Fractional diffusion limit of a linear kinetic equation in a bounded domain

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Abstract. A version of fractional diffusion on bounded domains, subject to 'homogeneous Dirichlet boundary conditions' is derived from a kinetic transport model with homogeneous inflow boundary conditions. For nonconvex domains, the result differs from standard formulations. It can be interpreted as the forward Kolmogorow equation of a stochastic process with jumps along straight lines, remaining inside the domain.

Key words: Kinetic transport equations, linear Boltzmann operator, anomalous diffusion limit, fractional diffusion, asymptotic analysis

AMS subject classification: 76P05, 35B40, 26A33

Acknowledgment: This work has been supported by the PhD program *Dissipation and Dispersion in Nonlinear PDEs*, supported by the Austrian Science Fund, grant no. W1245. P.A.S acknowledges support from the Consejo Nacional de Ciencia y Tecnologia of Mexico, and the Vienna Science and Technology Fund, grant no. LS13-029.

1 Introduction

This work is an extension to bounded domains of earlier efforts [4, 19, 20] to derive fractional diffusion equations from kinetic transport models. This raises the issue of the inclusion of boundary effects, which can, however, not be reduced to boundary conditions since fractional diffusion is a nonlocal process. Our main result is the derivation of a new way of realizing 'homogeneous Dirichlet boundary conditions', coinciding on convex domains with an already established model, see e.g. [14].

Let $\Omega \subset \mathbb{R}^d$ denote a bounded domain with smooth boundary. We shall study the asymptotic behavior as $\varepsilon > 0$ tends to zero of the kinetic relaxation model

$$\varepsilon^\alpha \partial_t f_\varepsilon + \varepsilon v \cdot \nabla_x f_\varepsilon = Q(f_\varepsilon) := \int_{\mathbb{R}^d} M f'_\varepsilon - M' f_\varepsilon \, \mathrm{d}v' \,, \tag{1}$$

with $f_\varepsilon = f_\varepsilon(x, v, t)$, $(x, v, t) \in \Omega \times \mathbb{R}^d \times [0, \infty)$ (where the superscript $'$ denotes evaluation at v'), subject to zero inflow boundary conditions and well prepared initial data:

$$f_\varepsilon(x, v, t) = 0 \qquad \text{for } (x, v) \in \Gamma^- \,, \quad t > 0 \,, \tag{2}$$

$$f_\varepsilon(x, v, 0) = f^{in}(x, v) := \rho^{in}(x) M(v) \qquad \text{for } (x, v) \in \Omega \,, \tag{3}$$

with $\Gamma^\pm = \{(x, v) |\, x \in \partial\Omega,\ \mathrm{sign}(v \cdot \nu(x)) = \pm 1\}$, where ν denotes the unit outward normal along $\partial\Omega$. We assume a 'fat-tailed' equilibrium distribution M, satisfying

$$M(v) = 1/|v|^{d+\alpha} \qquad \text{for } |v| \geq 1 \,, \text{ with } 0 < \alpha < 2 \,, \tag{4}$$

$$M(v) > 0, \qquad M(v) = M(-v) \qquad \text{for all } v \in \mathbb{R}^d \,, \tag{5}$$

$$M \in L^\infty(\mathbb{R}^d) \,, \qquad \text{and} \qquad \int_{\mathbb{R}^d} M(v) \, \mathrm{d}v = 1 \,. \tag{6}$$

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Note that these assumptions imply that M does not have finite second order moments.

The translation of homogeneous Dirichlet boundary conditions to fractional diffusion induce a certain behaviour of solutions close to the boundary. The domain of the fractional diffusion operator, we shall derive, contains test functions in

$$\mathcal{D}_\Omega := \{\varphi \in C_0^\infty(\overline{\Omega} \times [0,\infty)) : \delta(x)^{-2}\varphi(x,t) \text{ bounded}\}\,, \tag{7}$$

where $\delta(x) := \text{dist}(x, \partial\Omega)$ denotes the distance of a point $x \in \Omega$ to the boundary.

A convenient functional analytic setting for the main result of this paper is the L^2-space $L^2_{M^{-1}}(\Omega \times \mathbb{R}^d)$ of functions of (x,v) with weight $1/M(v)$.

