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THE SPHERICAL HARMONICS EXPANSION MODEL COUPLED TO THE POISSON EQUATION

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Abstract. The Spherical Harmonics Expansion (SHE) assumes a momentum distribution function only depending on the microscopic kinetic energy. The SHE-Poisson system describes carrier transport in semiconductors with self-induced electrostatic potential. Existence of weak solutions to the SHE-Poisson system subject to periodic boundary conditions is established, based on appropriate a priori estimates and a Schauder fixed point procedure. The long time behavior of the one-dimensional Dirichlet problem with well prepared boundary data is studied by an entropy-entropy dissipation method. Strong convergence to equilibrium is proven. In contrast to earlier work, the analysis is carried out without the use of the derivation from a kinetic problem.

1. **Introduction.** The *Spherical Harmonics Expansion (SHE) model* describes ensembles of particles, whose time dependent distribution $f(x,k,t)$ in phase space (position $x \in \mathbb{R}^d$, momentum $k \in \mathbb{R}^3$, time t) depends on the momentum only through the kinetic energy $\varepsilon_c(k)$, i.e. $f(x,k,t) = F(x,\varepsilon_c(k),t)$. It can be derived as a singular limit of a kinetic transport equation for f, under the assumption of dominant elastic scattering of the particles at a nonmoving background medium. If

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This work is dedicated to the memory of Naoufel BenAbdallah.

the particles are also subject to a position and time dependent gradient force field $-\nabla_x\Phi(x,t)$, then the SHE model takes the form

$$N(\varepsilon)\partial_t F - \tilde{\nabla}\cdot(D(\varepsilon)\tilde{\nabla}F) = 0\,, \tag{1}$$

with the modified gradient operator $\tilde{\nabla}$ defined by

$$\tilde{\nabla} = \nabla_x - \nabla_x\Phi\,\partial_\varepsilon\,. \tag{2}$$

The *density of states* $N(\varepsilon) \geq 0$ has to be chosen such that the *macroscopic particle density* is given by

$$\varrho_F(x,t) := \int_{\mathcal{R}(\varepsilon_c)} F(x,\varepsilon,t)N(\varepsilon)d\varepsilon = \int_{\mathbb{R}^3} F(x,\varepsilon_c(k),t)dk\,, \tag{3}$$

and the diffusivity $D(\varepsilon) \geq 0$ depends on the details of the elastic scattering mechanism. The SHE model has been derived as a description of charge transport in semiconductors (see, e.g. [2, 15, 9, 14]), and it owes its name to its first derivation by a moment method. The present work is also motivated by the application to semiconductors, and the particles are the electrons in the conduction band of a semiconductor crystal. Although not very essential for the results of this work, we make two simplifying assumptions: First, the crystal structure will be neglected by considering an isotropic background. This is the justification for using a scalar diffusivity $D(\varepsilon)$. A related assumption is to replace the exact band structure of the conduction band by the so called *parabolic band* approximation $\varepsilon_c(k) = (2\pi)^{2/3}|k|^2$, where the factor in this dimensionless representation is chosen to produce (according to 3) the density of states

$$N(\varepsilon) = \sqrt{\varepsilon}\,.$$

For the diffusivity, the form

$$D(\varepsilon) = \varepsilon^\alpha\,, \qquad \alpha > 0\,,$$

will be assumed. The cases $\alpha = 1/2$ [7] and $\alpha = 1$ [13] occur in the literature (see also [4]).

The force field is the electric field due to the self-consistent *electrostatic potential* Φ, satisfying the Poisson equation

$$-\Delta_x\Phi = \varrho_F - C\,, \tag{4}$$

where C denotes the *doping profile,* a fixed given distribution of charges built into the semiconductor crystal, here assumed constant and nonnegative; see, e.g. [1, 5, 16] for related models. In [17] the author studied the diffusion approximation of an initial-boundary value problem for a Boltzmann-Poisson system with an elastic operator modeling electron-impurity collision. He proved the convergence of the renormalized solutions to weak solutions of the SHE-Poisson system, obtaining a result analogous to our Theorem 2.1. Although using similar analytical tools, our work provides a "direct" proof of existence of solutions to the SHE-Poisson model, and explores techniques for dealing with the non-parabolic structure of the SHE equation. Moreover, the analysis of the long-time behavior of the system with well-prepared Dirichlet boundary data on a one-dimensional position domain is new.

