Elastic and Drift-Diffusion Limits of Electron-Phonon Interaction in Semiconductors

Christian Schmeiser and Alexander Zwirchmayr[1]

Abstract. This paper is concerned with electron transport in semiconductors when electron-phonon interaction is considered. Smallness of the mean free path compared to a characteristic length scale and of the phonon energy compared to the thermal energy of the crystal are assumed. The corresponding limits in the transport problem are carried out and shown not to commute. An intermediate limit leads to a new macroscopic model.

Key words: Semiconductors, kinetic equations, phonons, drift-diffusion limit

AMS subject classification: 82A70, 78A35

Acknowledgement: This work has been supported by the Austrian Fonds zur Förderung der wissenschaftlichen Forschung under grant Nr. P11308-MAT.

[1]Institut für Angewandte und Numerische Mathematik, TU Wien, Austria.

1 Introduction

This paper is concerned with the derivation of macroscopic (or drift-diffusion) limits of transport equations modelling the flow of electrons driven by a prescribed electric field and influenced by scattering with phonons of constant energy. We consider a model problem motivated by semiclassical models for charge transport in semiconductors. Our basic physical assumptions are that of a parabolic energy band and that all scattering processes except with one type of phonons are neglected.

We are interested in situations where the Knudsen number is small, i.e., the mean free path is small compared to the length scales relevant for the application of interest. Furthermore, we assume that the phonon energy is small compared to the thermal energy corresponding to the lattice temperature. In this case the scattering mechanism is almost elastic, i.e., electrons suffer during a collision a change in energy small compared to the thermal energy. Thus, we are confronted with a problem with two small parameters: The Knudsen number and the scaled phonon energy. After presenting the problem in section 2 and proving some important properties of the collision operator in section 3, we rigorously carry out the limits as the two small parameters tend to zero in different orders in sections 4–7.

In section 4 the elastic limit of the transport problem is derived. Formally, it just amounts to setting the phonon energy equal to zero in the collision operator. Section 5 contains a rigorous justification of the drift-diffusion limit (Knudsen number to zero) of the elastic transport equation. Formally the limiting procedure has first been carried out in [2], see also [1]. The limit of the distribution function depends locally just on the energy and satisfies an equation called the "Spherical Harmonic Expansion Model" in [1], since it can also be derived by a low order expansion of the distribution function in terms of spherical harmonics.

In section 6 the drift-diffusion limit of the full (inelastic) transport problem is carried out. This has already been done in [7], however under very involved assumptions on the scattering data, which are strongly reduced here. The limiting distribution function locally belongs to the kernel of the scattering operator consisting of products of the Maxwellian distribution with periodic functions of the energy, where the period is the phonon energy [3]. In the (weak) limit as the phonon energy tends to zero (performed in section 7) a local Maxwellian is obtained with a density satisfying the standard drift-diffusion equation of semiconductor theory (see [5]).

The main result of sections 4–7 is that the two limits do not commute. An intermediate limit, where the small parameters are related, is rigorously carried out in section 8. The result is a new model having the form of the Spherical Harmonic Expansion Model with a differential operator in the energy direction as collision term.

Possible extensions of our results include more general band structures as well as nonlinear collision operators taking into account the Pauli exclusion principle. The limits of sections 5 and 6 corresponding to these extensions have been formally carried out in [1] and [4]. A forthcoming publication [8] will be concerned with the extension of the new model from section 8.

2 The Kinetic Model

We describe the motion of electrons by a distribution function $f(x,v,t)$ depending on the space coordinates $x \in \mathbb{R}^3$, velocity $v \in \mathbb{R}^3$ and time $t \geq 0$. The distribution function is determined by the (scaled) Boltzmann equation

$$\alpha^2 \frac{\partial f}{\partial t} + \alpha \left(v \cdot \nabla_x f + \frac{1}{2} \nabla_x \Phi \cdot \nabla_v f \right) = Q_\varepsilon(f) \tag{2.1}$$

and the initial condition

$$f(x,v,0) = f_I(x,v)\,. \tag{2.2}$$

The (given) electric field is written in terms of a time and space dependent electrostatic potential $\Phi(x,t)$. The collision operator is assumed to be of the form

$$\begin{aligned} Q_\varepsilon(f)(t,x,v) \;\; = \;\; & \int_{\mathbf{R}^3} [e^\varepsilon W_\varepsilon(v,v') + W_\varepsilon(v',v)]\, f(t,x,v')\, dv' \\ & -f(t,x,v) \int_{\mathbf{R}^3} [e^\varepsilon W_\varepsilon(v',v) + W_\varepsilon(v,v')]\; dv' \end{aligned} \tag{2.3}$$

with

$$W_\varepsilon(v,v') = k(v,v')\, \delta\left(|v'|^2 - |v|^2 - \varepsilon \right)\,.$$

The equations are dimensionless. A reference length L has been chosen and the inverse of the relaxation time τ has been taken out of the collision integral.

The reference velocity is the thermal velocity $v_{th} = \sqrt{2k_B T/m}$ with the Boltzmann constant k_B, the lattice temperatur T and the electron mass m. The dimensionless parameter

$$\alpha = v_{th}\tau/L$$

is the Knudsen number.

The collision operator Q_ε models interaction of electrons with phonons of constant energy. The dimensionless parameter

$$\varepsilon = \frac{\hbar\omega}{k_B T}$$

is the ratio of the phonon energy to the thermal energy. The scaled energy of an electron is $|v|^2$ reflecting the fact that we use the parabolic band assumption. The consequent rotational invariance is also required of the overlap factor $k(v,v')$ assumed to be positive, continuous on $\mathbb{R}^3 \times \mathbb{R}^3$ and satisfying

$$k(Sv, Sv') = k(v,v') \qquad \text{for every orthogonal matrix } S.$$

Typical models have the form

$$k(v,v') = \left(|v-v'|^2 + \beta\right)^{-n}, \qquad \beta, n > 0\,. \tag{2.4}$$

We introduce the variable transformation

$$v = \sqrt{u}\,\boldsymbol{\omega}\,, \qquad u = |v|^2 \geq 0\,, \quad \boldsymbol{\omega} = \frac{v}{|v|} \in S^2\,,$$

where S^2 denotes the unit sphere in $\mathbb{R}^3$. Because of the rotational invariance, k depends only on u, u' and $s = \boldsymbol{\omega}\cdot\boldsymbol{\omega}'$ and we assume

$$k \in C\left([0,\infty)^2 \times [-1,1]\right)\,, \qquad k(u,u',s) = k(u',u,s) > 0. \tag{2.5}$$

In the following we shall also need the assumptions

$$\sqrt{u+\varepsilon}\int_{-1}^{1} k(u,u+\varepsilon,\sigma)d\sigma \leq \varrho\,, \qquad \sqrt{u+\varepsilon}\,k(u,u+\varepsilon,s) \geq b(u) > 0$$
$$\text{for } u>0,\ -1 \leq s \leq 1, \qquad \text{with } b(u) \geq \kappa\sqrt{u}\,e^{(\delta-1)u}, \tag{2.6}$$

with the positive constants ϱ, κ and δ independent of $\varepsilon \in [0,\varepsilon_0]$. A simple computation shows that (2.6) is satisfied for the model (2.4) with $\beta > 0$ and $n \geq 1/2$.