Theorem 1. *Let $\rho^{in} \in L^2(\Omega)$, and let f_ε be the solution of (1)–(3). Then, for any $T > 0$, there exists $\rho \in L^\infty(0,T;L^2(\Omega))$ such that $f_\varepsilon(x,v,t) \to \rho(x,t)M(v)$ as $\varepsilon \to 0$, in $L^\infty(0,T;L^2_{M^{-1}}(\Omega \times \mathbb{R}^d))$ weak-$\star$, and ρ satisfies*

$$\int_\Omega \rho^{in}\varphi(t=0)dx + \int_0^\infty \int_\Omega \rho\,\partial_t\varphi\,dx\,dt = \int_0^\infty \int_\Omega \rho(h_\alpha\varphi - \mathcal{L}_\alpha(\varphi))dx\,dt, \tag{8}$$

for all $\varphi \in \mathcal{D}_\Omega$, with

$$\mathcal{L}_\alpha(\varphi)(x,t) = \Gamma(\alpha+1)\text{P.V.}\int_{\{w\in\mathbb{R}^d:[x,x+w]\subset\Omega\}} \frac{\varphi(x+w,t)-\varphi(x,t)}{|w|^{d+\alpha}}\,dw,$$

and

$$h_\alpha(x) = \int_{\mathbb{R}^d} \frac{1}{|w|^{d+\alpha}}\, e^{-\frac{|x-x_0(x,w)|}{|w|}}\,dw\,, \tag{9}$$

where $[x,y]$, $x,y \in \mathbb{R}^d$, denotes the straight line segment connecting x and y, and $x_0(x,w)$ is the point closest to x in the intersection of $\partial\Omega$ with the ray starting at x in the direction w.

The function h_α is well defined by (9) and converges to ∞ when $x \to \partial\Omega$, see Proposition 1 in Section 4.

Remark 1. *Theorem 1 remains true with slightly modified proofs for generalized versions of the model. For example, (4) may be replaced by the more general condition*

$$M(v) \sim 1/|v|^{d+\alpha} \qquad \text{as } |v| \to \infty. \tag{10}$$

An example coming from stochastic analysis is the probability density function of an α-stable process, see [6].

Remark 2. *Another possible generalization is to permit a more general collision operator, satisfying the micro-reversibility principle:*

$$Q(f) = \int_{\mathbb{R}^d} [\sigma(v,v')M(v)f(v') - \sigma(v',v)M(v')f(v)]\,dv'$$

where the cross-section σ is symmetric, i.e. $\sigma(v,v') = \sigma(v',v)$, v, v' in $\mathbb{R}^d$, and bounded from above and away from zero:

$$0 < \nu_1 \le \sigma(v,v') \le \nu_2 < \infty\,.$$

The derivation of macroscopic limits from kinetic equations when the collision kernel has a Maxwellian as an equilibrium distribution is a classical problem studied in the pioneering works [25], [15], and [18]. Here the essential properties of the equilibrium distribution are vanishing mean velocity and finite second order moments. In the case where the equilibrium distribution is heavy-tailed, the problem was first studied for relaxation type collision operators in [20], [19] and [4], from an analytical point of view and in [16] with a probabilistic approach, obtaining as a macroscopic limit a fractional heat equation. These are results on whole space, and they have recently been extended to collision operators of fractional Fokker-Planck type [8] and to the derivation of fractional diffusion with drift [1, 2, 3]. The proofs of most of these results are based on the moment method introduced in [19], which will also be used here.

To find an appropriate definition of fractional diffusion in a bounded domain is not obvious since it describes the probability distribution of a jump process. The formulation of appropriate models as macroscopic limits of kinetic equations is the subject of this work and of the very recent contribution [7], where the problem of deriving a fractional heat equation from a kinetic fractional-Fokker-Planck equation is tackled with zero inflow and specular reflection boundary conditions, where the spatial domain is a circle. The main differences between this work and [7] are that we use a relaxation type collision operator, we only consider inflow boundary conditions, but we permit general, in particular nonconvex, position domains.

There are several equivalent definitions of the fractional Laplacian in the whole domain (see [17]), however, for bounded domains there are different definitions, depending on the details of the underlying stochastic process. For instance, if we consider the stochastic process consisting of a fractional Brownian motion with an $\alpha/2-$stable subordinator and killed upon leaving the domain it has as infinitesimal generator the restricted fractional Laplacian (see [14])

$$-(-\Delta|_\Omega)^{\alpha/2}\varphi(x) := c_{d,\alpha}\ \text{P.V.}\int_{\mathbb{R}^d} \frac{\varphi(y)\mathbf{1}_\Omega(y) - \varphi(x)}{|x-y|^{d+\alpha}}\,\mathrm{d}y\,, \qquad c_{d,\alpha} > 0\,. \tag{11}$$

This operator has also been derived in [7] as macroscopic limit of a kinetic equation in a circle, subject to zero inflow boundary conditions. The macroscopic operator of Theorem 1 can be written in the similar form,

$$-h_\alpha\varphi + \mathcal{L}_\alpha(\varphi) = \Gamma(\alpha+1)\,\text{P.V.}\int_{\mathbb{R}^d} \frac{\varphi(y)\mathbf{1}_{\mathcal{S}_\Omega(x)}(y) - \varphi(x)}{|x-y|^{d+\alpha}}dy\,, \tag{12}$$

where $\mathcal{S}_\Omega(x)$ denotes the biggest star-shaped subdomain of Ω with center in x. Obviously, (11) and (12) coincide for convex Ω (the situation of [7]). The difference in the stochastic process interpretations of (11) and (12) is that in the latter jumps are only permitted along straight lines, which do not leave the domain.

