for the selection of the equilibria, the regularity questions are important. We separately check the case ε^* finite and ε^* infinite.

Theorem 4.2 *Suppose that $\mathcal{R}(\varepsilon_c)=[0,\varepsilon^*]$ with ε^* finite and that Hypothesis 2 holds. Let g be a given integrable function on $[0,\varepsilon^*]$ such that the functions*

$$\frac{1}{D_{\mathcal{E}}(\varepsilon)}\int_0^{\varepsilon} g(u)du \quad \text{and} \quad \frac{1}{D_{\mathcal{E}}(\varepsilon)}\int_{\varepsilon}^{\varepsilon^*} g(u)du\,,$$

are integrable at $\varepsilon=0$ and ε^*, respectively. Then, the equation

$$Q_{\mathcal{E}}(F)=g\,, \tag{4.11}$$

admits a bounded solution F such that $D_{\mathcal{E}}F' \to 0$ as $\varepsilon \to \{0, \varepsilon^*\}$, if and only if

$$\int_0^{\varepsilon^*} g \, d\varepsilon = 0 \, .$$

This solution is unique under the additional constraint

$$\int_0^{\varepsilon^*} FN \, d\varepsilon = 0 \, .$$

Proof: Equation (4.11) is written

$$(D_{\mathcal{E}}F' + SF)' = g \, , \tag{4.12}$$

Integration of (4.12) gives

$$D_{\mathcal{E}}F' + SF = r + C \, , \tag{4.13}$$

where C is a constant and

$$r(\varepsilon) = \int_0^{\varepsilon} g(u) du \, .$$

For F to be bounded at 0 with $D_{\mathcal{E}}F' \to 0$ as $\varepsilon \to 0$, we obviously need to choose $C = 0$. Then, the general solution of (4.13) is

$$F(\varepsilon) = \int_0^{\varepsilon} U(u, \varepsilon) \left(\frac{r}{D_{\mathcal{E}}} \right) (u) du + C M_{\mathcal{E}} \, , \tag{4.14}$$

where U, defined in (4.6), is solution of the homogeneous equation, $M_{\mathcal{E}}$ is the equilibrium (4.8) and C is an arbitrary constant. We now examine under which condition the solution (4.14) satisfies the requirements of the theorem near $\varepsilon = \varepsilon^*$. Define

$$s(\varepsilon) = \int_{\varepsilon}^{\varepsilon^*} g(u) du \, , \quad G = \int_0^{\varepsilon^*} g(u) du = r + s \, . \tag{4.15}$$

We write, for an arbitrary $\varepsilon_0 \in (0, \varepsilon^*)$,

$$\begin{aligned} F(\varepsilon) \;\; &= \;\; - \int_{\varepsilon_0}^{\varepsilon} U(u, \varepsilon) \left(\frac{s}{D_{\mathcal{E}}} \right) (u) du \\ & \quad + G \int_{\varepsilon_0}^{\varepsilon} U(u, \varepsilon) \left(\frac{1}{D_{\mathcal{E}}} \right) (u) du + C' M_{\mathcal{E}} \, . \end{aligned} \tag{4.16}$$

The first term F_1 at the r.h.s. of (4.16) is obviously bounded and such that $D_{\mathcal{E}} F_1' \to 0$ as $\varepsilon \to \varepsilon^*$. The second term F_2 is a solution of the equation

$$D_{\mathcal{E}} F' + SF = G\,,$$

which satisfies these requirements only if $G = 0$. ∎

To handle the case $\varepsilon^* = \infty$, we need additional hypotheses:

Hypothesis 3: *(i) If $\varepsilon^* = \infty$, we assume that S is monotone in the neighbourhood of ∞, such that*

$$\frac{\partial}{\partial \varepsilon}\left(\frac{1}{S}\right)(\varepsilon) = o\Big(\frac{1}{D_{\mathcal{E}}}(\varepsilon)\Big)\,,$$

as $\varepsilon \to \infty$ and that the integral

$$\int_{\varepsilon_0}^{\infty} U(u, \varepsilon_0) \left| \frac{\partial}{\partial \varepsilon}\left(\frac{1}{S}\right)(u) \right| du$$

diverges.
(ii) In addition

$$\frac{\partial}{\partial \varepsilon}\left(\frac{N D_{\mathcal{E}}}{S}\right)(\varepsilon) = o(N(\varepsilon))\,,$$

and

$$\left(\frac{N D_{\mathcal{E}}}{S}\right)(\varepsilon)\, U(\varepsilon_0, \varepsilon) \to 0\,,$$

as $\varepsilon \to \infty$.