3 Properties of the Collision Operator

Carrying out the integration in the u' direction in (2.3), the collision operator reads

$$\begin{aligned} Q_\varepsilon(f)(\sqrt{u}\,\boldsymbol{\omega}) &= e^\varepsilon \frac{\sqrt{u+\varepsilon}}{2} \int_{S^2} k(u, u+\varepsilon, \boldsymbol{\omega}\cdot\boldsymbol{\omega}')\, f(\sqrt{u+\varepsilon}\,\boldsymbol{\omega}')\, d\boldsymbol{\omega}' \\ &+ \frac{\sqrt{(u-\varepsilon)_+}}{2} \int_{S^2} k(u, (u-\varepsilon)_+, \boldsymbol{\omega}\cdot\boldsymbol{\omega}')\, f(\sqrt{(u-\varepsilon)_+}\,\boldsymbol{\omega}')\, d\boldsymbol{\omega}' \\ &- \lambda_\varepsilon(u) f(\sqrt{u}\,\boldsymbol{\omega}) . \end{aligned} \qquad (3.1)$$

(The subscript '+' denotes the positive part.) The scattering frequency λ_ε is given by

$$\begin{aligned} \lambda_\varepsilon(u) &= \frac{\sqrt{u+\varepsilon}}{2} \int_{S^2} k(u, u+\varepsilon, \boldsymbol{\omega}\cdot\boldsymbol{\omega}')\, d\boldsymbol{\omega}' \\ &\quad + e^\varepsilon \frac{\sqrt{(u-\varepsilon)_+}}{2} \int_{S^2} k(u, (u-\varepsilon)_+, \boldsymbol{\omega}\cdot\boldsymbol{\omega}')\, d\boldsymbol{\omega}' \\ &= \pi\sqrt{u+\varepsilon} \int_{-1}^{1} k(u, u+\varepsilon, s)\, ds \\ &\quad + e^\varepsilon \pi \sqrt{(u-\varepsilon)_+} \int_{-1}^{1} k(u, (u-\varepsilon)_+, s)\, ds . \end{aligned}$$

Setting formally $\varepsilon = 0$ in (3.1) we obtain an elastic collision operator

$$Q_0(f)(\sqrt{u}\,\boldsymbol{\omega}) = \sqrt{u} \int_{S^2} k(u, u, \boldsymbol{\omega}\cdot\boldsymbol{\omega}') f(\sqrt{u}\,\boldsymbol{\omega}')\, d\boldsymbol{\omega}' - \lambda_0(u) f(\sqrt{u}\,\boldsymbol{\omega}) \qquad (3.2)$$

with

$$\lambda_0(u) = 2\pi\sqrt{u} \int_{-1}^{1} k(u, u, s)\, ds .$$

The operators Q_ε for positive ε and Q_0 have been analyzed in [3], [7] and, respectively, [1]. They have strongly different properties. Here we shall prove the most important results under the weaker (compared to [7]) assumptions (2.6) on the overlap factor.

The collision operators are symmetric with respect to the space

$$\mathcal{X} \;:=\; L^2\left(\mathbb{R}^3; e^{|v|^2} dv\right) , \qquad \langle f_1, f_2 \rangle_{\mathcal{X}} = \int_{\mathbb{R}^3} f_1 f_2 e^{|v|^2}\, dv .$$

A simple computation gives

$$\langle Q_\varepsilon(f_1), f_2\rangle_{\mathcal{X}} = -\int_0^\infty \int_{S^2}\int_{S^2} \frac{\sqrt{u(u+\varepsilon)}}{4} e^u k(u, u+\varepsilon, \boldsymbol{\omega}\cdot\boldsymbol{\omega}')$$

$$[e^\varepsilon f_1(\sqrt{u+\varepsilon}\,\boldsymbol{\omega}') - f_1(\sqrt{u}\,\boldsymbol{\omega})][e^\varepsilon f_2(\sqrt{u+\varepsilon}\,\boldsymbol{\omega}') - f_2(\sqrt{u}\,\boldsymbol{\omega})]d\boldsymbol{\omega}' d\boldsymbol{\omega} du\,. \tag{3.3}$$

The corresponding formula for Q_0 is obtained by setting $\varepsilon = 0$ in the above.

Lemma 3.1 *With the assumption (2.6), Q_ε is a bounded operator on $\mathcal{X}$ uniformly with respect to $\varepsilon \in [0, \varepsilon_0]$*

Proof: By the symmetry of Q_ε it is sufficient to estimate

$$\begin{aligned}
&|\langle Q_\varepsilon(f), f\rangle_{\mathcal{X}}| \le \int_0^\infty \int_{S^2}\int_{S^2} \frac{\sqrt{u(u+\varepsilon)}}{2} e^u k(u, u+\varepsilon, \boldsymbol{\omega}\cdot\boldsymbol{\omega}') \\
&[e^{2\varepsilon} f(\sqrt{u+\varepsilon}\,\boldsymbol{\omega}')^2 + f(\sqrt{u}\,\boldsymbol{\omega})^2] d\boldsymbol{\omega}' d\boldsymbol{\omega} du \\
&= \pi \int_0^\infty \int_{S^2} \sqrt{u(u+\varepsilon)} e^u \int_{-1}^1 k(u, u+\varepsilon, s) ds \\
&[e^{2\varepsilon} f(\sqrt{u+\varepsilon}\,\boldsymbol{\omega})^2 + f(\sqrt{u}\,\boldsymbol{\omega})^2] d\boldsymbol{\omega} du \\
&\le \pi \varrho \int_0^\infty \int_{S^2} \sqrt{u} e^u [e^{2\varepsilon} f(\sqrt{u+\varepsilon}\,\boldsymbol{\omega})^2 + f(\sqrt{u}\,\boldsymbol{\omega})^2] d\boldsymbol{\omega} du \\
&\le 2\pi\varrho(e^\varepsilon + 1)\|f(v)\|_{\mathcal{X}}^2\,. \quad \blacksquare
\end{aligned}$$

Substitution in (3.1) shows that

$$\mathcal{S}_\varepsilon = \{e^{-u} P_\varepsilon(u) :\ P_\varepsilon(u+\varepsilon) = P_\varepsilon(u),\ P_\varepsilon \in L^2((0,\varepsilon))\}$$

is a subset of the kernel of Q_ε, $\varepsilon > 0$. All ε-periodic functions of u are collision invariants. Accordingly,

$$\mathcal{S}_0 = \{f \in \mathcal{X} :\ f = f(u)\}$$

is a subset of the kernel of Q_0. We shall prove in the following that these sets *are* the nullspaces.