For completeness we also mention the spectral fractional Laplacian defined as follows: The operator $-\Delta$ subject to homogeneous Dirichlet boundary conditions along $\partial\Omega$ has positive eigenvalues $0 < \lambda_1 \le \lambda_2 \ldots$ with corresponding normalized eigenfunctions $\{e_k\}_{k\ge 1}$. The spectral fractional Laplacian (subject to homogeneous Dirichlet boundary conditions) is defined by

$$(-\Delta_\Omega)^{\alpha/2}\varphi(x) := \sum_{i=1}^\infty \lambda_i^{\alpha/2} e_i(x)\int_\Omega e_i(y)\varphi(y)dy\,. \tag{13}$$

It can also be interpreted as generating a stochastic process (see [9]). A representation formula similar to (11) and (12) has been derived in [23]:

$$(-\Delta_\Omega)^{\alpha/2}\varphi(x) = c_{d,\alpha}\ \text{P.V.}\int_\Omega [\varphi(x)-\varphi(y)]J(x,y)\,\mathrm{d}y + c_{d,\alpha}\,\kappa(x)\varphi(x), \qquad \text{for } x\in\Omega$$

where the functions J and κ and the constant $c_{d,\alpha}$ satisfy (with positive constants C_1, C_2 and C_3)

$$C_1\delta(x)\delta(y) \le J(x,y) \le C_2\min\left(\frac{1}{|x-y|^{d+\alpha}}, \frac{\delta(x)\delta(y)}{|x-y|^{d+2+\alpha}}\right),$$

and

$$C_3^{-1}\delta^{-\alpha}(x) \le \kappa(x) \le C_3\delta^{-\alpha}(x).$$

In [22] it is proven that the two operators $(-\Delta_\Omega)^{\alpha/2}$ and $(-\Delta|_\Omega)^{\alpha/2}$ are different since, for instance, the eigenfunctions of the former are smooth up to the boundary whereas the eigenfunctions of the latter are no better than Hölder continuous up to the boundary. In recent years fractional Laplace operators have been extensively used since they seem to be more suitable for the description of phenomena such as contaminants propagating in water [5], plasma physics [12], among many others (see [24] and [21]). However, there is some literature where for the fractional Laplacian on bounded domains the definitions (11) and (13) are used interchangeably, thus leading to false results.

2 Uniform estimates and modified test functions

It is a standard result of kinetic theory that the initial-boundary value problem (1)–(3) with an equilibrium distribution M satisfying (4)–(6) and an initial position density $\rho^{in} \in L^1(dx)$ has a unique solution, which is nonnegative, if the same holds for ρ^{in} (see, e.g. [10], Chapter XXI). This will be assumed in the following, where we always denote by dx, dv, and dt the Lebesgue measures on Ω, $\mathbb{R}^d$, and, respectively, $(0,\infty)$. We start with standard estimates:

Lemma 1. *Let $\rho^{in} \in L^2_+(dx)$. Then the solution f_ε of (1)–(3) satisfies*

$$f_\varepsilon \in L^\infty(dt,\, L^2_+(dx\,dv/M)) \qquad \textit{uniformly as } \varepsilon \to 0\,,$$

and, with $\rho_\varepsilon := \rho_{f_\varepsilon}$,

$$f_\varepsilon - \rho_\varepsilon M = O(\varepsilon^{\alpha/2}) \qquad \textit{in } L^2(dx\,dv\,dt/M)\,, \qquad \textit{as } \varepsilon \to 0\,.$$

Proof. Multiplication of (1) by f_ε/M, integration with respect to x and v, the divergence theorem, and the boundary condition (2) yield

$$\begin{aligned} \frac{\varepsilon^\alpha}{2}\frac{d}{dt}\int_\Omega\int_{\mathbb{R}^d}\frac{f_\varepsilon^2}{M}dv\,dx + \varepsilon\int_{\Gamma^+} v\cdot\nu\frac{f_\varepsilon^2}{2M}dv\,dx &= \int_\Omega\int_{\mathbb{R}^d} Q(f_\varepsilon)\frac{f_\varepsilon}{M}dv\,dx \\ &= -\|f_\varepsilon - \rho_\varepsilon M\|^2_{L^2(dx\,dv/M)}\,, \end{aligned} \tag{14}$$

where the second equality is a well known fact and the result of a straightforward computation (see, e.g. [11]). The nonnegativity of the second term and an integration with respect to t over $(0,T)$ give

$$\frac{\varepsilon^\alpha}{2}\|f_\varepsilon(\cdot,\cdot,T)\|^2_{L^2(dx\,dv/M)} + \int_0^T \|f_\varepsilon - \rho_\varepsilon M\|^2_{L^2(dx\,dv/M)}dt \le \frac{\varepsilon^\alpha}{2}\|\rho^{in}\|^2_{L^2(dx)}\,,$$

