A note on the anisotropic generalizations of the Sobolev and Morrey embedding theorems

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Abstract. We make a contribution to the theory of embeddings of anisotropic Sobolev spaces into L^p-spaces (Sobolev case) and spaces of Hölder continous functions (Morrey case). In the case of bounded domains the generalized embedding theorems published so far pose quite restrictive conditions on the domain's geometry (in fact, the domain must be "almost rectangular"). Motivated by the study of some evolutionary PDEs, we introduce the so-called "semirectangular setting", where the geometry of the domain is compatible with the vector of integrability exponents of the various partial derivatives, and show that the validity of the embedding theorems can be extended to this case. Second, we discuss the a-priori integrability requirement of the Sobolev anisotropic embedding theorem and show that under a purely algebraic condition on the vector of exponents, this requirement can be weakened. Lastly, we present a counterexample showing that for domains with general shapes the embeddings indeed do not hold.

Key words: Anisotropic Sobolev spaces, Embedding Theorems, Gagliardo–Nirenberg–Sobolev inequality, Morrey inequality.

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Introduction

For a bounded $C^{0,1}$-domain $\Omega \subset \mathbb{R}^n$, the Sobolev space $W^{1,p}(\Omega)$ with $p \geq 1$ consists of functions $u \in L^p(\Omega)$ with first order distributional derivatives in $L^p(\Omega)$, i.e.,

$$W^{1,p}(\Omega) = \left\{ u \in L^p(\Omega);\ \frac{\partial u}{\partial x_i} \in L^p(\Omega) \text{ for } i = 1, \dots, n \right\} ,$$

with the norm

$$\|u\|_{W^{1,p}(\Omega)} = \left(\|u\|_p^p + \sum_{i=1}^{n} \left\| \frac{\partial u}{\partial x_i} \right\|_p^p \right)^{1/p} ,$$

where here and in the following the $L^p(\Omega)$-norm is denoted by $\|\cdot\|_p$. For $p < n$, we have the Gagliardo-Nirenberg-Sobolev inequality

$$\|u\|_q \leq c \|u\|_{W^{1,p}(\Omega)} ,$$

with $q = \frac{np}{n-p}$ and with a constant c independent of u (see [11] for the original exposition by S. L. Sobolev). Consequently, the space $W^{1,p}(\Omega)$ is continuously embedded into $L^q(\Omega)$:

$$W^{1,p}(\Omega) \hookrightarrow L^q(\Omega) .$$

Moreover, for each $1 \leq q' < q$ the embedding into $L^{q'}(\Omega)$ is compact (see, e.g., [4]).

For $p = n$, the space $W^{1,n}(\Omega)$ is continuously and compactly embedded into each $L^{q'}(\Omega)$ for $1 \leq q' < \infty$, but generally $W^{1,n}(\Omega) \nsubseteq L^\infty(\Omega)$.

Finally, for $p > n$, the Morrey inequality (see [8], [9] for the original proof) says that

$$\|u\|_{C^{0,\beta}(\Omega)} \leq c \|u\|_{W^{1,p}(\Omega)}$$

with $\beta = 1 - \frac{n}{p}$ and with a constant c independent of u. Therefore, $W^{1,p}(\Omega)$ is continuously embedded into the space of β-Hölder-continuous functions:

$$W^{1,p}(\Omega) \hookrightarrow C^{0,\beta}(\Omega) .$$

For each $1 \leq \beta' < \beta$ the embedding into $C^{0,\beta'}(\Omega)$ is compact.