Two different settings with bounded position domains $\Omega \subset \mathbb{R}^d$, $d = 1, 2$ or 3, will be considered: Either the periodic case with $\Omega = \mathbb{T}^d$, the d-dimensional torus, and $C > 0$; or Ω a bounded Lipschitz domain, Dirichlet boundary conditions, and $C = 0$.

The system 1–4 is considered subject to the initial condition

$$F(x,\varepsilon,0) = F^0(x,\varepsilon) \geq 0\,, \tag{5}$$

where the initial datum satisfies

A1. *Boundedness and bounded total mass:* $F^0 \in L^\infty_{x,\varepsilon}$, $F^0 \in L^1_{x,\varepsilon}(N(\varepsilon))$.
A2. *Bounded total kinetic energy:* $\varepsilon F^0 \in L^1_{x,\varepsilon}(N(\varepsilon))$.
A3. *Global charge neutrality:* Only in the periodic case $\Omega = \mathbb{T}^d$,

$$\int_{\mathbb{T}^d} (\varrho_{F^0} - C)dx = 0\,.$$

Here and in the following, in the notation for function spaces, the subscripts indicate the domains $x \in \Omega$, $\varepsilon \in [0,\infty)$, $t \in [0,\infty)$, and weight functions are written in parantheses.

By the charge conservation law

$$\partial_t \varrho_F + \nabla_x \cdot j_F = 0\,, \quad \text{with } j_F = -\int_0^\infty D\tilde{\nabla} F\, d\varepsilon\,, \tag{6}$$

global charge neutrality is propagated in time, which is necessary and sufficient for the solvability of the Poisson equation 4 subject to periodic boundary conditions. The setting with Dirichlet boundary conditions will be detailed in Section 3.

The paper is organized as follows: In Section 2, we study the existence of weak solutions to the SHE-Poisson system 1–5 in the periodic setting $x \in \mathbb{T}^d$. We first consider the SHE equation with a prescribed potential $\Phi \in L^\infty_t(W^{1,\infty}_x)$ (Section 2.1). Since this regularity cannot be expected for the solution of the coupled system, we first consider a regularized problem, which is solved by an application of the Schauder fixed point theorem, followed by removal of the regularization (Section 2.2). In Section 3 we study the long time behavior of the Dirichlet problem for the SHE-Poisson system on a one-dimensional position domain with well prepared boundary data. Under appropriate assumptions, we prove that the solution converges to equilibrium in the L^2-sense.

2. **Existence of solutions.** In this section the existence of weak solutions to the SHE-Poisson system 1–5 for the case $x \in \mathbb{T}^d$ will be proved.

We say that 1, 5 is satisfied in the weak sense, if

$$\begin{aligned}
&\int_{\mathbb{T}^d}\int_0^\infty F^0 \varphi(t=0) N(\varepsilon) d\varepsilon\, dx + \int_0^\infty \int_{\mathbb{T}^d}\int_0^\infty F \partial_t \varphi N(\varepsilon) d\varepsilon\, dx\, dt \\
&= \int_0^\infty \int_{\mathbb{T}^d}\int_0^\infty D(\varepsilon) \tilde{\nabla} F \cdot \tilde{\nabla}\varphi\, d\varepsilon\, dx\, dt\,,
\end{aligned} \tag{7}$$

for all $\varphi \in C^\infty_{0,t,\varepsilon}([0,\infty)\times[0,\infty);\, C^\infty_x(\mathbb{T}^d))$.

Theorem 2.1. *Let* $1/2 \leq \alpha \leq 3/2$ *and* **A1**–**A3** *hold. Then the SHE-Poisson system 1–5 has a weak solution* (F,Φ) *with*

$$F \geq 0\,, \qquad F \in L^\infty_{t,x,\varepsilon}\,, \qquad \tilde{\nabla} F \in L^2_{t,x,\varepsilon}\,, \qquad \Phi \in L^\infty_t(W^{2,5/3}_x)\,.$$

Moreover, it satisfies local mass conservation 6 with

$$\varrho_F \in L^\infty_t(L^{5/3}_x)\,, \qquad j_F \in L^2_t(L^{p_\alpha}_x)\,, \quad p_\alpha = \frac{15}{11+2\alpha}\,.$$

The proof is carried out in several steps: First, we prove the existence of solutions of the SHE equation with a prescribed potential $\Phi \in L_t^\infty(W_x^{1,\infty})$, based on suitable apriori estimates (Section 2.1). Then, we couple it to the Poisson equation, using the Schauder fixed point theorem (Section 2.2).