completing the proof. ∎

For the proof of Theorem 1 we employ the moment method introduced in [19], which relies on test functions solving a suitably chosen adjoint problem. For given $\varphi \in \mathcal{D}_\Omega$ the function $\chi_\varepsilon(x,v,t)$ is the solution of the stationary kinetic equation

$$\chi_\varepsilon - \varepsilon v\cdot\nabla_x\chi_\varepsilon = \varphi\,, \tag{15}$$

subject to the inflow boundary condition

$$\chi_\varepsilon = 0 \qquad \text{on } \Gamma^+\,. \tag{16}$$

Note that the left hand side of (15) is an adjoint version of a part of (1), where only the loss term of the collision operator and the transport operator have been kept.

We can readily solve (15), (16) via the method of characteristics, obtaining

$$\chi_\varepsilon(x,v,t) = \int_0^{r(x,v)/\varepsilon} \mathrm{e}^{-s}\varphi(x+\varepsilon sv,t)ds\,, \qquad \text{where} \quad r(x,v) = \frac{|x-x_0(x,v)|}{|v|}\,, \tag{17}$$

and $x_0(x,v)$ is the point closest to x in the intersection of $\partial\Omega$ and the ray starting at x with direction v. In the following a different representation will be convenient:

$$\chi_\varepsilon(x,v,t) = \varphi(x,t)\left(1-\mathrm{e}^{-r(x,v)/\varepsilon}\right) + \int_0^{r(x,v)/\varepsilon} \mathrm{e}^{-s}[\varphi(x+\varepsilon sv,t)-\varphi(x,t)]ds\,. \tag{18}$$

This already shows the main difference to the whole space situation [19], which is the boundary layer correction in the parenthesis on the right hand side of (18).

In the following we shall need a uniform boundedness result.

Lemma 2. *Let $\varphi\in\mathcal{D}_\Omega$ and let χ_ε be given by (17). Then*

$$\|\chi_\varepsilon\|_{L^2(M\,dx\,dv\,dt)} \le \|\varphi\|_{L^2(dx\,dt)}\,, \qquad \|\partial_t\chi_\varepsilon\|_{L^2(M\,dx\,dv\,dt)} \le \|\partial_t\varphi\|_{L^2(dx\,dt)}\,.$$

Proof. Multiplication of (15) by $M\chi_\varepsilon$ and integration with respect to v gives

$$\|\chi_\varepsilon\|^2_{L^2(M\,dv)} - \frac{\varepsilon}{2}\nabla_x\cdot\int_{\mathbb{R}^d} vM\chi_\varepsilon^2 dv = \varphi\int_{\mathbb{R}^d} M\chi_\varepsilon dv \le |\varphi|\,\|\chi_\varepsilon\|_{L^2(M\,dv)}\,,$$

where the Cauchy-Schwarz inequality and the normalization of M has been used. Integration with respect to x and t, the divergence theorem, and the boundary condition (16) for χ_ε lead to

$$\|\chi_\varepsilon\|^2_{L^2(M\,dx\,dv\,dt)} - \frac{\varepsilon}{2}\int_0^\infty\int_{\Gamma^-} \nu\cdot vM\chi_\varepsilon^2 dv\,d\sigma\,dt \le \|\varphi\|_{L^2(dx\,dt)}\|\chi_\varepsilon\|_{L^2(M\,dx\,dv\,dt)}\,,$$

completing the proof of the first inequality. The proof of the second is analogous after differentiation of (15) with respect to t. ∎

3 Proof of Theorem 1

With $\varphi\in\mathcal{D}_\Omega$ and χ_ε defined by (17), multiplication of (1) by χ_ε and integration with respect to x, v and t gives

$$\begin{aligned}
-\int_0^\infty\int_{\mathbb{R}^d}\int_\Omega f_\varepsilon\partial_t\chi_\varepsilon dx\,dv\,dt - \int_{\mathbb{R}^d}\int_\Omega \rho^{in}M\chi_\varepsilon(t=0)dx\,dv& \\
= \varepsilon^{-\alpha}\int_0^\infty\int_{\mathbb{R}^d}\int_\Omega (\rho_\varepsilon M\chi_\varepsilon - f_\varepsilon\chi_\varepsilon + f_\varepsilon\varepsilon v\cdot\nabla_x\chi_\varepsilon)dx\,dv\,dt& \\
= \int_0^\infty\int_\Omega \rho_\varepsilon\left(\varepsilon^{-\alpha}\int_{\mathbb{R}^d} M(\chi_\varepsilon-\varphi)dv\right)dx\,dt\,.&
\end{aligned} \tag{19}$$

In the sequel we shall need the following notation: For $x,y\in\mathbb{R}^d$ we denote by $[x,y]$ the line segment connecting x and y. Furthermore, we denote by $\mathcal{S}_\Omega(x)$ the largest star shaped subdomain of Ω with center x, i.e.

$$\mathcal{S}_\Omega(x) := \{y\in\Omega:\ [x,y]\subset\Omega\}$$

The heart of our analysis is the asymptotics for the term in parantheses on the right hand side of (19).