In many applications it is natural to work with the concept of anisotropic Sobolev spaces, where the various weak partial derivatives of u are integrable with different exponents, collected in the vector $\vec{p} = (p_1, \ldots, p_n)$ such that $1 \le p_1 \le p_2 \le \ldots \le p_n \le \infty$. However, it is not obvious how to replace the a-priori integrability requirement $u \in L^p(\Omega)$ from the definition of $W^{1,p}(\Omega)$. We shall consider the two "extremal" cases and define

$$\begin{aligned} \underline{W}^{\vec{p}}(\Omega) &:= \left\{ u \in L^{p_1}(\Omega),\ \tfrac{\partial u}{\partial x_i} \in L^{p_i}(\Omega) \text{ for } i = 1, \ldots, n \right\}, \\ \overline{W}^{\vec{p}}(\Omega) &:= \left\{ u \in L^{p_n}(\Omega),\ \tfrac{\partial u}{\partial x_i} \in L^{p_i}(\Omega) \text{ for } i = 1, \ldots, n \right\}, \end{aligned} \tag{1}$$

with the respective norms

$$\|u\|_{\underline{W}^{\vec{p}}(\Omega)} = \sum_{i=1}^{n} \left\| \frac{\partial u}{\partial x_i} \right\|_{p_i} + \|u\|_{p_1}, \tag{2}$$

$$\|u\|_{\overline{W}^{\vec{p}}(\Omega)} = \sum_{i=1}^{n} \left\| \frac{\partial u}{\partial x_i} \right\|_{p_i} + \|u\|_{p_n}. \tag{3}$$

Obviously, for a bounded Ω, it is $\overline{W}^{\vec{p}}(\Omega) \subset \underline{W}^{\vec{p}}(\Omega)$. The assumption that the components of $\vec{p}$ are ordered, which will always be made in the following, is not a restriction of generality (if, in the general case, in the above definitions p_1 and p_n are replaced by the minimum and, respectively, the maximum of $\{p_1, \ldots, p_n\}$). It is worth pointing out that in the classical textbooks on function spaces, such as [7], an even more general notion of anisotropy is usual, considering (nonmixed) partial derivatives of different orders in different directions.

It is suggestive to study the possibilities of extending the classical Gagliardo-Nirenberg-Sobolev and Morrey inequalities for the spaces $\overline{W}^{\vec{p}}(\Omega)$ and $\underline{W}^{\vec{p}}(\Omega)$. This leads naturally to anisotropic generalizations of the embedding theorems, such as presented in [12] for functions defined on the whole $\mathbb{R}^n$ and in [10] for functions defined on bounded domains $\Omega \subset \mathbb{R}^n$. Best constants and extremal functions for the Sobolev embedding on $\mathbb{R}^n$ were studied recently in [3]. In the case of a bounded domain, which we will be interested in, its geometry plays a crucial role. The direct approach leads to embedding theorems that hold only for functions defined on rectangular domains. The reason is that the so-called extension theorem, which is used in the classical case, "mixes" the derivatives, such that the extended function no more belongs to the given anisotropic Sobolev space. In [10], a slightly more general results are achieved; roughly speaking, the shape of Ω has to be well suited to include cubes of finite size. But since the definition of the admissible domains is quite technical and not really relevant for applications, we cite the two key theorems from [10] with validity limited to rectangular domains:

Theorem 1 (Anisotropic Sobolev embedding) *Let $\Omega \subset \mathbb{R}^n$ be a rectangular domain and $\vec{p} = (p_1, \ldots, p_n)$ with*

$$\sum_{i=1}^{n} \frac{1}{p_i} \geq 1 \,. \tag{4}$$

Let $\mathbf{q}(\vec{p})$ be defined as

$$\mathbf{q}(\vec{p}) = \frac{n}{\sum_{i=1}^{n} \frac{1}{p_i} - 1} \quad \text{if} \quad \sum_{i=1}^{n} \frac{1}{p_i} > 1$$

or chosen arbitrarily from the interval $[1, \infty)$ otherwise. Then the space $\overline{W}^{\vec{p}}(\Omega)$ is continuously embedded into $L^{\mathbf{q}(\vec{p})}(\Omega)$.