Obviously, $\mathcal{S}_\varepsilon \subset \mathcal{S}_0$ holds. The orthogonal projection onto the closed subspace $\mathcal{S}_0$ of $\mathcal{X}$ is given by

$$(\Pi_0 f)(u) = \frac{1}{4\pi} \int_{S^2} f(\sqrt{u}\,\boldsymbol{\omega})\, d\boldsymbol{\omega}\,,$$

implying the orthogonal decomposition $f = \Pi_0 f + \Pi_0^\perp f$, with $\int_{S^2} \Pi_0^\perp f \, d\boldsymbol{\omega} = 0$ for all $f \in \mathcal{X}$. In the further decomposition

$$f = \Pi_\varepsilon f + \Pi_\varepsilon^\perp f + \Pi_0^\perp f \tag{3.4}$$

Π_ε denotes the projection onto $\mathcal{S}_\varepsilon$ and is determined by (see [6])

$$e^w (\Pi_\varepsilon f)(w) = \frac{\sum_{j=0}^\infty \sqrt{w+\varepsilon j}\,(\Pi_0 f)(w+\varepsilon j)}{\sum_{j=0}^\infty \sqrt{w+\varepsilon j}\, e^{-w-\varepsilon j}},$$

for $w \in (0,\varepsilon)$ and by periodic continuation of the right hand side elsewhere. The elements of the orthogonal complement of $\mathcal{S}_\varepsilon$ in $\mathcal{S}_0$ satisfy

$$\sum_{j=0}^\infty \sqrt{w+\varepsilon j}\,(\Pi_\varepsilon^\perp f)(w+\varepsilon j) = 0\,, \qquad w \in (0,\varepsilon)\,. \tag{3.5}$$

The most important technical tool for the rest of the paper will be the following coercivity result for Q_ε. Similar results can be found in [1] and [7]. The main differences to [7] are the weaker assumptions (2.6) on the overlap factor and detailed results concerning the dependence on ε.

Lemma 3.2 *Assume $f \in \mathcal{X}$ and $0 \le \varepsilon \le \varepsilon_0$. Then*

$$-\langle Q_\varepsilon(f), f\rangle_{\mathcal{X}} \ge 2\pi \int_{\mathbf{R}^3} (\Pi_0^\perp f)^2 e^{|v|^2} b(|v|^2)dv + \varepsilon^2 c \int_{\mathbf{R}^3} (\Pi_\varepsilon^\perp f)^2 |v| dv\,,$$

with an ε-independent constant $c > 0$.

Proof: Setting $e^u f(\sqrt{u}\,\boldsymbol{\omega}) = g(u,\boldsymbol{\omega})$, using (3.3) and (2.6), we obtain

$$\begin{aligned} &-\langle Q_\varepsilon(f), f\rangle_{\mathcal{X}} \\ &\ge \int_0^\infty \frac{\sqrt{u}}{4} e^{-u} b(u) \int_{S^2}\int_{S^2} [g(u+\varepsilon,\boldsymbol{\omega}') - g(u,\boldsymbol{\omega})]^2 d\boldsymbol{\omega}' d\boldsymbol{\omega}\, du\,. \end{aligned} \tag{3.6}$$

The decomposition (3.4) of f induces a corresponding decomposition $g = \Pi_\varepsilon g + \Pi_\varepsilon^\perp g + \Pi_0^\perp g$ where by an abuse of notation we use the same symbols for the projections. Note that $\Pi_\varepsilon g$ is an ε-periodic function of u. After expanding the square and carrying out the integrations over the unit ball, the right hand side of (3.6) can be written as

$$\begin{aligned} &4\pi^2 \int_0^\infty \sqrt{u}\, e^{-u} b(u) [(\Pi_\varepsilon^\perp g)(u+\varepsilon) - (\Pi_\varepsilon^\perp g)(u)]^2 du \\ &+ 2\pi \int_{\mathbf{R}^3} e^{-u} b(u) [(\Pi_0^\perp g)(u+\varepsilon,\boldsymbol{\omega}')^2 + (\Pi_0^\perp g)(u,\boldsymbol{\omega})^2] dv = A + B\,. \end{aligned}$$

The estimate

$$B \geq 2\pi \int_{\mathbf{R}^3} (\Pi_0^\perp f)^2 e^{|v|^2} b(|v|^2) dv$$

is obvious. In the term A we introduce the abbreviations $h(u) = (\Pi_\varepsilon^\perp g)(u)$ and $s(u) = \sqrt{u}\, e^{-u}$.

In the relations

$$h_k = h_l + \begin{cases} \sum_{j=l}^{k-1} (h_{j+1} - h_j)\,, & l < k, \\ \sum_{j=k}^{l-1} (h_j - h_{j+1})\,, & l > k, \end{cases} \tag{3.7}$$

and in the following we use the abbreviation $F_j(w) = F(w + \varepsilon j)$ for functions $F(u)$ and $w \in (0, \varepsilon)$. With this notation (3.5) reads $\sum_{l=0}^\infty s_l h_l = 0$. Multiplication of (3.7) by s_l and summation gives

$$h_k \sum_{l=0}^\infty s_l = \sum_{l=0}^{k-1} s_l \sum_{j=l}^{k-1} (h_{j+1} - h_j) + \sum_{l=k+1}^\infty s_l \sum_{j=k}^{l-1} (h_j - h_{j+1})\,.$$

By an application of the Cauchy-Schwarz inequality the estimate

$$|h_k| \sum_{l=0}^\infty s_l \leq (A_k + B_k) \sqrt{\sum_{j=0}^\infty s_j b_j (h_{j+1} - h_j)^2} \tag{3.8}$$

is obtained with

$$A_k = \sum_{l=0}^{k-1} s_l \sqrt{\sum_{j=l}^{k-1} \frac{1}{s_j b_j}}\,, \qquad B_k = \sum_{l=k+1}^\infty s_l \sqrt{\sum_{j=k}^{l-1} \frac{1}{s_j b_j}}\,.$$

With the assumption (2.6) and $s_l/s_j \leq e^{\varepsilon(j-l)}$ for $l < j$ the further estimates

$$s_k A_k \leq c\varepsilon^{-3/2} \sqrt{w + \varepsilon k}\, e^{-\varepsilon\delta k/2}\,, \qquad s_k B_k \leq c\varepsilon^{-3/2} e^{-\varepsilon k}$$

can be shown (c denotes generic constants independent of ε and w). Here we employed that $\sum_{j=0}^\infty F_j = O(\varepsilon^{-1})$ holds for $F \in L^1((0,\infty))$. This we also use when we now multiply (3.8) by s_k, take the square, sum over k and estimate:

$$\sum_{k=0}^\infty s_k^2 h_k^2 \leq \frac{c}{\varepsilon^2} \sum_{j=0}^\infty s_j b_j (h_{j+1} - h_j)^2\,.$$

Integrating this inequality with respect to w from 0 to ε, we obtain the desired estimate for the quantity A introduced above:

$$A \geq \varepsilon^2 \frac{c}{2} \int_0^\infty s(u)^2 h(u)^2 du = \varepsilon^2 c \int_{\mathbf{R}^3} (\Pi_\varepsilon^\perp f)^2 |v| dv \,. \quad \blacksquare$$