Theorem 4.3 *Suppose that $\mathcal{R}(\varepsilon_c) = [0, \infty]$ and that Hypotheses 2 and 3 (i) hold. Let g be an integrable function on $[0, \varepsilon^*]$ such that*

$$\frac{1}{D_{\mathcal{E}}(\varepsilon)} \int_0^{\varepsilon} g(u) du\,,$$

is integrable at 0. Then, the equation (4.11) admits a locally bounded solution F such that $D_{\mathcal{E}} F' \to 0$ as $\varepsilon \to 0$, $SF \to 0$ and $D_{\mathcal{E}} F' \to 0$ as $\varepsilon \to \infty$ if and only if

$$\int_0^{\infty} g d\varepsilon = 0\,.$$

If, in addition, hypothesis 3 (ii) is satisfied, and if

$$\left(\frac{N}{S}\right)(\varepsilon)\int_{\varepsilon}^{\infty} g(u)du$$

is integrable at ∞*, then,* $\int_0^\infty NF d\varepsilon$ *is finite and the solution is unique under the additional constraint*

$$\int_0^\infty FN d\varepsilon = 0\,.$$

Proof: The proof starts like that of theorem 4.2. To satisfy the requirements at $\varepsilon = 0$, the solution must be given by (4.14). We now investigate the behaviour of F at ∞. For $\varepsilon_0 > 0$ arbitrary, formula (4.16) is true provided that s and G are defined by (4.15) with $\varepsilon^* = \infty$. We have

$$\int_{\varepsilon_0}^{\varepsilon} U(u,\varepsilon)\left(\frac{1}{D_{\mathcal{E}}}\right)(u)du = U(\varepsilon_0,\varepsilon)\int_{\varepsilon_0}^{\varepsilon} U(u,\varepsilon_0)\left(\frac{1}{D_{\mathcal{E}}}\right)(u)du\,.$$

An integration by parts gives:

$$\begin{aligned}\int_{\varepsilon_0}^{\varepsilon} U(u,\varepsilon_0)\left(\frac{1}{D_{\mathcal{E}}}\right)(u)du &= \frac{1}{S(\varepsilon)}U(\varepsilon,\varepsilon_0) - \frac{1}{S(\varepsilon_0)} \\ &\quad - \int_{\varepsilon_0}^{\varepsilon} U(u,\varepsilon_0)\frac{\partial}{\partial\varepsilon}\left(\frac{1}{S}\right)(u)du\,,\end{aligned}$$

Because of hypothesis 3 (i) and classical comparison arguments, we have

$$\int_{\varepsilon_0}^{\varepsilon} U(u,\varepsilon_0)\frac{\partial}{\partial\varepsilon}\left(\frac{1}{S}\right)(u)du = o\left(\int_{\varepsilon_0}^{\varepsilon} U(u,\varepsilon_0)\left(\frac{1}{D_{\mathcal{E}}}\right)(u)du\right)\,,$$

and both integrals diverge as $\varepsilon \to \infty$. We deduce that F_2 (see definition in the proof of theorem 4.2) is such that $SF_2 \to G$ as $\varepsilon \to \infty$. Since $s \to 0$, we also deduce $SF_1 \to 0$ thus proving $SF \to 0$ if and only if $G = 0$. Then, we also have $D_{\mathcal{E}}F' = s - SF \to 0$, ending the existence part of the proof.

We now suppose that hypothesis 3(ii) is satisfied and investigate the integrability of NF, which is that of

$$K = \int_{\varepsilon_0}^{\infty} N(\varepsilon)\left(\int_{\varepsilon_0}^{\varepsilon} U(u,\varepsilon)\left(\frac{s}{D_{\mathcal{E}}}\right)(u)du\right)d\varepsilon\,.$$

We write

$$K = \int_{\varepsilon_0}^{\infty} \left(\frac{s}{D\mathcal{E}}\right)(u)\, U(u,\varepsilon_0) \left(\int_u^{\infty} U(\varepsilon_0,\varepsilon) N(\varepsilon) d\varepsilon\right) du\,.$$

By an integration by parts and hypothesis 3(ii), we have:

$$\int_u^{\infty} U(\varepsilon_0,\varepsilon)N(\varepsilon)d\varepsilon = \left(\frac{ND\mathcal{E}}{S}\right)(u)U(\varepsilon_0,u) + \int_u^{\infty} \frac{\partial}{\partial\varepsilon}\left(\frac{ND\mathcal{E}}{S}\right)(\varepsilon)\, U(\varepsilon_0,\varepsilon)d\varepsilon\,.$$

By hypothesis 3(ii) and classical comparison arguments, we obtain

$$\int_u^{\infty} U(\varepsilon_0,\varepsilon)N(\varepsilon)d\varepsilon \sim \left(\frac{ND\mathcal{E}}{S}\right)(u)U(\varepsilon_0,u)\,.$$