Theorem 2 (Anisotropic Morrey embedding) *Let $\Omega \subset \mathbb{R}^n$ be a rectangular domain. Let $\vec{p} = (p_1, \ldots, p_n)$ satisfy*

$$\sum_{i=1}^{n} \frac{1}{p_i} < 1 \,. \tag{5}$$

Then the space $\underline{W}^{\vec{p}}(\Omega)$ is continuously embedded into $C^{0,\beta}(\Omega)$ with

$$0 < \beta = \frac{\alpha}{n/p_1 + \alpha} \leq 1 \,, \qquad \alpha = 1 - \sum_{i=1}^{n} \frac{1}{p_i} \,. \tag{6}$$

In our exposition, we use these two theorems as a starting point for further considerations. First, we show that their validity can be extended for non-rectangular domains which are "compatible" with the vector of exponents $\vec{p}$:

Definition 1 (Semirectangular restriction) *If the set of the n elements of the vector $\vec{p}$ consists of L distinct values ($1 \leq L \leq n$), let us denote the multiplicity of each of the values in $\vec{p}$ by n_i, $i = 1, \ldots, L$, such that $n_1 + \ldots + n_L = n$. We say that a bounded domain $\Omega \subset \mathbb{R}^n$ satisfies the semirectangular restriction related to the vector $\vec{p}$, if there exist bounded $C^{0,1}$ domains $\Omega_i \subset \mathbb{R}^{n_i}$, $i = 1, \ldots, L$, such that $\Omega = \Omega_1 \times \ldots \times \Omega_L$.*

Let us remark that this is a typical situation in the existence and regularity considerations of evolutionary PDEs, as for example in [2], [1] and [6]. Solution of an evolutionary problem is a function $f = f(t,x)$ of time $t \in [0,T]$ and space $x \in \Omega \subset \mathbb{R}^d$. Typically, one is able to obtain a-priori information of the form $\frac{\partial f}{\partial t} \in L^{p_t}$ and $\frac{\partial f}{\partial x_i} \in L^{p_x}$, $i = 1,\dots,d$, but usually $p_t \neq p_x$. Here the domain $[0,T] \times \Omega$ obviously satisfies the semirectangular condition with respect to the vector of exponents $\vec{p} = (p_t, p_x, \dots, p_x)$.

Second, we take a closer look at the integrability assumption of Theorem 1. Indeed, due to the a-priori requirement $u \in L^{p_n}(\Omega)$, the theorem reduces to a trivial tautology if $p_n \geq \mathbf{q}(\vec{p})$. We show that under an algebraic condition on the vector $\vec{p}$, the integrability assumption can be weakened to $u \in L^{p_1}(\Omega)$. Notice that in the Morrey case, Theorem 2 poses just the minimal condition $u \in L^1(\Omega)$.

Lastly, to our best knowledge, no counterexample exists in the literature showing that the statement of Theorems 1 and 2 does not hold for general shapes of Ω. We fill this gap in the last section.

The semirectangular embeddings

The key ingredient is the slight generalization of the classical extension theorem (see [4]), which we give below. Let us recall from Definition 1 that for a given $\vec{p}$, we denote the number of its distinct entries by L and the multiplicity of each distinct value by n_i, $i = 1,\dots,L$, such that $n_1 + \ldots + n_L = n$. Then we are able to extend Theorems 1 and 2 for domains satisfying the semirectangular restriction (or simply "compatible domains"), i.e., of the type $\Omega = \Omega_1 \times \ldots \times \Omega_L$, where each Ω_i is a bounded $C^{0,1}$ domain in $\mathbb{R}^{n_i}$ of arbitrary geometry.