Our claims that $\mathcal{S}_\varepsilon$ and $\mathcal{S}_0$ are the nullspaces of Q_ε, $\varepsilon > 0$, and, respectively, Q_0 are corollaries of Lemma 3.2. Obviously, the ranges of Q_ε, $\varepsilon > 0$, and of Q_0 are subsets of the orthogonal complements of $\mathcal{S}_\varepsilon$ and, respectively, $\mathcal{S}_0$ in $\mathcal{X}$. In [7], range$(Q_\varepsilon) = \mathcal{S}_\varepsilon^\perp$ for $\varepsilon > 0$ has been proven under assumptions on the overlap factor stronger than (2.6). For the elastic case $\varepsilon = 0$ the situation is much simpler. Since there is no coupling in the u-direction, different values of u can be treated separately and range$Q_0 = \mathcal{S}_0^\perp$ follows from the Fredholm alternative. For the analysis of the following sections a complete characterization of the range is not necessary. We only need:

Lemma 3.3 **(1)** *The equation $Q_\varepsilon(h_\varepsilon) = e^{-|v|^2} v P(|v|^2)$ with $P(u+\varepsilon) = P(u)$ has a unique solution in $\mathcal{S}_\varepsilon^\perp$, given by $h_\varepsilon(\sqrt{u}\,\boldsymbol{\omega}) = -e^{-u}\mu_\varepsilon(u)P(u)\boldsymbol{\omega}$ where μ_ε solves*

$$-p_\varepsilon(u)\mu_\varepsilon(u+\varepsilon) + [r_\varepsilon(u) + e^\varepsilon r_\varepsilon(u-\varepsilon)]\mu_\varepsilon(u) - e^\varepsilon p_\varepsilon(u-\varepsilon)\mu_\varepsilon(u-\varepsilon) = u \,,$$

with

$$\begin{aligned} r_\varepsilon(u) &= \pi\sqrt{u(u+\varepsilon)} \int_{-1}^1 k(u, u+\varepsilon, s) ds \,, \\ p_\varepsilon(u) &= \pi\sqrt{u(u+\varepsilon)} \int_{-1}^1 s k(u, u+\varepsilon, s) ds \,, \end{aligned}$$

for $u \geq 0$ and $r_\varepsilon(u) = p_\varepsilon(u) = 0$ for $u < 0$. The function μ_ε can be estimated by

$$\int_0^\infty \mu_\varepsilon(u)^2 \sqrt{u}\, e^{-u} b(u) du \leq \frac{1}{4\pi^2\kappa\delta^2} \,. \tag{3.9}$$

(2) *The unique solution of $Q_0(h_0) = v g(|v|^2)$ in $\mathcal{S}_0^\perp$ is given by*

$$h_0(\sqrt{u}\,\boldsymbol{\omega}) = -\mu_0(u) g(u) \boldsymbol{\omega} \,, \qquad \text{with } \mu_0(u) = \frac{1}{2\pi \int_{-1}^1 (1-s) k(u,u,s) ds} \,.$$

Proof: **(1)** Uniqueness is a consequence of Lemma 3.2. Also the estimate (3.9) follows from the coercivity result and a straightforward computation using the assumption (2.6) on the overlap factor.

For proving existence we define the bilinear form

$$a(\mu_1,\mu_2) = -\langle Q_\varepsilon(e^{-u}\mu_1\omega_i), e^{-u}\mu_2\omega_i\rangle_{\mathcal{X}} \,.$$

Then the problem $Q_\varepsilon(-e^{-u}\mu_\varepsilon\boldsymbol{\omega}) = e^{-|v|^2}v$ is formally equivalent to

$$a(\mu_\varepsilon,\mu) = \frac{2\pi}{3}\int_0^\infty ue^{-u}\mu(u)du\,, \tag{3.10}$$

for all appropriate μ. With (3.3) we compute

$$\begin{aligned} a(\mu_1,\mu_2) \;\; = \;\; & \frac{2\pi}{3}\int_0^\infty e^{-u}[r_\varepsilon(u)(\mu_1(u+\varepsilon)\mu_2(u+\varepsilon)+\mu_1(u)\mu_2(u)) \\ & -p_\varepsilon(u)(\mu_1(u+\varepsilon)\mu_2(u)+\mu_1(u)\mu_2(u+\varepsilon))]du \end{aligned}$$

A Hilbert space $\mathcal{Y}$ of functions on $(0,\infty)$ is induced by the scalar product

$$\langle\mu_1,\mu_2\rangle_{\mathcal{Y}} = \int_0^\infty e^{-u}r_\varepsilon(u)(\mu_1(u+\varepsilon)\mu_2(u+\varepsilon)+\mu_1(u)\mu_2(u))du\,.$$

As a regularization of a we define

$$a_n(\mu_1,\mu_2) \;\; = \;\; a(\mu_1,\mu_2) + \frac{1}{n}\langle\mu_1,\mu_2\rangle_{\mathcal{Y}}\,.$$

Then boundedness and coercivity of a_n is easily shown. Therefore a unique solution $\mu_{\varepsilon,n}\in\mathcal{Y}$ of the problem

$$a_n(\mu_{\varepsilon,n},\mu) = \frac{2\pi}{3}\int_0^\infty ue^{-u}\mu(u)du\,, \qquad \forall\,\mu\in\mathcal{Y},$$

exists. By the construction of a_n it satisfies the estimate (3.9), which is uniform in n. Therefore we can pass to the limit $n\to\infty$ and obtain the existence of a solution of (3.10).

(2) The result is shown by a simple computation. ∎

4 The Elastic Limit of the Boltzmann Equation

In this section we carry out the limit $\varepsilon \to 0$ in (2.1), (2.2). The formal limit is given by

$$\begin{aligned} &\alpha^2 \frac{\partial f}{\partial t} + \alpha \left(v \cdot \nabla_x f + \frac{1}{2} \nabla_x \Phi \cdot \nabla_v f \right) = Q_0(f) \\ &f(x,v,0) = f_I(x,v)\,. \end{aligned} \tag{4.1}$$

For the electric potential and for the initial data we assume

$$\Phi(x,t) \in W^{1,\infty}(\mathbb{R}^3 \times (0,\infty)) \tag{4.2}$$

and

$$f_I \in L^2(\mathbb{R}^6; e^{|v|^2} dv dx)\,. \tag{4.3}$$

Multiplication of the Boltzmann equation (2.1) by $f e^{|v|^2 - \Phi}$ and integration gives formally

$$\begin{aligned} &\frac{\alpha^2}{2} \frac{d}{dt} \int_{\mathbf{R}^3} \|f\|_{\mathcal{X}}^2 e^{-\Phi} dx - \int_{\mathbf{R}^3} \langle Q_\varepsilon(f), f \rangle_{\mathcal{X}} e^{-\Phi} dx \\ &= -\frac{\alpha^2}{2} \int_{\mathbf{R}^3} \|f\|_{\mathcal{X}}^2 \frac{\partial \Phi}{\partial t} e^{-\Phi} dx \\ &\leq \left\| \frac{\partial \Phi}{\partial t} \right\|_{L^\infty(\mathbf{R}^3 \times (0,\infty))} \frac{\alpha^2}{2} \int_{\mathbf{R}^3} \|f\|_{\mathcal{X}}^2 e^{-\Phi} dx\,. \end{aligned} \tag{4.4}$$

The definiteness of Q_ε and the Gronwall lemma imply

$$\int_{\mathbf{R}^3} \|f\|_{\mathcal{X}}^2 dx \leq e^{K(t)} \int_{\mathbf{R}^3} \|f_I\|_{\mathcal{X}}^2 dx\,, \tag{4.5}$$

with $K(t) = t \|\frac{\partial \Phi}{\partial t}\|_{L^\infty(\mathbf{R}^3 \times (0,\infty))} + 2\|\Phi\|_{L^\infty(\mathbf{R}^3 \times (0,\infty))}$. Standard transport theory provides the existence of a unique mild solution of (2.1), (2.2) satisfying (4.5). Note that this estimate is uniform in both ε and α.