The method of proof does not provide uniqueness of solutions, although uniqueness is expected to hold. The authors are not aware of (and do not know how to prove) a uniqueness result.

2.1. **The SHE equation with prescribed potential.** This section is devoted to the proof of the following theorem.

Theorem 2.2. *Let the field $\nabla_x \Phi \in L^\infty_{t,x}$ be given. Then there exists a unique weak solution F of the SHE equation 7, satisfying*

$$F \in L^\infty_{t,x,\varepsilon} \cap L^\infty_t(L^1_{x,\varepsilon}(N(\varepsilon))) \,, \qquad D(\varepsilon)\tilde\nabla F \in L^2_{t,loc}(L^{p_\alpha}_x(L^1_\varepsilon)) \,,$$

with the consequence $j_F \in L^2_{t,loc}(L^{p_\alpha}_x)$. Moreover, $\varrho_F \in L^\infty_{t,loc}(L^{5/3}_x)$ holds as well as (in the distibutional sense) local mass conservation:

$$\partial_t \varrho_F + \nabla_x \cdot j_F = 0 \,. \tag{8}$$

The main difficulty when solving the SHE equation lies in the fact that it is not strictly parabolic in three ways. First, the diffusivity $D(\varepsilon)$ vanishes for $\varepsilon = 0$. Second and more importantly, there is diffusion only in d directions of the $(d+1)$-dimensional (x,ε)-space. Finally, the density of states vanishes for $\varepsilon = 0$.

Therefore, the proof is based on a regularization of the SHE equation 7:

$$\begin{aligned}
&\int_{\mathbb{T}^d}\int_0^\infty F^0 \varphi(t=0) N(\varepsilon+\delta) d\varepsilon\, dx + \int_0^\infty \int_{\mathbb{T}^d}\int_0^\infty F_\delta \partial_t \varphi N(\varepsilon+\delta) d\varepsilon\, dx\, dt \\
&= \int_0^\infty \int_{\mathbb{T}^d}\int_0^\infty D(\varepsilon+\delta)(\tilde\nabla F_\delta \cdot \tilde\nabla \varphi + \delta\, \partial_\varepsilon F_\delta\, \partial_\varepsilon \varphi) d\varepsilon\, dx\, dt \,,
\end{aligned} \tag{9}$$

for all $\varphi \in C^\infty_{0,t,\varepsilon}([0,\infty)\times[0,\infty); C^\infty_x(\mathbb{T}^d))$. Note that this includes the no-flux boundary condition $D(\delta)(\tilde\nabla F_\delta \cdot \nabla_x \Phi - \delta \partial_\varepsilon F_\delta) = 0$ for $\varepsilon = 0$.

We start by defining the weighted Hilbert spaces

$$\begin{aligned}
\mathcal{H}_\delta \quad &:= \quad \left\{ u \in L^1_{loc}(\mathbb{T}^d \times [0,\infty)) : \int_{\mathbb{T}^d}\int_0^\infty u^2 N(\varepsilon+\delta) d\varepsilon\, dx < \infty \right\} \,, \\
\mathcal{V}_\delta \quad &:= \quad \left\{ u \in \mathcal{H}_\delta ; \int_{\mathbb{T}^d}\int_0^\infty D(\varepsilon+\delta) |\nabla_{x,\varepsilon} u|^2 d\varepsilon\, dx < \infty \right\} \,,
\end{aligned}$$