Theorem 3 *Let $\vec{p}$ be the vector of exponents with the corresponding multiplicities $n_1,\dots,n_L$. Let $\Omega = \Omega_1 \times \ldots \times \Omega_L$ be a domain compatible (in the sense of Definition 1) with $\vec{p}$ and for each $i = 1,\dots,L$ let $\tilde{\Omega}_i$ be a* rectangular *extension of Ω_i, i.e., $\Omega_i \subset \tilde{\Omega}_i$. Denote $\tilde{\Omega} = \tilde{\Omega}_1 \times \ldots \times \tilde{\Omega}_L$. Then there exists a bounded linear operator*

$$E : \overline{W}^{\vec{p}}(\Omega) \to \overline{W}^{\vec{p}}(\tilde{\Omega})$$

such that $Eu = u$ almost everywhere in Ω and

$$\|Eu\|_{\overline{W}^{\vec{p}}(\tilde{\Omega})} \leq C \, \|u\|_{\overline{W}^{\vec{p}}(\Omega)} \, , \tag{7}$$

the constant C depending only on $\vec{p}$, Ω and $\tilde{\Omega}$.

Proof: We construct the "partial" extension operator E_1, which extends the functions from Ω to $\tilde{\Omega}_1 \times \Omega_2 \times \ldots \times \Omega_L$, the rest is then achieved inductively. Let us denote $\Omega_{2\ldots L} = \Omega_2 \times \ldots \times \Omega_L$. We refer to the classical proof [4] for the details of the construction of E_1; we only need to observe that the extended function $E_1 u$ belongs to the space $\overline{W}^{\vec{p}}(\tilde{\Omega}_1 \times \Omega_{2\ldots L})$ and check the boundedness of E_1. The key point is the fact that the extension of u can be written explicitely as $E_1 u(x,y) = \sum_{i=1}^{s} \xi_i(x) u(\psi_i(x), y)$ for some $s \in \mathbb{N}$, where $x \in \tilde{\Omega}_1$, $y \in \Omega_{2\ldots L}$, with each $\xi_i \in C_0^\infty(\tilde{\Omega}_1)$ and each ψ_i being a C^1 diffeomorphism mapping $\tilde{\Omega}_1$ into Ω_1. Moreover, the functions ξ_i and ψ_i depend only on Ω_1 and $\tilde{\Omega}_1$. From this we can immediately infer the inequality

$$\|E_1 u\|_{\overline{W}^{\vec{p}}(\tilde{\Omega}_1 \times \Omega_{2\ldots L})} \leq C \, \|u\|_{\overline{W}^{\vec{p}}(\Omega)} ,$$

the constant C being independent of u. ■

Now it is easy to prove the semirectangular Sobolev and, resp., Morrey embedding theorems.

Theorem 4 *Let $\Omega \subset \mathbb{R}^n$ be a domain satisfying the semirectangular restriction with respect to a vector of exponents $\vec{p}$, such that*

$$\sum_{i=1}^{n} \frac{1}{p_i} \geq 1 .$$

Then $\overline{W}^{\vec{p}}(\Omega)$ equipped with the norm (3) is continuously embedded into $L^{\mathbf{q}(\vec{p})}(\Omega)$.

Theorem 5 *Let $\Omega \subset \mathbb{R}^n$ be a domain satisfying the semirectangular restriction with respect to a vector of exponents $\vec{p}$, such that*

$$\sum_{i=1}^{n} \frac{1}{p_i} < 1 .$$

Then the space $\underline{W}^{\vec{p}}(\Omega)$ equipped with the norm (2) is continuously embedded into $C^{0,\beta}(\Omega)$ with

$$0 < \beta = \frac{\alpha}{n/p_1 + \alpha} \leq 1 , \qquad \alpha = 1 - \sum_{i=1}^{n} \frac{1}{p_i} .$$

Proof: We use the extension operator E described in Theorem 3 to extend our functions from Ω onto the rectangular domain $\tilde{\Omega}$. For $\overline{W}^{\vec{p}}(\tilde{\Omega})$ and, resp., $\underline{W}^p\vec{p}(\tilde{\Omega})$ we apply the embeddings from Theorems 1 and, resp., 2. ∎

Notice that for the isotropic case $L = 1$ and $p_1 = \ldots = p_n$, the statements of the above two theorems reduce to the classical embeddings without any restriction on the domain geometry. On the other hand, in the "totally anisotropic" case $L = n$ and $p_i \neq p_j$ if $i \neq j$, the only compatible domain Ω is an n-dimensional rectangle.