Theorem 4.1 *Let the assumptions (2.5), (2.6), (4.2), (4.3) hold. Then for fixed $\alpha > 0$ the solution of (2.1), (2.2) converges as $\varepsilon \to 0$ in $L^\infty_{loc}([0,\infty);$ $L^2(\mathbb{R}^6;\ e^{|v|^2} dv dx))$ weak* to the unique solution of (4.1).*

Proof. Convergence of the solution of (2.1), (2.2) for a suitable subsequence is a consequence of the estimate (4.5). It follows from a standard argument that we can go to the limit in the left hand side of the Boltzmann equation (2.1) in the sense of distributions.

For a test function $\varphi(x,v,t) \in C_0^\infty(\mathbb{R}^6 \times [0,\infty))$ we set $\overline{\varphi} = \varphi e^{-|v|^2} \in C_0^\infty(\mathbb{R}^6 \times [0,\infty))$. Then the symmetry of the collision operator Q_ε implies

$$\int_0^\infty \int_{\mathbf{R}^6} Q_\varepsilon(f)\varphi \, dvdxdt = \int_0^\infty \int_{\mathbf{R}^3} \langle Q_\varepsilon(\overline{\varphi}), f\rangle_{\mathcal{X}} dxdt\,.$$

By the continuity of the overlap factor, $Q_\varepsilon(\overline{\varphi})$ converges to $Q_0(\overline{\varphi})$ in $C(\mathbb{R}^6 \times [0,\infty))$ as $\varepsilon \to 0$. Thus, we can go to the limit in the right hand side of the above equation implying the convergence in the sense of distributions of the collision term in the Boltzmann equation.

The observation that the solution of the limiting problem (4.1) is unique by an estimate analogous to (4.5) completes the proof. ∎

5 The Drift-Diffusion Limit of the Elastic Transport Equation

Here we are concerned with the limit $\alpha \to 0$ in the initial value problem (4.1) for the Boltzmann equation with the elastic collision operator.

Analogously to (4.4) in the preceding section we derive

$$\begin{aligned} &\frac{\alpha^2}{2}\frac{d}{dt}\int_{\mathbf{R}^3} \|f\|_{\mathcal{X}}^2 e^{-\Phi} dx - \int_{\mathbf{R}^3} \langle Q_0(f), f\rangle_{\mathcal{X}} e^{-\Phi} dx \\ &\le \left\| \frac{\partial \Phi}{\partial t} \right\|_{L^\infty(\mathbf{R}^3\times(0,\infty))} \frac{\alpha^2}{2} \int_{\mathbf{R}^3} \|f\|_{\mathcal{X}}^2 e^{-\Phi} dx\,. \end{aligned} \tag{5.1}$$

Thus, the solution of (4.1) also satisfies (4.5). The estimate being uniform with respect to α, we have weak* convergence in $L^\infty_{loc}([0,\infty);\ L^2(\mathbb{R}^6;\ e^{|v|^2})dvdx))$ of f to f_0 as $\alpha \to 0$ for a suitable subsequence.

On the other hand, with the coercivity of Q_0 (Lemma 3.2) we deduce from (5.1)

$$\int_{\mathbf{R}^6} (f - \Pi_0 f)^2 e^{|v|^2} b(|v|^2)\, dvdx = O(\alpha^2)\,,$$

uniformly in $t \in (0,T)$, where $b(|v|^2)$ is the function from assumption (2.6). Therefore the function

$$R := \frac{f - \Pi_0 f}{\alpha}$$

is uniformly bounded with respect to α in $L^\infty_{loc}([0,\infty);\ L^2(\mathbb{R}^6;\ e^{|v|^2)}b(|v|^2)dvdx))$. Again extracting a subsequence we have weak convergence of R to R_0 in that space.

Going to the limit in $f - \Pi_0 f = \alpha R$ gives $f_0 = \Pi_0 f_0$ and, thus, f_0 depends on v only through $|v|^2 = u$: $f_0 = f_0(x,u,t)$.

In order to evaluate R_0 we use $Q_0(\Pi_0 f) = 0$ for rewriting the Boltzmann equation (4.1):

$$\alpha \frac{\partial f}{\partial t} + v \cdot \nabla_x f + \frac{1}{2} \nabla_x \Phi \cdot \nabla_v f = Q_0(R)\,.$$

Going to the limit in the sense of distributions we obtain

$$Q_0(R_0) = v \cdot \nabla_x f_0 + \frac{1}{2} \nabla_x \Phi \cdot \nabla_v f_0 = v \cdot \left(\nabla_x f_0 + \nabla_x \Phi \frac{\partial f_0}{\partial u} \right)\,.$$

Because of $\Pi_0 R_0 = 0$, Lemma 3.3 (2) implies

$$R_0 = -\mu_0(u) \boldsymbol{\omega} \cdot \left(\nabla_x f_0 + \nabla_x \Phi \frac{\partial f_0}{\partial u} \right)\,. \tag{5.2}$$

Now we multiply the Boltzmann equation by a test function $\vartheta(x,|v|^2,t) \in C_0^\infty(\mathbb{R}^3 \times [0,\infty)^2)$ and integrate by parts:

$$\int_0^\infty \int_{\mathbf{R}^6} \left[f \frac{\partial \vartheta}{\partial t} + \frac{f}{\alpha} v \cdot \left(\nabla_x \vartheta + \nabla_x \Phi \frac{\partial \vartheta}{\partial u} \right) \right] dvdxdt = - \int_{\mathbf{R}^6} f_I \vartheta(t=0) dvdx\,.$$

The term involving the collision operator vanishes because ϑ is a collision invariant. Since $\int_{\mathbf{R}^3} v g(|v|^2) dv = 0$ holds for any g, the above equation is equivalent to

$$\int_0^\infty \int_{\mathbf{R}^6} \left[f \frac{\partial \vartheta}{\partial t} + R v \cdot \left(\nabla_x \vartheta + \nabla_x \Phi \frac{\partial \vartheta}{\partial u} \right) \right] dvdxdt = - \int_{\mathbf{R}^6} f_I \vartheta(t=0) dvdx\,.$$