with the natural scalar products

$$\begin{aligned}
\langle u, v\rangle_{\mathcal{H}_\delta} \quad &:= \quad \int_{\mathbb{T}^d}\int_0^\infty uv\, N(\varepsilon+\delta) d\varepsilon\, dx \,, \\
\langle u, v\rangle_{\mathcal{V}_\delta} \quad &:= \quad \langle u, v\rangle_{\mathcal{H}_\delta} + \int_{\mathbb{T}^d}\int_0^\infty D(\varepsilon+\delta) \nabla_{x,\varepsilon} u \cdot \nabla_{x,\varepsilon} v\, d\varepsilon\, dx \,,
\end{aligned}$$

and the corresponding norms $\|\cdot\|_{\mathcal{H}_\delta}$ and $\|\cdot\|_{\mathcal{V}_\delta}$. It is easily checked that $\mathcal{V}_\delta$ and $\mathcal{H}_\delta$ are Hilbert spaces and that $\mathcal{V}_\delta \hookrightarrow \mathcal{H}_\delta$ continuously and densely and $\mathcal{H}_\delta \subset {\mathcal{V}_\delta}'$, where ${\mathcal{V}_\delta}'$ is the dual space of $\mathcal{V}_\delta$.

For each $t \geq 0$, we define the bilinear form a_t on $\mathcal{V}_\delta \times \mathcal{V}_\delta$ by

$$a_t(u,v) := \int_{\mathbb{T}^d}\int_0^\infty D(\varepsilon+\delta)(\tilde\nabla u \cdot \tilde\nabla v + \delta \partial_\varepsilon u\, \partial_\varepsilon v) d\varepsilon\, dx \,.$$

The bilinear form a_t is continuous

$$a_t(u,v) \le C\,\|u\|_{\mathcal{V}_\delta}\,\|v\|_{\mathcal{V}_\delta} \quad \text{for all } u,v\in\mathcal{V}_\delta\,,$$

and coercive in the sense

$$a_t(u,u) \ge \min\left\{\frac{1}{2}, \frac{\delta}{1+2\|\nabla_x\Phi\|_\infty^2}\right\}\left(\|u\|_{\mathcal{V}_\delta}^2 - \|u\|_{\mathcal{H}_\delta}^2\right) \quad \text{for all } u\in\mathcal{V}_\delta\,,$$

where the coefficient is a coercivity constant for the quadratic form $|a-b\nabla_x\Phi|^2+\delta b^2$. Then, an application of a lemma by J.L. Lions (see [6], page 218) allows us to deduce, for every $\delta>0$ and $t\in[0,T]$, the existence of a mild solution F_δ of 9 with

$$F_\delta \in L^2((0,T);\mathcal{V}_\delta)\cap C([0,T];\mathcal{H}_\delta) \quad \text{and} \quad \partial_t F_\delta \in L^2((0,T);{\mathcal{V}_\delta}')\,.$$

Its macroscopic density and flux will be denoted by

$$\varrho_\delta = \int_0^\infty F_\delta N(\varepsilon+\delta)d\varepsilon\,,\quad j_\delta = -\int_0^\infty D(\varepsilon+\delta)\tilde\nabla F_\delta d\varepsilon\,. \tag{10}$$

To pass to the limit $\delta\to 0$ in 9, we need several uniform a-priori estimates.

Lemma 2.3 (Mass conservation, maximum principle)**.** *The solution F_δ of 9 is nonnegative and bounded,*

$$\|F_\delta(\cdot,\cdot,t)\|_{L^\infty_{x,\varepsilon}} \le \left\|F^0\right\|_{L^\infty_{x,\varepsilon}}\,, \qquad t\ge 0\,, \tag{11}$$

and satisfies local mass conservation,

$$\partial_t\varrho_\delta + \nabla_x\cdot j_\delta = 0\,, \tag{12}$$

in the distributional sense and, consequentially, global mass conservation,

$$\int_{\mathbb{T}^d}\varrho_\delta(x,t)dx = M_\delta := \int_{\mathbb{T}^d}\int_0^\infty F^0(x,\varepsilon)N(\varepsilon+\delta)d\varepsilon\,dx\,, \qquad t\ge 0\,.$$

Proof. Local mass conservation follows by approximating functions depending only on time and position by test functions in 9.