Then, the behaviour of K is similar to that of

$$K' = \int_{\varepsilon_0}^{\infty} \left(\frac{Ns}{S}\right)(u)du\,,$$

which converges due to the hypothesis of theorem 4.3. ∎

It is a simple matter to check that, in the above examples (Lyumkis and Chen coefficients with parabolic or Kane band diagram), the functions (3.16)–(3.17) satisfy the requirements of theorem 4.3. We shall now discuss the computation of the transport coefficients of the continuity equation (3.18) in the Lyumkis example (4.2) for a parabolic band and show that the resulting p.d.e. is parabolic. We recall

$$N(\varepsilon) = \sqrt{\varepsilon}\,, \quad S(\varepsilon) = \varepsilon\,, \quad D(\varepsilon) = \varepsilon Id\,, \quad \varepsilon \in \mathcal{R}(\varepsilon_c) = [0,\infty)\,, \qquad (4.17)$$

$$M_E(\varepsilon) = \frac{2}{\sqrt{\pi}\,T_E^{3/2}} e^{-\varepsilon/T_E}\,, \quad \mu(E) = \frac{2}{\sqrt{\pi T_E}}\,, \quad \text{with } T_E = 1 + E^2\,.$$

We now compute the term

$$\int_0^{\infty} \varepsilon \frac{\partial F_0^{\perp}}{\partial\varepsilon} d\varepsilon = -\int_0^{\infty} F_0^{\perp} d\varepsilon\,,$$

in (3.6). The equation (3.10) can be written as

$$Q_E(F_0^{\perp}) \;=\; \frac{2n}{\sqrt{\pi}}\frac{\partial}{\partial t}\left(\frac{1}{T_E}\right)\left(\frac{3}{2}\sqrt{\frac{\varepsilon}{T_E}} - \left(\frac{\varepsilon}{T_E}\right)^{3/2}\right) e^{-\varepsilon/T_E}$$

$$
\begin{aligned}
&+\nabla_x \cdot (n\mathcal{E}) \frac{2}{\sqrt{\pi}\, T_E^{3/2}} \left(2\sqrt{\frac{eps}{\pi T_E}} - \frac{\varepsilon}{T_E} \right) e^{-\varepsilon/T_E} \\
&+n\mathcal{E} \cdot \nabla_x \left(\frac{1}{T_E} \right) \frac{2}{\sqrt{\pi T_E}} \left(\sqrt{\frac{\varepsilon}{\pi T_E}} + \left(\frac{\varepsilon}{T_E} \right)^2 - \frac{5\varepsilon}{2T_E} \right) e^{-\varepsilon/T_E} \\
&+\mathcal{E} \cdot \nabla_x \left(n \frac{\partial}{\partial \varepsilon} (DM_E) \right) .
\end{aligned}
$$

Lemma 4.1 *Suppose that N, D and S are given by (4.17) and that g satisfies the hypotheses of theorem 4.3 and in particular, that $\int_0^\infty g(\varepsilon) d\varepsilon = 0$. Then the unique solution of problem (3.3) satisfies*

$$
-\int_0^\infty \varepsilon \frac{\partial F}{\partial \varepsilon} = \int_0^\infty F \, d\varepsilon = T_E \int_0^\infty G(u) \int_u^\infty g(vT_E) dv \, du \,,
$$

with

$$
G(u) = \frac{2}{\sqrt{\pi}} \int_0^\infty \frac{e^{-t}}{\sqrt{t+u} + \sqrt{t}} dt \,.
$$

Proof: Integration of $Q_E(F) = g$ gives

$$
\frac{\partial F}{\partial \varepsilon} + \frac{F}{T_E} = -r(\varepsilon) := -\frac{1}{T_E \varepsilon} \int_\varepsilon^\infty g(s) ds \,. \tag{4.18}
$$

The explicit computation of the solution gives

$$
F(0) = T_E \int_0^\infty \hat{G}(u) r(uT_E) du \,,
$$

with

$$
\hat{G}(u) = \frac{2}{\sqrt{\pi}} \int_u^\infty \sqrt{v}\, e^{u-v} dv = \frac{2}{\sqrt{\pi}} \int_0^\infty \sqrt{t+u}\, e^{-t} dt \,.
$$

Then, integration of (4.18) gives

$$
\int_0^\infty F \, d\varepsilon = T_E \int_0^\infty \frac{\hat{G}(u) - 1}{u} \int_u^\infty g(vT_E) dv \, du \,.
$$

The proof is completed by the observation that $\hat{G}(0) = 2\Gamma(3/2)/\sqrt{\pi} = 1$ and the computation

$$
\frac{\hat{G}(u) - \hat{G}(0)}{u} = \frac{2}{u\sqrt{\pi}} \int_0^\infty (\sqrt{t+u} - \sqrt{t}) e^{-t} dt = G(u) \,. \qquad \blacksquare
$$