In the limit of this equation we use the formula (5.2) for R_0 and make the coordinate transformation $v = \sqrt{u}\,\boldsymbol{\omega}$. A straightforward computation (using $\int_{S^2} \omega_i \omega_j d\boldsymbol{\omega} = \frac{4\pi}{3}\delta_{ij}$) gives

$$\int_0^\infty \int_{\mathbb{R}^3} \int_0^\infty \left[\sqrt{u} f_0 \frac{\partial \vartheta}{\partial t} - \frac{u\mu_0(u)}{3} \left(\nabla_x f_0 + \nabla_x \Phi \frac{\partial f_0}{\partial u} \right) \right. \tag{5.3}$$
$$\left. \cdot \left(\nabla_x \vartheta + \nabla_x \Phi \frac{\partial \vartheta}{\partial u} \right) \right] du dx dt = - \int_{\mathbb{R}^3} \int_0^\infty \sqrt{u} (\Pi_0 f_I) \vartheta(t=0)\, du dx \,.$$

This is a weak formulation of the equation

$$\sqrt{u} \frac{\partial f_0}{\partial t} = \left\{ \nabla_x + \nabla_x \Phi \frac{\partial}{\partial u} \right\} \cdot \left(\frac{u\mu_0(u)}{3} \left\{ \nabla_x + \nabla_x \Phi \frac{\partial}{\partial u} \right\} f_0 \right) , \tag{5.4}$$

subject to the initial condition

$$f_0(t=0) = \Pi_0 f_I \,.$$

Note that the differential operator $\nabla_x + \nabla_x \Phi \frac{\partial}{\partial u}$ is a gradient along surfaces of constant total energy $u - \Phi$. Thus, the equation (5.4) describes diffusion along such surfaces. This can be made even more transparent for the case of a time independent electric potential $\Phi = \Phi(x)$. Then, using $z = u - \Phi(x)$ instead of u as a coordinate, (5.4) becomes

$$\sqrt{z + \Phi}\, \frac{\partial f_0}{\partial t} = \nabla_x \cdot \left(\frac{(z+\Phi)\mu_0(z+\Phi)}{3} \nabla_x f_0 \right) \,.$$

In this case, z is only a parameter in the equation, and surfaces of different total energy are decoupled.

Finally, we collect our results in a theorem:

Theorem 5.1 *Let the assumptions (2.5), (2.6), (4.2), (4.3) hold. Then, as* $\alpha \to 0$*, a subsequence of the solutions of (4.1) converges to a solution of (5.3) in* $L^\infty_{loc}([0,\infty);\ L^2(\mathbb{R}^6;\ e^{|v|^2} dv dx))$ *weak*.*

6 The Drift-Diffusion Limit of the Inelastic Transport Equation

The limit $\alpha \to 0$ in the inelastic problem (2.1), (2.2) has been carried out in [7]. Here, we only sketch the proof based on the assumptions (2.6) on the overlap factor. The procedure is similar to the preceding section.

We recall that (2.1), (2.2) has a unique solution f satisfying the uniform estimate (4.5). From (4.4) and Lemma 3.2 we conclude

$$2\pi \int_{\mathbf{R}^6} (\Pi_0^\perp f)^2 e^{|v|^2} b(|v|^2)dvdx + \varepsilon^2 c \int_{\mathbf{R}^6} (\Pi_\varepsilon^\perp f)^2 |v| dvdx = O(\alpha^2)\,. \tag{6.1}$$

Thus, $R = (f - \Pi_\varepsilon f)/\alpha$ is bounded uniformly in α with respect to the norm induced by the left hand side of (6.1). This also implies for the limit f_0 of f as $\alpha \to 0$ that $f_0 = \Pi_\varepsilon f_0 \in \mathcal{S}_\varepsilon$ holds. Therefore a function $N(x,u,t)$, ε-periodic in u, exists such that $f_0(x, \sqrt{u}\,\boldsymbol{\omega}, t) = e^{-u} N(x,u,t)$.

For the distributional limit R_0 of R as $\alpha \to 0$ we obtain the equation

$$Q_\varepsilon(R_0) = e^{-u} v \cdot \left[\nabla_x N + \nabla_x \Phi \left(\frac{\partial N}{\partial u} - N\right)\right] \tag{6.2}$$

with the solution (Lemma 3.3)

$$R_0 = -e^{-u} \mu_\varepsilon \boldsymbol{\omega} \cdot \left[\nabla_x N + \nabla_x \Phi \left(\frac{\partial N}{\partial u} - N\right)\right]\,.$$

For a distributional formulation of the initial value problem (2.1), (2.2) we use a test function $\vartheta(x,u,t)$, ε-periodic in u. The term originating from the collision operator vanishes since ϑ is a collision invariant. The limit as $\alpha \to 0$ reads

$$\int_0^\infty \int_{\mathbf{R}^6} \left[e^{-u} N \frac{\partial \vartheta}{\partial t} + R_0 v \cdot \left(\nabla_x \vartheta + \nabla_x \Phi \frac{\partial \vartheta}{\partial u}\right)\right] dvdxdt$$
$$= - \int_{\mathbf{R}^6} f_I \vartheta(t=0) dvdx\,. \tag{6.3}$$

Substitution of the above formula for R_0 and some straightforward manipulations give

$$\int_0^\infty \int_{\mathbf{R}^3} \int_0^\varepsilon C(u) N \frac{\partial \vartheta}{\partial t} dudxdt - \int_0^\infty \int_{\mathbf{R}^3} \int_0^\varepsilon D(u)$$
$$\left[\nabla_x N + \nabla_x \Phi \left(\frac{\partial N}{\partial u} - N\right)\right] \cdot \left[\nabla_x \vartheta + \nabla_x \Phi \frac{\partial \vartheta}{\partial u}\right] dudxdt \tag{6.4}$$
$$= - \int_{\mathbf{R}^3} \int_0^\varepsilon C(u) e^u (\Pi_\varepsilon f_I) \vartheta(t=0) dudx\,,$$

with

$$
\begin{aligned}
C(u) &= \varepsilon \sum_{k=0}^{\infty} \sqrt{u+\varepsilon k}\, e^{-u-\varepsilon k}\,, \\
D(u) &= \frac{\varepsilon}{3} \sum_{k=0}^{\infty} (u+\varepsilon k) e^{-u-\varepsilon k} \mu_\varepsilon(u+\varepsilon k)\,.
\end{aligned}
$$

Summarizing we have the following result:

Theorem 6.1 *Let the assumptions (2.5), (2.6), (4.2), (4.3) hold. Then, as $\alpha \to 0$, a subsequence of the solutions of (2.1), (2.2) converges to a solution of (6.4) in $L^\infty_{loc}([0,\infty);\ L^2(\mathbb{R}^6;\ e^{|v|^2}dvdx))$ weak*.*