For proving boundedness we use the method of Stampacchia [6], and choose

$$H(s) = \begin{cases} 0 & \text{for } s\le 0\,,\\ s^2 & \text{for } s>0\,,\end{cases} \qquad K = \left\|F^0\right\|_{L^\infty_{x,\varepsilon}}\,,$$

and

$$\psi(t) = \int_{\mathbb{T}^d}\int_0^\infty N(\varepsilon+\delta)H(F_\delta-K)d\varepsilon\,dx\,,$$

satisfying $\psi(t)\ge 0$, $\psi(0)=0$, and

$$\begin{aligned}\psi'(t) \;&=\; -\int_{\mathbb{T}^d}\int_0^\infty D(\varepsilon+\delta)|\tilde\nabla F_\delta|^2 H''(F_\delta-K)d\varepsilon\,dx\\ &\quad -\delta\int_{\mathbb{T}^d}\int_0^\infty D(\varepsilon+\delta)(\partial_\varepsilon F_\delta)^2 H''(F_\delta-K)d\varepsilon\,dx \le 0\,.\end{aligned}$$

Therefore, $H(F_\delta-K)=0$ almost everywhere and the assertion follows. For proving nonnegativity, we use

$$\psi(t) = \int_{\mathbb{T}^d}\int_0^\infty N(\varepsilon+\delta)H(-F_\delta)d\varepsilon\,dx\,.$$

$\square$

Lemma 2.4 (Energy estimate)**.** *1) The global kinetic energy, defined as*

$$\mathcal{E}_\delta[F_\delta](t) = \int_{\mathbb{T}^d}\int_0^\infty \varepsilon F_\delta(x,\varepsilon,t)N(\varepsilon+\delta)d\varepsilon\, dx$$

is bounded locally in time, uniformly with respect to $\delta > 0$, and the bound depends on the $L^\infty_{t,x}$-norm of $\nabla_x\Phi$.
2) For every $T \geq 0$,

$$\int_0^T\int_{\mathbb{T}^d}\int_0^\infty D(\varepsilon+\delta)\frac{|\tilde\nabla F_\delta|^2}{F_\delta}\,d\varepsilon\,dx\,dt \\ +\delta\int_0^T\int_{\mathbb{T}^d}\int_0^\infty D(\varepsilon+\delta)\frac{(\partial_\varepsilon F_\delta)^2}{F_\delta}\,d\varepsilon\,dx\,dt \leq C_T\,,$$

with C_T independent from δ.
3)

$$\int_0^\infty\int_{\mathbb{T}^d}\int_0^\infty D(\varepsilon+\delta)|\tilde\nabla F_\delta|^2\,d\varepsilon\,dx\,dt \leq \frac12\|F^0\|_{L^\infty_{x,\varepsilon}}M_\delta\,.$$

Proof. There exist many mathematical entropies. For every smooth convex $\eta(F)$,

$$\frac{d}{dt}\int_{\mathbb{T}^d}\int_0^\infty \eta(F_\delta)N(\varepsilon+\delta)d\varepsilon\,dx = -\int_{\mathbb{T}^d}\int_0^\infty \eta''(F_\delta)D(\varepsilon+\delta)|\tilde\nabla F_\delta|^2 d\varepsilon\,dx \leq 0$$

holds, and the choice $\eta(F) = F^2/2$ proves 3).

The proof of the remaining statements relies on showing boundedness of the quasi entropy functional

$$H[F] = \int_{\mathbb{T}^d}\int_0^\infty(\varepsilon+\log F)FN(\varepsilon+\delta)d\varepsilon\,dx + M_\delta + C\,,$$

with the total mass M_δ and the constant

$$C = |\mathbb{T}^d|\int_0^\infty e^{-1-\varepsilon/2}(1+\varepsilon/2)N(\varepsilon+\delta)d\varepsilon\,.$$

The choice of C is motivated by the estimate

$$\int_0^\infty F\log F\,N(\varepsilon+\delta)d\varepsilon \\ \geq \int_{e^{-1-\varepsilon/2}<F<1}F\log F\,N(\varepsilon+\delta)d\varepsilon + \int_{F<e^{-1-\varepsilon/2}}F\log F\,N(\varepsilon+\delta)d\varepsilon \\ \geq -\int_0^\infty(1+\varepsilon/2)F\,N(\varepsilon+\delta)d\varepsilon - \int_0^\infty e^{-1-\varepsilon/2}(1+\varepsilon/2)N(\varepsilon+\delta)d\varepsilon\,,$$