Lemma 3. *Let* $\varphi \in \mathcal{D}_\Omega$ *and let* χ_ε *be given by* (17). *Then*

$$\lim_{\varepsilon\to 0} \varepsilon^{-\alpha} \int_{\mathbb{R}^d} M(\chi_\varepsilon - \varphi)dv = -h_\alpha \varphi + \mathcal{L}_\alpha(\varphi) \tag{20}$$

locally uniformly in x *and* t*, where*

$$h_\alpha(x) = \int_{\mathbb{R}^d} \frac{1}{|v|^{d+\alpha}} e^{-\frac{|x-x_0(x,v)|}{|v|}} dv \,,$$

$$\mathcal{L}_\alpha(\varphi)(x,t) = \Gamma(\alpha+1)\,\mathrm{P.V.} \int_{\mathcal{S}_\Omega(x)} \frac{\varphi(y,t)-\varphi(x,t)}{|y-x|^{d+\alpha}} dy \,.$$

Proof. The representation (18) of χ_ε induces the splitting

$$\varepsilon^{-\alpha} \int_{\mathbb{R}^d} M(\chi_\varepsilon - \varphi)dv = -h_\alpha^\varepsilon \varphi + \mathcal{L}_\alpha^\varepsilon(\varphi) \,,$$

with

$$\begin{aligned} h_\alpha^\varepsilon(x) &= -\varepsilon^{-\alpha} \int_{\mathbb{R}^d} M(v) e^{-r(x,v)/\varepsilon} dv \,, \\ \mathcal{L}_\alpha^\varepsilon(\varphi)(x,t) &= \varepsilon^{-\alpha} \int_{\mathbb{R}^d} \int_0^{r(x,v)/\varepsilon} M(v) e^{-s} [\varphi(x+\varepsilon s v, t) - \varphi(x,t)] ds\, dv \,. \end{aligned}$$

We shall consider these parts separately. In both cases we shall start by proving that the small velocities do not contribute to the limit. This splits the rest of the proof into 4 steps.

Step 1: We consider the contribution to h_α^ε coming from the small velocities. For $|v| \le 1$ we have $r(x,v) \ge \delta(x)$. Therefore

$$\varepsilon^{-\alpha} \int_{|v|\le 1} M(v) e^{-r(x,v)/\varepsilon} dv \le \varepsilon^{-\alpha} e^{-\delta(x)/\varepsilon} \le c \frac{\varepsilon^{2-\alpha}}{\delta(x)^2} \,,$$

since the map $z \mapsto z^2 e^{-z}$, $z \ge 0$, is bounded.

Step 2: The previous step implies that h_α^ε is asymptotically equivalent to

$$\varepsilon^{-\alpha} \int_{|v|>1} |v|^{-d-\alpha} e^{-r(x,v)/\varepsilon} dv \,. \tag{21}$$

In this integral we make the coordinate transformation $w = \varepsilon v$. Observing that

$$\frac{r(x,w/\varepsilon)}{\varepsilon} = \frac{|x - x_0(x,w/\varepsilon)|}{|w|} = r(x,w) \,,$$

since $x_0(x,w/\varepsilon) = x_0(x,w)$, the expression in (21) is equal to

$$\int_{|w|>\varepsilon} |w|^{-d-\alpha} e^{-r(x,w)} dw \,.$$

For proving that this converges to $h_\alpha(x)$, we need to estimate

$$\begin{aligned} \int_{|w|\le\varepsilon} |w|^{-d-\alpha} e^{-r(x,w)} dw &\le \int_{|w|\le\varepsilon} |w|^{-d-\alpha} e^{-\delta(x)/|w|} dw = |S^d| \delta(x)^{-\alpha} \int_{\delta(x)/\varepsilon}^\infty s^{\alpha-1} e^{-s} ds \\ &\le |S^d| \frac{\varepsilon^{2-\alpha}}{\delta(x)^2} \sup_{\gamma\ge 0} \left(\gamma^{2-\alpha} \int_\gamma^\infty s^{\alpha-1} e^{-s} ds \right) \,. \end{aligned}$$

The supremum is finite since the integrand is bounded and decays exponentially as $s \to \infty$.

Combining this result with Step 1 shows that

$$|h_\alpha^\varepsilon(x) - h_\alpha(x)| \le c\frac{\varepsilon^{2-\alpha}}{\delta(x)^2}\,,$$

implying pointwise convergence of h_α^ε to h_α in Ω. Since $|\varphi(x,t)| \le c\,\delta(x)^2$, the convergence of $h_\alpha^\varepsilon\varphi$ to $h_\alpha\varphi$ is uniform in (x,t).