From the lemma we obtain

$$\begin{aligned}\int_0^\infty F_0^\perp d\varepsilon &= A_1 n T_E \frac{\partial}{\partial t}\left(\frac{1}{T_E}\right) - A_2\sqrt{\frac{1}{T_E}}\nabla_x \cdot (n\mathcal{E}) \\ &\quad - A_3 n\sqrt{T_E}\mathcal{E}\cdot\nabla_x \frac{1}{T_E} + \frac{1}{\sqrt{\pi}}\mathcal{E}\cdot\nabla_x\left(\frac{n}{\sqrt{T_E}}\right),\end{aligned}$$

with

$$\begin{aligned} A_1 &= \frac{2}{\sqrt{\pi}}\int_0^\infty G(u)\int_u^\infty \left(\frac{3}{2}\sqrt{v} - v^{3/2}\right) e^{-v} dv\, du\,, \\ A_2 &= \frac{2}{\sqrt{\pi}}\int_0^\infty G(u)\int_u^\infty \left(\frac{2}{\sqrt{\pi}}\sqrt{v} - v\right) e^{-v} dv\, du\,, \\ A_3 &= \frac{2}{\sqrt{\pi}}\int_0^\infty G(u)\int_u^\infty \left(\frac{1}{\sqrt{\pi}}\sqrt{v} + v^2 - \frac{5}{2}v\right) e^{-v} dv\, du\,. \end{aligned}$$

Finally, the continuity equation becomes

$$\begin{aligned}&\frac{\partial n}{\partial t} - \nabla_x\cdot(n\mathcal{E}\mu^0) = \alpha\nabla_x\cdot\Bigg[\overline{D}(\mathcal{E})\nabla_x n + \frac{n}{\sqrt{\pi(1+E^2)}}\nabla_x E^2 \\ &- \frac{n\mathcal{E}}{(1+E^2)^{3/2}}\Bigg(A_1\sqrt{1+E^2}\frac{\partial E^2}{\partial t} - A_2(1+E^2)\nabla_x\cdot\mathcal{E} \\ &\qquad - \left(\frac{1}{2\sqrt{\pi}} - A_3\right)\mathcal{E}\cdot\nabla_x E^2\Bigg)\Bigg],\end{aligned} \tag{4.19}$$

with

$$\overline{D}(\mathcal{E}) = \frac{1}{\sqrt{1+E^2}}\left(\frac{2}{\sqrt{\pi}}(1+E^2)\mathrm{Id} + \left(A_2 - \frac{1}{\sqrt{\pi}}\right)\mathcal{E}\otimes\mathcal{E}\right).$$

We conclude this section by showing that (4.19) is parabolic:

Lemma 4.2 *The matrix $\overline{D}(\mathcal{E})$ is positive definite for every $\mathcal{E}\in\mathbb{R}^3$.*

Proof: It suffices to prove

$$\frac{2}{\sqrt{\pi}} + A_2 - \frac{1}{\sqrt{\pi}} \geq 0\,.$$

It is easily shown that

$$\int_u^\infty \left(v - \frac{2}{\sqrt{\pi}}\sqrt{v} \right) e^{-v} dv \geq 0$$

and $0 < G(u) \leq G(0) = \Gamma(1/2)/\sqrt{\pi} = 1$ hold. This implies

$$\begin{aligned} 0 < -A_2 &\leq \frac{2}{\sqrt{\pi}} \int_0^\infty \int_u^\infty \left(v - \frac{2}{\sqrt{\pi}}\sqrt{v} \right) e^{-v} dv\, du \\ &= \frac{2}{\sqrt{\pi}} \int_0^\infty \left(u^2 - \frac{2}{\sqrt{\pi}} u^{3/2} \right) e^{-u} du \\ &= \frac{2}{\sqrt{\pi}} \left(\Gamma(3) - \frac{2}{\sqrt{\pi}} \Gamma(5/2) \right) = \frac{1}{\sqrt{\pi}} . \qquad \blacksquare \end{aligned}$$

5 A Relaxation Time Model

In this section, the inelastic operator Q_{inel} is assumed to be a relaxation-time-type operator which drives the distribution function towards a Maxwellian with the lattice temperature. We start with the general form

$$Q_{inel}(f)(k) = \int_B s(k, k')[f(k')M(\varepsilon_c(k)) - f(k)M(\varepsilon_c(k'))]\, dk' ,$$

where $M(\varepsilon) = e^{-\varepsilon}$ is the Maxwellian with scaled lattice temperature equal to 1 and $s(k, k')$ is a nonnegative function symmetric in k, k'. Consequently, in the SHE model, the operator $Q(F)$ reads, for a function F depending only on the energy $\varepsilon(k)$,

$$Q(F)(\varepsilon) = \int_{\mathcal{R}(\varepsilon_c)} S(\varepsilon, \varepsilon')[F(\varepsilon')M(\varepsilon) - M(\varepsilon')F(\varepsilon)]d\varepsilon'$$

and

$$S(\varepsilon, \varepsilon') = \int_B \int_B \delta(\varepsilon_c(k) - \varepsilon)\delta(\varepsilon_c(k') - \varepsilon')s(k, k')dk'dk$$