(which uses the monotonicity of $F\log F$ for $F < 1/e$), implying

$$H[F] \geq \frac12\mathcal{E}_\delta[F] \tag{13}$$

A straightforward computation gives

$$\frac{dH[F_\delta]}{dt} = -\int_{\mathbb{T}^d}\nabla_x\Phi\cdot j_\delta\,dx - \delta\int_{\mathbb{T}^d}\int_0^\infty D(\varepsilon+\delta)\partial_\varepsilon F_\delta\,d\varepsilon\,dx \\ -\int_{\mathbb{T}^d}\int_0^\infty D(\varepsilon+\delta)\frac{|\tilde\nabla F_\delta|^2}{F_\delta}\,d\varepsilon\,dx - \delta\int_{\mathbb{T}^d}\int_0^\infty D(\varepsilon+\delta)\frac{(\partial_\varepsilon F_\delta)^2}{F_\delta}\,d\varepsilon\,dx$$

With the boundedness of $\nabla_x\Phi$ and the Hölder inequality we obtain

$$\begin{aligned}\left|\int_{\mathbb{T}^d}\nabla_x\Phi\cdot j_\delta\,dx\right| &\le c\left(\int_{\mathbb{T}^d}\int_0^\infty D(\varepsilon+\delta)\frac{|\tilde\nabla F_\delta|^2}{F_\delta}\,d\varepsilon\,dx\right)^{1/2} M_\delta^{1/(2p)}\\ &\quad\times\left(\int_{\mathbb{T}^d}\int_0^\infty\left(\frac{D(\varepsilon+\delta)}{N(\varepsilon+\delta)^{1/p}}\right)^{1/p'}F_\delta\,d\varepsilon\,dx\right)^{1/(2p')}.\end{aligned}$$

Now the explicit forms of $N(\varepsilon)=\sqrt{\varepsilon}$ and of $D(\varepsilon)=\varepsilon^\alpha$, first with $1/2<\alpha<3/2$, are used. With the choice $p=(3/2-\alpha)^{-1}>1$, the last factor above is equal to $\mathcal{E}_\delta[F_\delta]^{\alpha/2-1/4}$, leading to

$$\left|\int_{\mathbb{T}^d}\nabla_x\Phi\cdot j_\delta\,dx\right|\le\frac12\int_{\mathbb{T}^d}\int_0^\infty D(\varepsilon+\delta)\frac{|\tilde\nabla F_\delta|^2}{F_\delta}\,d\varepsilon\,dx+c\,H[F_\delta]^{\alpha-1/2},$$

with a constant c independent from δ. It is easily seen that this estimate also holds for $\alpha=1/2,3/2$. Analogously, we estimate

$$\left|\int_{\mathbb{T}^d}\int_0^\infty D(\varepsilon+\delta)\partial_\varepsilon F_\delta\,d\varepsilon\,dx\right|\le\frac12\int_{\mathbb{T}^d}\int_0^\infty D(\varepsilon+\delta)\frac{(\partial_\varepsilon F_\delta)^2}{F_\delta}\,d\varepsilon\,dx+c\,H[F_\delta]^{\alpha-1/2},$$

with the consequence

$$\begin{aligned}\frac{dH[F_\delta]}{dt} &\le c\,H[F_\delta]^{\alpha-1/2}-\frac12\int_{\mathbb{T}^d}\int_0^\infty D(\varepsilon+\delta)\frac{|\tilde\nabla F_\delta|^2}{F_\delta}\,d\varepsilon\,dx\\ &\quad-\frac{\delta}{2}\int_{\mathbb{T}^d}\int_0^\infty D(\varepsilon+\delta)\frac{(\partial_\varepsilon F_\delta)^2}{F_\delta}\,d\varepsilon\,dx,\end{aligned}$$

implying local in time boundedness of $H[F_\delta]$ which, by 13 completes the proof of 1). Integration with respect to time then gives 2). □