The Sobolev case: Discussion of the integrability assumption

To show that the question whether the a-priori assumption $u \in L^{p_n}(\Omega)$ in Theorems 1 and 4 can be weakened is not trivial, let us consider the following example: For a function $u \in W^{1,1}(\Omega_2)$ with Ω_2 being a rectangle in $\mathbb{R}^2$, define $\tilde{u} \in \underline{W}^{\vec{p}}(\Omega_3)$ with $\vec{p} = (1, 1, \infty)$ and $\Omega_3 = \Omega_2 \times (0, 1)$ such that

$$\tilde{u}(x_1, x_2, x_3) = u(x_1, x_2) \qquad \text{for each } (x_1, x_2, x_3) \in \Omega_3 .$$

Now, if the assertion of theorem 1 held also for $\underline{W}^{\vec{p}}(\Omega)$, it would assert that $\tilde{u} \in L^3(\Omega_3)$ and, straightforwardly, $u \in L^3(\Omega_2)$. This would mean that $W^{1,1}(\Omega_2)$ is a subset of $L^3(\Omega_2)$, a contradiction. Notice that here the case $p_n > \mathbf{q}(\vec{p})$ occurs and the contradiction is due to the lowest two exponents $p_1 = p_2 = 1$. In the forthcoming analysis we show that such argument can always be applied when $p_n > \mathbf{q}(\vec{p})$. On the other hand, the $L^{p_n}(\Omega)$-integrability assumption can be weakened to $L^{p_1}(\Omega)$-integrability in cases when $p_n \leq \mathbf{q}(\vec{p})$.

Definition 2 *For $1 \leq k \leq n$ we define*

$$q^k(\vec{p}) = \begin{cases} \frac{k}{\sum_{i=1}^k 1/p_i - 1} & \text{if } \sum_{i=1}^k 1/p_i > 1\,, \\ \infty & \text{else}\,, \end{cases}$$

and $\underline{q}(\vec{p}) = \min\{q^1(\vec{p}), \ldots, q^n(\vec{p}) = \mathbf{q}(\vec{p})\}$.

Lemma 1 *Let $\vec{p} = (p_1, \ldots, p_n)$ satisfy*

$$\sum_{i=1}^n \frac{1}{p_i} > 1\,.$$

Then one of the following two cases holds:

A) $q^1(\vec{p}) > \ldots > q^n(\vec{p})$ *and* $p_{k+1} < q^k(\vec{p})$ *for all* $k = 1, \ldots, n-1$*, and, consequently,* $\underline{q}(\vec{p}) = \mathbf{q}(\vec{p}) = q^n(\vec{p})$*.*

B) There exists an index K*,* $2 \leq K \leq n-1$*, such that* $q^1(\vec{p}) > \ldots > q^K(\vec{p}) \leq \ldots \leq q^n(\vec{p})$ *and, thus,* $\underline{q}(\vec{p}) = q^K(\vec{p}) \leq p_{K+1}$*.*

Proof: Since $q^1(\vec{p}) = \infty$, the sequence $\{q^k(\vec{p})\}$ always starts decreasing. A simple computation shows that $q^k(\vec{p}) > q^{k+1}(\vec{p})$ is equivalent to $q^k(\vec{p}) > p_{k+1}$. If this is true for all $k = 1, \ldots, n-1$, Case A holds. If not, there exists a smallest $K \geq 2$ such that $q^K(\vec{p}) \leq q^{K+1}(\vec{p})$. Again a simple computation shows that this is equivalent to $q^{K+1}(\vec{p}) \leq p_{K+1} \leq p_{K+2}$, where the second inequality is due to the ordering of the components of $\vec{p}$. This, however, implies $q^{K+1}(\vec{p}) \leq q^{K+2}(\vec{p})$, showing that, if the sequence $\{q^k(\vec{p})\}$ starts to increase, it continues to do so. This completes the proof. ∎