In order to obtain a relaxation-time model for inelastic scattering, we take

$$s(k, k') = H(\varepsilon_c(k))H(\varepsilon_c(k')) ,$$

with a given nonnegative function H (to be specified later). This implies

$$S(\varepsilon, \varepsilon') = N(\varepsilon)H(\varepsilon)N(\varepsilon')H(\varepsilon') .$$

Consequently we have

$$
\begin{aligned}
Q_{\mathcal{E}}(F) &= \frac{\partial}{\partial\varepsilon}\left(\mathcal{E}^{tr} D\mathcal{E}\frac{\partial F}{\partial\varepsilon}\right) + NHM(\varepsilon)\int_{\mathcal{R}(\varepsilon_c)} NHF(\varepsilon')d\varepsilon' \\
&\quad -NHF(\varepsilon)\int_{\mathcal{R}(\varepsilon_c)} NHM(\varepsilon')d\varepsilon'
\end{aligned}
$$

In this model $\tau(\varepsilon) = 1/H(\varepsilon)$ can be interpreted as an energy dependent relaxation time. To simplify the computations, we assume

$$H(\varepsilon) = \varepsilon^{-1/2-\alpha}\,, \qquad \alpha < 1\,,$$

and the parabolic band assumptions

$$\mathcal{R}(\varepsilon_c) = [0,\infty)\,, \qquad N(\varepsilon) = \sqrt{\varepsilon}\,, \qquad D(\varepsilon) = \varepsilon I_{3\times 3}\,.$$

Therefore

$$Q_E(F) = E^2\frac{\partial}{\partial\varepsilon}\left(\varepsilon\frac{\partial F}{\partial\varepsilon}\right) + \varepsilon^{-\alpha}e^{-\varepsilon}\int_0^\infty F(u)u^{-\alpha}\,du - \varepsilon^{-\alpha}F(\varepsilon)\Gamma(1-\alpha)\,. \quad (5.1)$$

The following lemma shows that the kernel of Q_E has dimension 1.

Lemma 5.1 *Let $E \geq 0$. Then the equation $Q_E(F) = 0$, where the operator Q_E is defined by (5.1), has a unique positive solution F_E such that*

$$\int_0^\infty F_E(u)u^{-\alpha}du = 1\,.$$

Proof: The result is trivial for $E = 0$, with the kernel of Q_0 spanned by the Maxwellian. For $E > 0$, we first introduce the change of variables

$$\varepsilon = \left(\frac{E^2(1-\alpha)^2}{\Gamma(1-\alpha)}y\right)^{\frac{1}{1-\alpha}}\,, \qquad F_E = \frac{\Gamma(1-\alpha)}{E^2(1-\alpha)}G_E\,. \quad (5.2)$$

The new unknown G_E solves the problem

$$-\frac{d}{dy}\left(y\frac{dG_E}{dy}\right) + G_E = \delta_E(y)\,, \qquad \int_0^\infty G_E(y)\,dy = 1\,. \quad (5.3)$$

with

$$\delta_E(y) = \frac{E^2(1-\alpha)}{\Gamma(1-\alpha)^2}\exp\left[-\left(\frac{E^2(1-\alpha)^2}{\Gamma(1-\alpha)}y\right)^{\frac{1}{1-\alpha}}\right]\,. \quad (5.4)$$

It is readily seen that

$$\int_0^\infty \delta_E(y)\,dy = 1\,, \qquad \lim_{E\to\infty} \delta_E = \delta$$

where δ is the Dirac measure.

In order to prove the existence and uniqueness of G_E, we first investigate the homogeneous equation

$$-\frac{d}{dy}\left(y\frac{dG}{dy}\right) + G = 0\,, \tag{5.5}$$

which can be transformed to the Bessel equation of order zero by the transformation $z = 2i\sqrt{y}$. The general solution can thus be written in terms of the zeroth order Bessel function and the Hankel function $H_0^{(1)}$ [8]. However, only the solution

$$G_\infty(y) = i\pi H_0^{(1)}(2i\sqrt{y}) = 2\int_1^\infty \frac{e^{-2s\sqrt{y}}}{\sqrt{s^2-1}}\,ds \tag{5.6}$$

is integrable. This and the positivity of G_∞ imply uniqueness of G_E. The asymptotic behaviour of G_∞ can be deduced from the behaviour of $H_0^{(1)}$ [8]:

$$\begin{aligned} G_\infty(y) &\sim \sqrt{\pi} y^{-1/4} e^{-2\sqrt{y}}\,, \qquad y\to\infty\,, \\ G_\infty(y) &\sim -\ln y\,, \qquad y \to 0^+\,. \end{aligned} \tag{5.7}$$

We shall also need the asymptotic behaviour of dG_∞/dy, given by the derivatives of the right hand sides of (5.7).