Step 3: We analyze the contributions from the small velocities to $\mathcal{L}_\alpha^\varepsilon(\varphi)$. For the test function difference, we apply the Taylor expansion:

$$\begin{aligned}&\left|\varepsilon^{-\alpha}\int_{|v|\le 1}\int_0^{r(x,v)/\varepsilon} M(v)e^{-s}\left(\varepsilon s v\cdot\nabla_x\varphi(x,t) + \frac{\varepsilon^2 s^2}{2}v^{tr}\nabla_x^2\varphi(\hat{x},t)v\right)ds\,dv\right|\\ &\le \left|\varepsilon^{1-\alpha}\nabla_x\varphi(x,t)\cdot\int_{|v|\le 1} vM(v)\int_0^{r(x,v)/\varepsilon} se^{-s}ds\,dv\right| + \varepsilon^{2-\alpha}c\int_{|v|\le 1}|v|^2M(v)dv\int_0^\infty s^2e^{-s}ds\,.\end{aligned}$$

In the first term on the right hand side we change the order of integration:

$$\begin{aligned}&\int_{|v|\le 1} vM(v)\int_0^{r(x,v)/\varepsilon} se^{-s}ds\,dv = \int_0^\infty se^{-s}\int_{|v|\le 1,\,\varepsilon s\le r(x,v)} vM(v)dv\,ds\\ &= \int_0^{\delta(x)/\varepsilon} se^{-s}\int_{|v|\le 1} vM(v)dv\,ds + \int_{\delta(x)/\varepsilon}^\infty se^{-s}\int_{|v|\le 1,\,\varepsilon s\le r(x,v)} vM(v)dv\,ds\end{aligned}$$

In the first term on the right hand side, the restriction $\varepsilon s \le r(x,v)$ can be omitted, since it is automatically satisfied for $\varepsilon s \le \delta(x) \le r(x,v)$. As a consequence this term vanishes by M being even. The last term can be estimated by

$$\int_{\delta(x)/\varepsilon}^\infty se^{-s}ds\int_{|v|\le 1}|v|M(v)dv \le c\frac{\varepsilon}{\delta(x)}\sup_{\gamma\ge 0}\left(\gamma\int_\gamma^\infty se^{-s}ds\right)\,.$$

Since $\varphi\in\mathcal{D}_\Omega$ implies $|\nabla_x\varphi(x,t)| \le c\delta(x)$, we have the result

$$\varepsilon^{-\alpha}\int_{|v|\le 1}\int_0^{r(x,v)/\varepsilon} M(v)e^{-s}[\varphi(x+\varepsilon sv,t) - \varphi(x,t)]ds\,dv = O(\varepsilon^{2-\alpha})\,,$$

uniformly in (x,t).

Step 4: It remains to consider

$$\begin{aligned}&\varepsilon^{-\alpha}\int_{|v|>1}\int_0^{r(x,v)/\varepsilon}|v|^{-d-\alpha}e^{-s}[\varphi(x+\varepsilon sv,t)-\varphi(x,t)]ds\,dv\\ &= \int_{|w|>\varepsilon}\int_0^{r(x,w)}|w|^{-d-\alpha}e^{-s}[\varphi(x+sw,t)-\varphi(x,t)]ds\,dw\\ &= \int_0^\infty s^{d+\alpha}e^{-s}\int_{|w|>\varepsilon,\,s<r(x,w)}\frac{\varphi(x+sw,t)-\varphi(x,t)}{|sw|^{d+\alpha}}dw\,ds\end{aligned} \tag{22}$$

By the coordinate transformation $x + sw = y$ the condition $s < r(x,w)$ becomes s-independent:

$$|x-y| < |x - x_0(x,y-x)| \quad\Longleftrightarrow\quad y\in\mathcal{S}_\Omega(x)\,.$$

Therefore (22) is equal to

$$\int_0^\infty s^\alpha e^{-s} \int_{\mathcal{S}_\Omega(x)\setminus B_{\varepsilon s}(x)} \frac{\varphi(y,t)-\varphi(x,t)}{|y-x|^{d+\alpha}} dy\, ds\,,$$

where $B_r(x)$ denotes the ball with center x and radius r. In order to prove that this converges to $\mathcal{L}_\alpha(\varphi)$, we need to show that

$$\int_0^\infty s^\alpha e^{-s} \int_{\mathcal{S}_\Omega(x)\cap B_{\varepsilon s}(x)} \frac{(y-x)\cdot\nabla_x\varphi(x,t) + (y-x)^{tr}\nabla_x^2\varphi(\hat{x},t)(y-x)/2}{|y-x|^{d+\alpha}} dy\, ds$$

tends to zero. The second term involving the Hessian of the test function can be estimated by

$$c\int_0^\infty s^\alpha e^{-s} \int_{B_{\varepsilon s}(x)} |y-x|^{2-d-\alpha} dy\, ds = c\,\varepsilon^{2-\alpha}\int_0^\infty s^2 e^{-s} ds\,.$$