Lemma 2.5 (Estimates on the density and fluxes)**.**

$$\begin{aligned}&\varrho_\delta\in L^\infty_{t,loc}(L^{5/3}_x),\qquad D(\varepsilon+\delta)\tilde\nabla F_\delta\in L^2_{t,loc}(L^{p_\alpha}_x(L^1_\varepsilon)),\\ &D(\varepsilon+\delta)F_\delta\in L^\infty_{t,loc}(L^{p_\alpha/(2-p_\alpha)}_x(L^1_\varepsilon)),\qquad D'(\varepsilon+\delta)F_\delta\in L^\infty_{t,loc}(L^{5/3}_x(L^1_\varepsilon)),\end{aligned}$$

with $p_\alpha=15/(11+2\alpha)$ *and with bounds uniform with respect to* $\delta>0$ *and dependent on* $\|\nabla_x\Phi\|_{L^\infty_{x,t}}$.

Proof. Throughout this proof, N and D have to be understood with the argument $\varepsilon+\delta$. Using the maximum principle (Lemma 2.3), for every $R\ge0$, the density can be estimated by

$$\varrho_\delta=\int_0^R F_\delta N\,d\varepsilon+\int_R^\infty F_\delta N\,d\varepsilon\le\frac23(R+\delta)^{3/2}\|F^0\|_\infty+\frac{1}{R+\delta}e_\delta,$$

where

$$e_\delta=\int_0^\infty(\varepsilon+\delta)F_\delta N\,d\varepsilon$$

satisfies $\int_{\mathbb{T}^d}e_\delta dx=\mathcal{E}_\delta[F_\delta]+\delta M_\delta$ and therefore, by Lemma 2.4, $e_\delta\in L^\infty_{t,loc}(L^1_x)$. Now we distinguish between the cases $e_\delta<\delta^{5/2}\|F^0\|_\infty$, when $\varrho_\delta\le\frac53\delta^{3/2}\|F^0\|_\infty$ follows with $R=0$, and $e_\delta\ge\delta^{5/2}\|F^0\|_\infty$, when the choice $R=(e_\delta/\|F^0\|_\infty)^{2/5}-\delta$ gives $\varrho_\delta\le\frac53\|F^0\|_\infty^{2/5}e_\delta^{3/5}$, completing the proof of the first claim of the lemma.

Starting with the consequence

$$\|D\tilde\nabla F_\delta\|_{L^1_\varepsilon}^{p_\alpha} \le \left(\int_0^\infty D\frac{|\tilde\nabla F_\delta|^2}{F_\delta}d\varepsilon\right)^{p_\alpha/2} \|DF_\delta\|_{L^1_\varepsilon}^{p_\alpha/2},$$

of the Cauchy-Schwarz inequality, we integrate with respect to x and use the Hölder inequality with exponent $2/p_\alpha$ to obtain

$$\|D\tilde\nabla F_\delta\|^2_{L_x^{p_\alpha}(L^1_\varepsilon)} \le \int_{\mathbb{T}^d}\int_0^\infty D\frac{|\tilde\nabla F_\delta|^2}{F_\delta}d\varepsilon\, dx\, \|DF_\delta\|_{L_x^{p_\alpha/(2-p_\alpha)}(L^1_\varepsilon)},$$

Thus, the second claim of the lemma follows from Lemma 2.4 and from the third, which remains to be proved. This will again be achieved by repeated applications of the Hölder inequality, starting with

$$\|DF_\delta\|_{L^1_\varepsilon} \le \varrho_\delta^{1/q}\left(\int_0^\infty \left(\frac{D}{N}\right)^{q'} F_\delta d\varepsilon\right)^{1/q'},$$

where the choice $q = 3/(4-2\alpha)$ gives

$$\|DF_\delta\|_{L^1_\varepsilon}^{\frac{p_\alpha}{2-p_\alpha}} \le \varrho_\delta^{\frac{p_\alpha(4-2\alpha)}{3(2-p_\alpha)}} e_\delta^{\frac{p_\alpha(2\alpha-1)}{3(2-p_\alpha)}}$$

Integration with respect to x and another application of the Hölder inequality with exponent $\frac{5(2-p_\alpha)}{p_\alpha(4-2\alpha)}$ implies

$$\|DF_\delta\|_{L_x^{p_\alpha/(2-p_\alpha)}(L^1_\varepsilon)} \le \|\varrho_\delta\|_{L_x^{5/3}}^{(4-2\alpha)/3}\left(\mathcal{E}_\delta[F_\delta]+\delta M_\delta\right)^{(2\alpha-1)/3},$$

completing the proof of the third claim of the lemma by applying the first and Lemma 2.4.