The solution of (5.3) is obtained by the variation-of-constants ansatz $G_E(y) = G_\infty(y)\nu(y)$ with the result

$$G_E(y) = G_\infty(y)\int_0^y \frac{1}{tG_\infty(t)^2}\int_t^\infty G_\infty(s)\delta_E(s)\,ds\,dt\,. \tag{5.8}$$

From (5.7), we deduce that G_E is well defined. It remains to check the integrability of G_E and the validity of the second equation in (5.3). We compute

$$\begin{aligned} y\frac{dG_E}{dy}(y) &= y\frac{dG_\infty}{dy}(y)\int_0^y \frac{1}{tG_\infty(t)^2}\int_t^\infty G_\infty(s)\delta_E(s)\,ds\,dt \\ &\quad + \frac{1}{G_\infty(y)}\int_y^\infty G_\infty(s)\delta_E(s)\,ds\,. \end{aligned} \tag{5.9}$$

It is easily seen from (5.7) that (5.9) tends to zero as $y \to 0+$. The monotonicity of δ_E and G_∞ implies

$$\begin{aligned} \left| y \frac{dG_E}{dy}(y) \right| &\le \left| y \frac{dG_\infty}{dy}(y) \right| \int_0^y \frac{\delta_E(t)}{tG_\infty(t)^2} \int_t^\infty G_\infty(s)\, ds\, dt \\ &\quad + \int_y^\infty \delta_E(s)\, ds\,. \end{aligned} \tag{5.10}$$

The second term on the right hand side obviously tends to zero as $y \to \infty$. In the first term we use (5.7) and

$$\int_t^\infty s^r e^{-as^q} ds \sim \frac{t^{r-q+1}}{aq} e^{-at^q}, \qquad t \to \infty\,, \tag{5.11}$$

which holds for any $r \in \mathbb{R}$, $a, q > 0$, to show

$$\int_t^\infty G_\infty(s)\, ds \sim \sqrt{\pi} t^{1/4} e^{-2\sqrt{t}}, \qquad t \to \infty\,.$$

As a consequence we have

$$\begin{aligned} &\frac{\delta_E(t)}{tG_\infty(t)^2} \int_t^\infty G_\infty(s)\, ds = O\left(t^{-1/4}\delta_E(t) e^{2\sqrt{t}}\right) \\ &\quad = O\left(t^\beta \frac{d}{dt}\left(\delta_E(t) e^{2\sqrt{t}}\right)\right) = O\left(y^\beta \frac{d}{dt}\left(\delta_E(t) e^{2\sqrt{t}}\right)\right)\,, \end{aligned}$$

as $t \le y \to \infty$ for some $\beta \ge 0$. This estimate is sufficient for showing that also the first term on the right hand side of (5.10) and, thus, $y\, dG_E/dy$ tends to zero as $y \to \infty$. Now integration of the differential equation in (5.3) completes the proof. ∎

The lemma verifies Hypothesis 1 with

$$M_E(\varepsilon) = \frac{F_E(\varepsilon)}{\int_0^\infty \sqrt{z}\, F_E(z)\, dz}\,.$$

The mobility is given by

$$\mu(E) = \frac{\int_0^\infty F_E\, d\varepsilon}{\int_0^\infty \sqrt{\varepsilon}\, F_E\, d\varepsilon} = \left(\frac{\sqrt{\Gamma(1-\alpha)}}{E(1-\alpha)}\right)^{\frac{1}{1-\alpha}} \frac{\int_0^\infty G_E y^{\alpha/(1-\alpha)}\, dy}{\int_0^\infty G_E y^{(1+2\alpha)/2(1-\alpha)}\, dy}\,,$$

with the function G_E given by (5.8). The asymptotic behaviour as $E \to \infty$ is analyzed in the following:

Lemma 5.2 *As $E \to \infty$, $G_E(y)$ (given by (5.8)) converges to $G_\infty(y)$ (given by (5.6)) for $y > 0$. For all $\beta > -1$,*

$$\lim_{E\to\infty} \int_0^\infty y^\beta G_E(y)\, dy = \int_0^\infty y^\beta G_\infty(y)\, dy\,.$$

Before proving the lemma, we point out its obvious consequence,

$$\mu(E) \sim \mu_0 E^{-1/(1-\alpha)}\,, \qquad \text{as } E \to \infty\,,$$

Consequently $|v(\mathcal{E})| = O(E^{-\alpha/(1-\alpha)})$ as $E \to \infty$. Thus, the velocity saturates for large fields if $\alpha = 0$, it tends to zero for $0 < \alpha < 1$ and to ∞ for $\alpha < 0$.