The estimation of the first term is more subtle. Actually, the integral with respect to y has to be understood as a principal value for $\alpha \ge 1$. Since

$$\text{P.V.}\int_{B_r(x)} \frac{y-x}{|y-x|^{d+\alpha}} dy = 0\,, \qquad \text{for } r>0\,,$$

and $B_{\varepsilon s}(x) \subset \mathcal{S}_\Omega(x)$ for $\varepsilon s < \delta(x)$, we have

$$\begin{aligned} &\int_0^\infty s^\alpha e^{-s}\text{P.V.}\int_{\mathcal{S}_\Omega(x)\cap B_{\varepsilon s}(x)} \frac{(y-x)\cdot\nabla_x\varphi(x,t)}{|y-x|^{d+\alpha}} dy\, ds \\ &= \int_{\delta(x)/\varepsilon}^\infty s^\alpha e^{-s} \int_{(\mathcal{S}_\Omega(x)\cap B_{\varepsilon s}(x))\setminus B_{\delta(x)}} \frac{(y-x)\cdot\nabla_x\varphi(x,t)}{|y-x|^{d+\alpha}} dy\, ds\,, \end{aligned} \tag{23}$$

which can be estimated by

$$c\delta(x)\int_{\delta(x)/\varepsilon}^\infty s^\alpha e^{-s}\int_{B_{\varepsilon s}(x)\setminus B_{\delta(x)}} |y-x|^{1-d-\alpha} dy\, ds = c\delta(x)\int_{\delta(x)/\varepsilon}^\infty s^\alpha e^{-s}\int_{\delta(x)}^{\varepsilon s} r^{-\alpha} dr\, ds\,.$$

With

$$\int_{\delta(x)}^{\varepsilon s} r^{-\alpha} dr \le \begin{cases} c(\varepsilon s)^{1-\alpha}\,, & \alpha<1\,, \\ \log(\varepsilon s/\delta(x))\,, & \alpha = 1\,, \\ c\delta(x)^{1-\alpha}\,, & \alpha>1\,, \end{cases}$$

it is straightforward to obtain that (23) is $O(\varepsilon^{2-\alpha})$ for $\alpha \ne 1$ and $O(\varepsilon\log(1/\varepsilon))$ for $\alpha = 1$, uniformly in (x,t). This completes the proof of the uniform convergence of $\mathcal{L}_\alpha^\varepsilon(\varphi)$ to $\mathcal{L}_\alpha(\varphi)$. ∎

Corollary 1. *Let $\varphi \in \mathcal{D}_\Omega$ and let χ_ε be defined by* (17)*. Then*

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}^d} M(v)[\chi_\varepsilon(x,v,t) - \varphi(x,t)]dv = \lim_{\varepsilon\to 0}\int_{\mathbb{R}^d} M(v)[\partial_t\chi_\varepsilon(x,v,t) - \partial_t\varphi(x,t)]dv = 0, \tag{24}$$

uniformly with respect to $(x,t) \in \operatorname{supp}(\varphi)$.

Proof. The first statement is an immediate consequence of Lemma 3. The second statement follows, since $\varphi \in \mathcal{D}_\Omega$ implies $\partial_t\varphi \in \mathcal{D}_\Omega$ and since the map $\partial_t\varphi \mapsto \partial_t\chi_\varepsilon$ is the same as $\varphi \mapsto \chi_\varepsilon$. ∎

The remaining steps in the proof of Theorem 1 are rather standard. As a consequence of Lemma 1 and of the estimate

$$|\rho_\varepsilon| \le \|f_\varepsilon\|_{L^2(dv/M)} \qquad \Longrightarrow \qquad \|\rho_\varepsilon\|_{L^2(dx)} \le \|f_\varepsilon\|_{L^2(dx\,dv/M)}\,,$$

we obtain

$$\rho_\varepsilon \overset{*}{\rightharpoonup} \rho \quad \text{in } L^\infty(dt;\, L^2(dx))\,, \qquad f_\varepsilon \overset{*}{\rightharpoonup} \rho M \quad \text{in } L^\infty(dt;\, L^2(dx\,dv/M))\,,$$

when restricting to subsequences. Now we are ready for passing to the limit in (19). We decompose the first term by using

$$\int_{\mathbb{R}^d} f_\varepsilon \partial_t \chi_\varepsilon dv = \int_{\mathbb{R}^d} (f_\varepsilon - \rho_\varepsilon M)\partial_t \chi_\varepsilon dv + \rho_\varepsilon \int_{\mathbb{R}^d} M \partial_t \chi_\varepsilon dv\,.$$