The equation (6.4) is a weak formulation of the differential equation

$$
C(u)\frac{\partial N}{\partial t} = \left(\nabla_x + \nabla_x \Phi \frac{\partial}{\partial u}\right) \cdot \left[D(u)\left(\nabla_x N + \nabla_x \Phi \left(\frac{\partial N}{\partial u} - N\right)\right)\right]\,,
$$

subject to the initial condition

$$
N(x,u,0) = e^u (\Pi_\varepsilon f_I)(x,u)\,.
$$

Obviously, the coefficient $C(u)$ is positive. For an ε-periodic function $P(u) \geq 0$ we have

$$
\begin{aligned}
&\frac{3}{\varepsilon}\int_0^\varepsilon P(u)D(u)du = \int_0^\infty P(u)ue^{-u}\mu_\varepsilon(u)du \\
&= -\frac{1}{2\pi}\langle Q_\varepsilon(\sqrt{P}\, e^{-u}\mu_\varepsilon \boldsymbol{\omega}), \sqrt{P}\, e^{-u}\mu_\varepsilon \boldsymbol{\omega}\rangle_{\mathcal{X}} \geq \int_{\mathbb{R}^3} Pe^{-|v|^2} b\mu_\varepsilon^2 dv \\
&= 2\pi \int_0^\varepsilon P(u) \sum_{k=0}^{\infty} \sqrt{u+\varepsilon k}\, e^{-u-\varepsilon k} b(u+\varepsilon k)\mu_\varepsilon(u+\varepsilon k)^2\, du\,,
\end{aligned}
$$

and, thus,

$$
D(u) \geq \frac{2\pi\varepsilon}{3} \sum_{k=0}^{\infty} \sqrt{u+\varepsilon k}\, e^{-u-\varepsilon k} b(u+\varepsilon k)\mu_\varepsilon(u+\varepsilon k)^2 > 0\,.
$$

In the above estimate we have used Lemma 3.2 and $\sqrt{P}\, e^{-u}\mu_\varepsilon\boldsymbol{\omega} \in \mathcal{S}_0^\perp$.

7 The Elastic Limit of the Drift-Diffusion Equation

In this section the limit $\varepsilon \to 0$ in (6.4) is carried out. We shall deal with the equivalent system (6.2), (6.3) for N and $R_0 \in \mathcal{S}_0^\perp$. Note that the estimate (4.5) for f used in the preceding section is not only uniform in α but also in ε. After going to the limit $\alpha \to 0$, it gives a uniform-in-ε-estimate for N in $L^\infty_{loc}([0,\infty);\ L^2(\mathbb{R}^3 \times (0,\infty);\ \sqrt{u}\, e^{-u} du dx))$. Similarly, by $R_0 \in \mathcal{S}_0^\perp$, the estimate (6.1) for R gives a uniform estimate for R_0. Thus, for appropriate subsequences, R_0 and N have weak limits $\overline{R}_0$ and, respectively, $\overline{N}$ as $\varepsilon \to 0$. N being ε-periodic in u, the weak limit $\overline{N}$ is u-independent.

Now we choose in (6.3) an arbitrary test function $\vartheta(x,t) \in C_0^\infty(\mathbb{R}^3 \times (0,\infty))$ independent of u and go to the limit $\varepsilon \to 0$ in (6.2) and in (6.3). Going to the limit in the collision term in (6.2) has been justified in section 4. By Lemma 3.3, $\overline{R}_0$ can be computed from the limiting equation:

$$\overline{R}_0 = -e^{-u}\mu_0 \boldsymbol{\omega} \cdot (\nabla_x \overline{N} - \overline{N}\nabla_x \Phi)\,.$$

Substituting this representation in the limit of (6.3) gives

$$\begin{aligned} &\frac{\sqrt{\pi}}{2}\int_0^\infty \int_{\mathbf{R}^3} \overline{N}\frac{\partial \vartheta}{\partial t} dx dt - \int_0^\infty \int_{\mathbf{R}^3} \overline{D}(\nabla_x \overline{N} - \overline{N}\nabla_x \Phi)\cdot \nabla_x \vartheta dx dt \\ &= -\int_{\mathbf{R}^3} \vartheta(t=0) \int_0^\infty \sqrt{u}(\Pi_0 f_I) du dx\,, \end{aligned} \tag{7.1}$$

with

$$\overline{D} = \frac{1}{3}\int_0^\infty u e^{-u}\mu_0(u) du > 0\,.$$

This is the weak formulation of the drift-diffusion equation

$$\frac{\sqrt{\pi}}{2}\frac{\partial \overline{N}}{\partial t} = \overline{D}\nabla_x \cdot (\nabla_x \overline{N} - \overline{N}\nabla_x \Phi)\,,$$

subject to the initial condition

$$\overline{N}(t=0) = \frac{2}{\sqrt{\pi}}\int_0^\infty \sqrt{u}(\Pi_0 f_I) du\,.$$

Uniqueness of $\overline{N}$ follows from the standard theory for parabolic equations. Note that the coefficients $\frac{\sqrt{\pi}}{2}$ and $\overline{D}$ are the formal limits as $\varepsilon \to 0$ of C and, respectively, D, defined in the preceding section.

Theorem 7.1 *Let the assumptions (2.5), (2.6), (4.2), (4.3) hold. Then, as* $\varepsilon \to 0$*, the solutions of (6.4) constructed in Theorem 6.1 converge to the solution of (7.1) in* $L^\infty_{loc}([0,\infty);\ L^2(\mathbb{R}^3 \times (0,\infty);\ \sqrt{u}\, e^{-u} du dx))$ *weak*.*

8 An Intermediate Limit

In the last 4 sections the limits $\alpha \to 0$ and $\varepsilon \to 0$ in (2.1), (2.2) have been carried out. It has been shown in sections 4 and 5 that

$$\lim_{\alpha\to 0}\lim_{\varepsilon\to 0} f(x,v,t) = f_0(x,|v|^2,t)\,,$$

where f_0 solves (5.3), a problem modelling diffusion along surfaces of constant total energy. On the other hand, the result of sections 6 and 7 is that

$$\lim_{\varepsilon\to 0}\lim_{\alpha\to 0} f(x,v,t) = e^{-|v|^2}\overline{N}(x,t)\,,$$

where $\overline{N}$ solves the drift-diffusion problem (7.1).

Since the limits do not commute it is to be expected that a distinguished intermediate limit exists. This is the subject of the present section. We set $\varepsilon = \gamma\alpha$ in (2.1), (2.2) and carry out the limit $\alpha \to 0$.

In this section we need to differentiate the overlap factor. A differentiability assumption valid for models of the form (2.4) is the following: The overlap factor can be written as $k(u,u',s) = \hat{k}(\sqrt{u},\sqrt{u'},s)$ with

$$\hat{k} \in C^1\left([0,\infty)^2 \times [-1,1]\right)\,. \tag{8.1}$$

For a test function $\vartheta(u) \in C_0^\infty([0,\infty))$ we compute

$$\begin{aligned}\sqrt{u}Q_\varepsilon(\vartheta e^{-u}) &= e^{-u} r_\varepsilon(u)[\vartheta(u+\varepsilon) - \vartheta(u)] \\ &\quad - e^{\varepsilon-u} r_\varepsilon(u-\varepsilon)[\vartheta(u) - \vartheta(u-\varepsilon)]\,,\end{aligned} \tag{8.2}$$

with r_ε defined in Lemma 3.3. A simple computation gives the following result:

Lemma 8.1 *Assume (8.1). Then*

$$\lim_{\varepsilon\to 0}\frac{r_\varepsilon(u) - r_\varepsilon(u-\varepsilon)}{\varepsilon} = \frac{dr_0}{du}(u)\,,$$

uniformly for u *in compact subsets of* $[0,\infty)$.