Lemma 1 gives us a direct evidence that in the 'strong' version of Case B, where $p_{K+1} > q^K(\vec{p})$ for some K, we cannot expect the space $\underline{W}^{\vec{p}}(\Omega)$ to be embedded into $L^{\mathbf{q}(\vec{p})}(\Omega)$. Indeed, note that the sharp inequality implies $\mathbf{q}(\vec{p}) > q^K(\vec{p})$ and consider, for a function $u \in \underline{W}^{(p_1,\ldots,p_K)}((0,1)^K)$, its 'extension' $\tilde{u}$ to $\underline{W}^{\vec{p}}((0,1)^n)$ given by

$$\tilde{u}(x_1, \ldots, x_n) \quad = \quad u(x_1, \ldots, x_K) \quad \text{for } (x_1, \ldots, x_K, \ldots, x_n) \in (0,1)^n \,.$$

If the space $\underline{W}^{\vec{p}}((0,1)^n)$ was embedded into $L^{\mathbf{q}(\vec{p})}((0,1)^n)$, we would have $u \in L^{\mathbf{q}(\vec{p})}((0,1)^K)$ for each $u \in \underline{W}^{(p_1,\ldots,p_K)}((0,1)^K)$, which is false since the latter space can be expected to be embedded at most into $L^{q^K(\vec{p})}((0,1)^K)$. For the following, remember that $\underline{W}^{\vec{p}}(\Omega)$ is equipped with the natural norm

$$\|u\|_{\underline{W}^{\vec{p}}(\Omega)} = \sum_{i=1}^{n} \left\| \frac{\partial u}{\partial x_i} \right\|_{p_i} + \|u\|_{p_1} \,.$$

Theorem 6 *Let* $\vec{p}$ *and* Ω *be as in Theorem 4. Then, in Case A of Lemma 1, the space* $\underline{W}^{\vec{p}}(\Omega)$ *is continuously embedded into* $L^{\underline{q}(\vec{p})}(\Omega) = L^{\mathbf{q}(\vec{p})}(\Omega)$*. In Case B,* $\underline{W}^{\vec{p}}(\Omega)$ *is continuously embedded into every* $L^q(\Omega)$ *with* $q < \underline{q}(\vec{p})$*.*

Proof: Case A: We define $\vec{p}^k = (p_1^k, \ldots, p_n^k)$ by $p_i^k = \min\{p_i, p_k\}$, i.e., $\vec{p}^k = (p_1, \ldots, p_k, \ldots, p_k)$, $k = 1, \ldots, n$. We shall prove by induction that for every $k = 1, \ldots, n$, $\underline{W}^{\vec{p}^k}(\Omega) \hookrightarrow \overline{W}^{\vec{p}^k}(\Omega)$ and that, consequently (by Theorem 4),

$$\|u\|_{\mathbf{q}(\vec{p}^k)} \leq c\, \|u\|_{\underline{W}^{\vec{p}^k}(\Omega)} \qquad \text{for all } u \in \underline{W}^{\vec{p}^k}(\Omega)\,. \tag{8}$$

For $k = n$, this amounts to the statement of the theorem.

The claim is obviously satisfied for $k = 1$. Now assume it holds for a certain k. We construct a sequence $\{q_l^k\}_{l\geq 0}$ by $q_0^k = \mathbf{q}(\vec{p}^k)$ and $q_{l+1}^k = \mathbf{q}(p_1, \ldots, p_k, q_l^k, \ldots, q_l^k)$. The solution of this recursion can be explicitely given:

$$q_l^k = \left(\frac{1}{q^k(\vec{p})} + \left(1 - \frac{k}{n}\right)^{l+1} \left(\frac{1}{p_k} - \frac{1}{q^k(\vec{p})} \right) \right)^{-1} .$$

Since $p_k < q^k(\vec{p})$ (lemma 1), the sequence is strictly increasing and $\lim_{l\to\infty} q_l^k = q^k(\vec{p})$. By a second induction we claim that

$$\underline{W}^{(p_1,\ldots,p_k,q_l^k,\ldots,q_l^k)}(\Omega) \hookrightarrow \overline{W}^{(p_1,\ldots,p_k,q_l^k,\ldots,q_l^k)}(\Omega) ,$$

as long as $q_l^k \leq p_{k+1}$. Actually, since $q_0^k = \mathbf{q}(\vec{p}^k) \leq p_{k+1}$, (8) implies the above for $l = 0$. Again, the induction proceedes, repeatedly applying Theorem 4, in a bootstrapping way.