The first term on the right hand side tends to zero by

$$\left| \int_{\mathbb{R}^d} (f_\varepsilon - \rho_\varepsilon M)\partial_t \chi_\varepsilon dv \right| \leq \|f_\varepsilon - \rho_\varepsilon M\|_{L^2(dv/M)} \|\partial_t \chi_\varepsilon\|_{L^2(M\,dv)}\,,$$

Lemma 1, and Lemma 2. In the second term we may pass to the limit $\rho\,\partial_t\varphi$ by the weak$*$ convergence of ρ_ε and the strong convergence of $\int_{\mathbb{R}^d} M\partial_t\chi_\varepsilon dv$ (Corollary 1). The limit in the second term of (19) is a consequence of Corollary 1. Finally, passing to the limit in the right hand side of (19) is justified by the weak$\star$ convergence of ρ_ε and by Lemma 3. This completes the proof of Theorem 1.

4 Discussion

In this section we discuss properties of the fractional diffusion operator. First we show that the function h_α defined in (9) is well defined and tends to infinity at the boundary of Ω.

Proposition 1. *Let h_α be defined by* (9)*, then there exists $C > 0$ such that*

$$0 < h_\alpha(x) \leq C\delta(x)^{-\alpha}\,, \qquad x \in \Omega\,. \tag{25}$$

Proof. In order to prove (25) let us chose $x \in \Omega$ and note that $|x - x_0(x,w)| \geq \delta(x)$. Next, let us introduce a polar coordinates change of variables in the integral (9), and note the following:

$$h_\alpha(x) = \int_0^{2\pi} \int_0^\infty \frac{1}{\eta^{d+\alpha}} \mathrm{e}^{-|x-x_0(x,\sigma)|/\eta} \eta^{d-1}\, \mathrm{d}\eta\, \mathrm{d}\sigma$$

where η denotes the radial variable. Now, introducing the change of variables $r = \delta(x)\eta$ we obtain

$$\begin{aligned} h_\alpha(x) &\leq \int_0^{2\pi} \int_0^\infty \frac{1}{r^{d+\alpha}} \mathrm{e}^{-\delta(x)/r} r^{d-1}\, \mathrm{d}r\, \mathrm{d}\sigma \\ &\leq \int_0^{2\pi} \int_0^\infty \frac{1}{(\delta(x)\eta)^{1+\alpha}} \mathrm{e}^{-1/\eta} \delta(x)\, \mathrm{d}\eta\, \mathrm{d}\sigma \\ &= \frac{1}{\delta^\alpha(x)} \int_0^{2\pi} \int_0^\infty \frac{1}{\eta^{1+\alpha}} \mathrm{e}^{-1/\eta}\, \mathrm{d}\eta\, \mathrm{d}\sigma, \end{aligned}$$

from which (25) follows. In addition, we obtain that $h_\alpha(x)$ is finite for every $x \in \Omega$. ∎

In [13] it has been shown that the fractional heat equation

$$\begin{aligned} \partial_t u(x,t) &= -c_{d,\alpha}\, \text{P.V.} \int_{\mathbb{R}^d} \frac{u(x,t) - u(y,t)}{|x-y|^{d+\alpha}}\, \mathrm{d}y && \text{in } \Omega, t > 0, \\ u(x,t) &= 0 && \text{in } \mathbb{R}^d \setminus \Omega, \\ u(x,0) &= u^{in}(x) && \text{in } \Omega, \end{aligned}$$

has a unique solution such that for any fixed $t_0 > 0$ the following estimate holds

$$\sup_{t\geq t_0}\left\|\frac{u(\cdot,t)}{\delta^{\alpha/2}(\cdot)}\right\|_{C^\alpha(\overline{\Omega})} \leq C(t_0)\|u^{in}\|_{L^2(\Omega)}.$$

Therefore, for any fixed time $t > 0$, $u(x,t)$ behaves like $\delta^{\alpha/2}(x)$ when $x \to \partial\Omega$.

In this work we neither prove the uniqueness of weak solutions nor any Hölder regularity results, however, formally using $\varphi(x,t) = \rho(x,t)\mathbf{1}_{[0,T]}(t)$ in (8) yields

$$\frac{1}{2}\|\rho(\cdot,T)\|^2_{L^2(\Omega)} + \int_0^T\int_\Omega h_\alpha\rho^2 dx\,dt + \Gamma(\alpha+1)\int_0^T\int_{x,y:\,[x,y]\subset\Omega}\frac{(\varphi(x)-\varphi(y))^2}{|x-y|^{d+\alpha}}dx\,dy\,dt = \frac{1}{2}\|\rho^{in}\|^2_{L^2(\Omega)}.$$

This implies uniqueness at least formally. Also the boundedness of the second integral together with Proposition 1 induces results on the behaviour of ρ close to the boundary. In particular for $\alpha > 1$, as a consequence of Proposition 1, h_α is not integrable, implying some decay of $\rho(x,t)$ as $\delta(x) \to 0$.

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