This result and Taylor expansion of the remaining terms in the right hand side of (8.2) gives

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^2}Q_\varepsilon(\vartheta e^{-u}) = \frac{1}{\sqrt{u}}\frac{d}{du}\left(e^{-u}r_0\frac{d\vartheta}{du}\right) , \tag{8.3}$$

where the convergence is uniform in u. Now we consider a test function $\vartheta(v)\in C_0^\infty(\mathbb{R}^3)$. By $Q_\varepsilon(\Pi_0 f)\in\mathcal{S}_0$ and the symmetry of Q_ε we have

$$\begin{aligned}\frac{1}{\varepsilon}\int_{\mathbf{R}^3} Q_\varepsilon(\Pi_0 f)\vartheta dv &= \frac{1}{\varepsilon}\int_{\mathbf{R}^3} Q_\varepsilon(\Pi_0 f)(\Pi_0\vartheta)dv \\ &= \frac{1}{\varepsilon}\langle Q_\varepsilon(e^{-|v|^2}\Pi_0\vartheta),\Pi_0 f\rangle_{\mathcal{X}} \longrightarrow 0\,, \end{aligned}\tag{8.4}$$

as ε tends to zero by (8.3) and the uniform boundedness of f.

After these preparations, we proceed similarly to sections 5 and 6. We again use equation (4.4) for obtaining the uniform estimate (4.5) for f. Also Lemma 3.2 implies that $R=(f-\Pi_0 f)/\alpha$ is uniformly bounded. Therefore weak limits $f_0(x,|v|^2,t)$ and R_0 exist for a subsequence as $\alpha\to 0$.

The Boltzmann equation is written as

$$\alpha\frac{\partial f}{\partial t}+v\cdot\nabla_x f+\frac{1}{2}\nabla_x\Phi\cdot\nabla_v f = Q_{\gamma\alpha}(R)+\frac{1}{\alpha}Q_{\gamma\alpha}(\Pi_0 f)\,.$$

By (8.4) the last term tends to zero in the distributional sense. In the limit we obtain as in section 5

$$R_0 = -\mu_0(u)\boldsymbol{\omega}\cdot\left(\nabla_x f_0+\nabla_x\Phi\frac{\partial f_0}{\partial u}\right) . \tag{8.5}$$

Now we choose an arbitrary test function $\vartheta(x,u,t)\in C_0^\infty(\mathbb{R}^3\times[0,\infty)^2)$ and consider the weak form of (2.1), (2.2)

$$\begin{aligned}&\int_0^\infty\int_{\mathbf{R}^6}\left[f\frac{\partial\vartheta}{\partial t}+Rv\cdot\left(\nabla_x\vartheta+\nabla_x\Phi\frac{\partial\vartheta}{\partial u}\right)\right]dv\,dx\,dt \\ &+\frac{1}{\alpha^2}\int_0^\infty\int_{\mathbf{R}^3}\langle Q_{\gamma\alpha}(e^{-|v|^2}\vartheta),f\rangle_{\mathcal{X}}\,dx\,dt = -\int_{\mathbf{R}^6} f_I\vartheta(t=0)dv\,dx\,.\end{aligned}$$

With (8.3) we can go to the limit as usual and obtain

$$\int_0^\infty\int_{\mathbf{R}^3}\int_0^\infty\left[\sqrt{u}f_0\frac{\partial\vartheta}{\partial t}-\frac{u\mu_0(u)}{3}\left(\nabla_x f_0+\nabla_x\Phi\frac{\partial f_0}{\partial u}\right)\right.$$

$$\cdot \left(\nabla_x \vartheta + \nabla_x \Phi \frac{\partial \vartheta}{\partial u} \right) \Big] du dx dt + \gamma^2 \int_0^\infty \int_{\mathbf{R}^3} \int_0^\infty e^u f_0 \frac{\partial}{\partial u} \left(e^{-u} r_0 \frac{\partial \vartheta}{\partial u} \right) du dx dt$$
$$= - \int_{\mathbf{R}^3} \int_0^\infty \sqrt{u} (\Pi_0 f_I) \vartheta(t=0)\, du dx \,. \tag{8.6}$$

Theorem 8.1 *Let the assumptions (2.5), (2.6), (4.2), (4.3), (8.1) hold. Then, as $\alpha \to 0$, a subsequence of the solutions of (2.1), (2.2) with $\varepsilon = \gamma\alpha$ converges to a solution of (8.6) in $L^\infty_{loc}([0,\infty);\ L^2(\mathbb{R}^6;\ e^{|v|^2} dv dx))$ weak*.*

The problem (8.6) is a weak formulation of

$$\sqrt{u} \frac{\partial f_0}{\partial t} = \left\{ \nabla_x + \nabla_x \Phi \frac{\partial}{\partial u} \right\} \cdot \left(\frac{u \mu_0(u)}{3} \left\{ \nabla_x + \nabla_x \Phi \frac{\partial}{\partial u} \right\} f_0 \right) + \overline{Q}(f_0) \,,$$

with

$$\overline{Q}(f_0) = \gamma^2 \frac{\partial}{\partial u} \left(e^{-u} r_0 \frac{\partial}{\partial u} (e^u f_0) \right) \,,$$

subject to the initial condition

$$f_0(t=0) = \Pi_0 f_I \,.$$

We conjecture that the two different limits mentioned at the beginning of this section can be obtained by letting $\gamma \to 0$ and, respectively, $\gamma \to \infty$.

References

[1] N. Ben Abdallah, P. Degond, On a hierarchy of macroscopic models for semiconductors, preprint, Université Paul Sabatier, Toulouse, 1995.

[2] P. Dmitruk, A. Saul, L. Reyna, High electric field approximation to charge transporet in semiconductor devices, *Appl. Math. Letters* **5** (1992), pp. 99–102.

[3] A. Majorana, Space homogeneous solutions of the Boltzmann equation describing electron-phonon interactions in semiconductors, *Transp. Theory and Statist. Phys.* **20** (1991), pp. 261–279.

[4] P.A. Markowich, F. Poupaud, C. Schmeiser, Diffusion approximation of nonlinear electron-phonon collision mechanisms, *M^2AN* **29** (1995), pp. 857–869.

[5] P.A. Markowich, C. Ringhofer, C. Schmeiser, *Semiconductor equations*, Springer-Verlag, Wien, 1990.

[6] P.A. Markowich, C. Schmeiser, Relaxation approximation for electron-phonon interaction in semiconductors, *Math. Models and Meth. in Appl. Sci.* **5** (1995), pp. 519–527.

[7] P.A. Markowich, C. Schmeiser, The drift-diffusion limit for electron-phonon interaction in semiconductors, to appear in *Asymptotic Analysis*, 1996.

[8] C. Schmeiser, A. Zwirchmayr, A drift-diffusion model for almost elastic electron-phonon interaction in semiconductors, manuscript, TU Wien, 1996.