Proof: As a first step, we rewrite G_E as

$$G_E(y) = C_E G_\infty(y) + H_E(y)$$

with

$$\begin{aligned} H_E(y) &= G_\infty(y) \int_1^y \frac{1}{tG_\infty(t)^2} \int_t^\infty G_\infty(s)\delta_E(s)\, ds\, dt\,, \\ C_E &= \int_0^1 \frac{1}{tG_\infty(t)^2} \int_t^\infty G_\infty(s)\delta_E(s)\, ds\, dt = 1 - \int_0^\infty H_E(y)\, dy\,. \end{aligned}$$

The second representation of C_E follows from the fact that both the integrals of G_E and of G_∞ over $\mathbb{R}_+$ are equal to 1. Since $\lim_{E\to\infty} \delta_E = \delta$, we deduce

$$\lim_{E\to\infty} H_E(y) = 0\,, \qquad \forall y > 0\,.$$

On the other hand, from the monotinicity of G_∞, we have the estimate

$$|H_E(y)| \le G_\infty(y) \left| \int_1^y \frac{dt}{tG_\infty(t)} \right|\,.$$

The right hand side of the above inequality is a locally bounded function on $\mathbb{R}_+$ whose asymptotic behaviour as $y \to 0+$ is given by $|\log(y)\log(|\log(y)|)|$. Therefore $y^\beta H_E(y)$ is in $L^1(0,1)$ for $\beta > -1$ and the dominated convergence theorem yields

$$\lim_{E\to\infty} \int_0^1 y^\beta |H_E(y)|\, dy = 0\,.$$

Let us now prove that

$$\lim_{E\to\infty} \int_1^\infty y^\beta |H_E(y)|\, dy = 0\,.$$

By using the Fubini theorem, we obtain

$$\int_1^\infty y^\beta |H_E(y)|\, dy = \int_1^\infty \frac{G_1(t) G_{2,E}(t)}{t G_\infty(t)^2}\, dt\,,$$

where

$$G_1(t) = \int_t^\infty y^\beta G_\infty(y)\, dy\,, \qquad G_{2,E}(t) = \int_t^\infty G_\infty(s)\delta_E(s)\, ds\,.$$

Using (5.11) together with the inequality

$$G_{2,E}(t) \le \delta_E(t) \int_t^\infty G_\infty(s)\, ds\,,$$

we obtain the estimate

$$\frac{G_1(t) G_{2,E}(t)}{t G_\infty(t)^2} \le C_\beta t^{\beta'} \delta_E(t)\,, \qquad t > 1\,,$$

where C_β and β' only depend on β. Besides

$$\int_1^\infty \delta_E(t) t^{\beta'}\, dt = C_{\alpha,\beta'} E^{-2\beta'} \int_{\frac{E^2(1-\alpha)^2}{\Gamma(1-\alpha)}}^\infty e^{-u^{1/(1-\alpha)}} u^{\beta'}\, du\,.$$

Using (5.11) once more, we finally obtain

$$\lim_{E\to\infty} \int_1^\infty y^\beta |H_E(y)|\, dy = 0\,.$$

Noting the consequence $\lim_{E\to\infty} C_E = 1$ the proof is completed. ∎

6 Phonon scattering

In contrast to section 4, we now consider the interaction of electrons with phonons whose energy ε_{ph} is of the order of magnitude of the thermal energy. The corresponding scattering operator in the SHE-model is a difference operator. Its expression, for the parabolic band case with the Lyumkis coefficients (4.2) is

$$\begin{aligned} Q(F)(\varepsilon) &= e^{\varepsilon_p}\sqrt{\varepsilon(\varepsilon+\varepsilon_p)}\, F(\varepsilon+\varepsilon_p) + \sqrt{\varepsilon(\varepsilon-\varepsilon_p)_+}\, F(\varepsilon-\varepsilon_p) \\ &\quad - \left(\sqrt{\varepsilon(\varepsilon+\varepsilon_p)} + e^{\varepsilon_{ph}}\sqrt{\varepsilon(\varepsilon-\varepsilon_p)_+}\right) F(\varepsilon) \end{aligned}$$

with $\varepsilon_p = \varepsilon_{ph}/kT$.

We shall perform formal asymptotics of order 1 of Q_E as $E \to \infty$ and $E \to 0$ to compute the value of the mobility $\mu(E)$ at $E = 0$ and its behaviour as $E \to \infty$.