Eventually, q_{l+1}^k becomes bigger than p_{k+1} since $p_{k+1} < q^k(\vec{p})$ (Lemma 1). Then we obviously have $\underline{W}^{\vec{p}^{k+1}}(\Omega) \hookrightarrow \overline{W}^{\vec{p}^{k+1}}(\Omega)$, and the induction step (and therefore the proof for Case A) is completed.

Case B: We start in the same way as in Case A and prove (8) for $k = 1, \ldots, K-1$. Then, for $q < \underline{q}(\vec{p}) = q^K(\vec{p})$, we construct the sequence $\{q_l^K\}_{l\geq 0}$ as above and stop the second iteration process as soon as $q_{l+1}^K > q$. ■

Counterexamples for a domain with general shape

We present an example showing that the generalized Sobolev and Morrey embeddings cannot be easily extended for cases where the semirectangular condition is not met. Namely, we consider the two-dimensional setting with $\vec{p} = (p_1, p_2)$ such that $p_1 \neq p_2$. In this case the semirectangular condition states that the domain Ω must be rectangular. We show that the embeddings fail on non-rectangular domains.

We start with the Sobolev case. Let us consider, for $r > 1$, the domain $\Omega_r \subset \mathbb{R}^2$,

$$\Omega_r = \{(x_1, x_2) : -1 < x_2 < 1, |x_2|^r < x_1 < 1\} . \tag{9}$$

A simple calculation shows that for $0 < \gamma < 1/r$ the function $u(x_1, x_2) = x_1^{-\gamma}$ satisfies the relations

$$\frac{\partial u}{\partial x_1} \in L^{p_1}(\Omega_r)\,, \quad \frac{\partial u}{\partial x_2} \in L^{p_2}(\Omega_r)$$

with

$$1 \le p_1 < \frac{r+1}{r+r\gamma}\,, \quad 1 \le p_2 \le \infty\,.$$

Now, for $0 < \varepsilon < 1 - r\gamma$, we set

$$p_1 = \frac{r+1}{r+r\gamma+\varepsilon}\,, \qquad p_2 = \frac{r+1}{1-r\gamma-\varepsilon}\,.$$

Then we have $\frac{1}{p_1}+\frac{1}{p_2} = 1$ and, due to Theorem 1, $\mathbf{q}(p_1, p_2) < \infty$ can be chosen arbitrarily high. However, it is easy to check that u belongs to $L^q(\Omega)$ only if $q < \frac{r+1}{r\gamma}$, while the integrability assumption of Theorem 2 is not violated if we assume $r\gamma < 1/2$ and $\varepsilon < 1 - 2r\gamma$, since then $\max\{p_1, p_2\} < \frac{r+1}{r\gamma}$. Note that $\Omega_r \in C^{0,1}$ for each $r > 1$; moreover, when r is an even integer, the boundary of Ω_r is analytic close to the origin, where the problem arises. The strength of the allowed singularity tends to zero and the upper integrability bound $\frac{r+1}{r\gamma}$ tends to infinity as $r \to \infty$, when Ω_r approaches the rectangle $\Omega_\infty = (0, 1) \times (-1, 1)$.

We can use the same construction also for the Morrey case. Namely, with the choice

$$1 < p_1 < \frac{r+1}{r+r\gamma}\,, \quad p_2 = \infty\,,$$

the assumptions (6), (5) of Theorem 2 are satisfied, but u is obviously not Hölder continous.

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