We start with the limit $E \to \infty$. We introduce the small parameter $\delta = (\varepsilon_p/E)^2$ and the rescaling $\varepsilon = yE^2/\varepsilon_p$. Then the equation $Q_E(F) = 0$ is equivalent to

$$\delta \frac{d}{dy}\left(y\frac{dF}{dy}\right) + e^{\varepsilon_p}\sqrt{y(y+\delta)}\,F(y+\delta) + \sqrt{y(y-\delta)_+}\,F(y-\delta)$$
$$-\left(\sqrt{y(y+\delta)} + e^{\varepsilon_p}\sqrt{y(y-\delta)_+}\right)F(y) = 0\,.$$

Expansion around $\delta = 0$ and comparing coefficients of δ gives

$$\frac{d}{dy}\left[y\left(\frac{dF}{dy} + (e^{\varepsilon_p}-1)F\right)\right] = 0\,.$$

Solving this equation and transforming back to the original variables gives

$$M_E(\varepsilon) \sim \frac{2}{\sqrt{\pi}}\left(\frac{\varepsilon_p(e^{\varepsilon_p}-1)}{E^2}\right)^{3/2}\exp\left(-\frac{\varepsilon_p(e^{\varepsilon_p}-1)}{E^2}\varepsilon\right)\,,$$

and, thus,

$$\mu(E) \sim \frac{2}{\sqrt{\pi}}\frac{\sqrt{\varepsilon_p(e^{\varepsilon_p}-1)}}{E}\,, \qquad E \to \infty\,.$$

Note that this asymptotic behaviour with velocity saturation at large fields is qualitatively the same as that of section 4.

The computation of the low field mobility $\mu(0)$ is more subtle. This is due to the fact that the kernel of the phonon scattering operator Q is infinite dimensional: It is the set of all functions of the form $e^{-\varepsilon}P(\varepsilon)$ with P periodic with period ε_p [10]. Therefore the limit M_0 of M_E as $E \to 0$ is not determined uniquely by the formal limiting equation $Q(M_0) = 0$. We only have $M_0(\varepsilon) = e^{-\varepsilon}P(\varepsilon)$ with an arbitrary ε_p-periodic P. More information is obtained by using the fact that all ε_p-periodic functions R are collision invariants of Q [10]:

$$\int_0^\infty Q(F)R\,d\varepsilon = 0\,, \qquad \text{for arbitrary } F\,.$$

We multiply the equation $Q_E(M_E) = 0$ by R, integrate with respect to ε, divide the resulting equation by E^2 and let $E \to 0$:

$$\int_0^\infty (\varepsilon M_0')' R \, d\varepsilon = 0 \,, \qquad \text{for all } \varepsilon_p\text{-periodic } R \,.$$

An integration by parts and substitution of the form of M_0 shows that this equation is equivalent to

$$0 = -\int_0^{\varepsilon_{ph}} a(\varepsilon)(P' - P)R' \, d\varepsilon = \int_0^{\varepsilon_{ph}} [a(\varepsilon)(P' - P)]' R \, d\varepsilon \,, \qquad (6.1)$$

with

$$a(\varepsilon) = \sum_{j=0}^\infty (j\varepsilon_p + \varepsilon) e^{-j\varepsilon_p - \varepsilon} \,, \qquad 0 \le \varepsilon \le \varepsilon_p \,.$$

The second equality in (6.1) is due to $a(0) = a(\varepsilon_p)$. The summation of the series defining a can be carried out by deriving a differential equation for a. We obtain

$$a(\varepsilon) = \frac{e^{-\varepsilon}}{1 - e^{-\varepsilon_p}} \left(\varepsilon + \frac{\varepsilon_p}{e^{\varepsilon_p} - 1} \right)$$

Equation (6.1) is a weak formulation of the differential equation

$$[a(P' - P)]' = 0 \,, \qquad \text{in } (0, \varepsilon_p) \,,$$

with periodic boundary conditions. The solution is given by

$$P(\varepsilon) = c e^\varepsilon \left[\varepsilon_p - (1 - e^{-\varepsilon_p}) \ln \left(1 + (e^{\varepsilon_p} - 1) \frac{\varepsilon}{\varepsilon_p} \right) \right] \,, \qquad 0 \le \varepsilon \le \varepsilon_{ph} \,.$$

Note that the periodic continuation of P is in $C^{1,1}([0, \infty))$ but not in $C^2([0, \infty))$. The constant c has to be determined such that the condition

$$\int_0^\infty \sqrt{\varepsilon} \, e^{-\varepsilon} P(\varepsilon) \, d\varepsilon = 1$$

is satisfied. It is noteworthy that, as opposed to the examples in the previous sections, $M_0(\varepsilon) = e^{-\varepsilon} P(\varepsilon)$ is not a Maxwellian. The low field mobility is a function of the scaled phonon energy ε_p and given by

$$\mu(0) = \int_0^\infty M_0(\varepsilon) \, d\varepsilon = \frac{1}{1 - e^{-\varepsilon_p}} \int_0^{\varepsilon_p} e^{-\varepsilon} P(\varepsilon) \, d\varepsilon \,.$$

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