Finally, the estimate

$$\|D'F_\delta\|_{L^1_\varepsilon} \le \|F^0\|_\infty\int_0^1 D'd\varepsilon + \int_1^\infty D'F_\delta d\varepsilon \le \|F^0\|_\infty D(1+\delta)+\alpha\varrho_\delta,$$

completes the proof. □

Now we are ready to pass to the limit $\delta\to 0$ in 9 and 12 and to conclude the proof of Theorem 2.2. In the last regularization term in 9, we use

$$\int_0^\infty D\partial_\varepsilon F_\delta\partial_\varepsilon\varphi\, d\varepsilon = -\int_0^\infty (DF_\delta\partial_\varepsilon^2\varphi + D'F_\delta\partial_\varepsilon\varphi)d\varepsilon,$$

and Lemma 2.5.

2.2. **Coupling with the Poisson equation.** As a first step, the Poisson equation 4 is regularized:

$$-(1-\gamma\Delta_x)^2\Delta_x\Phi = \varrho_F - C, \qquad \gamma>0. \tag{14}$$

Theorem 2.6. *Let* **A1**–**A3** *hold and* $\gamma>0$. *Then the regularized SHE-Poisson system 7, 14 has a solution* (F_γ,Φ_γ) *satisfying* $\Phi_\gamma\in L_t^\infty(W_x^{1,\infty})$.

Proof. By elliptic regularity and Sobolev imbedding [10], the solution Φ of 14 satisfies

$$\|\Phi\|_{W_x^{1,\infty}} \le c_\gamma\|\varrho_F - C\|_{L^1_x} \tag{15}$$

Therefore, for $T>0$, we define a fixed point mapping Ψ_γ on the convex set

$$\mathcal{B}_\gamma := \left\{\Phi\in L^2_t((0,T);W_x^{1,\infty}):\ \|\Phi(\cdot,t)\|_{W_x^{1,\infty}}\le c_\gamma\left(M+|\mathbb{T}^d|C\right),\ 0\le t\le T\right\},$$

with the total mass $M = \int_{\mathbb{T}^d}\int_0^\infty F^0\,d\varepsilon\,dx$, by the following two steps:

- Given $\Phi \in \mathcal{B}_\gamma$, define F as the unique weak solution of the SHE equation 7.
- Solve the regularized Poisson equation

$$-(1-\gamma\Delta)^2\Delta\Psi_\gamma(\Phi) = \varrho_F - C\,, \qquad \text{on } \mathbb{T}^d\,. \tag{16}$$

By the results of the previous subsection, the first step is well defined. By mass conservation and 15, Ψ_γ maps $\mathcal{B}_\gamma$ into itself.

To prove the continuity of Ψ_γ, let us consider a sequence $\{\Phi_n\}_{n\in\mathbb{N}} \subset \mathcal{B}_\gamma$, converging to Φ in $L^2_t((0,T);W^{1,\infty}_x)$, and F_n the sequence of the corresponding solutions of the SHE equation, weakly convergent to F due to the a-priori estimates of Lemma 2.3. We need to verify that F is the (unique) solution of the SHE equation with the potential Φ. For a test function φ, $\nabla_x\varphi - \nabla_x\Phi_n\partial_\varepsilon\varphi$ converges strongly to $\tilde\nabla\varphi = \nabla_x\varphi - \nabla_x\Phi\partial_\varepsilon\varphi$ in $L^2_t((0,T);\,L^\infty_{x,\varepsilon})$. Due to Lemma 2.4, $\tilde\nabla_n F_n$ is uniformly bounded in $L^2_{t,x,\varepsilon}(D(\varepsilon))$, and, consequently, has a weakly converging subsequence. The limiting object is identified as $\tilde\nabla F$, using the strong convergence of $\nabla_x\Phi_n$ and the weak convergence of F_n in $\tilde\nabla_n F_n = \nabla_x F_n - \partial_\varepsilon(\nabla_x\Phi_n F_n)$. Consequently, we can pass to the limit $n\to\infty$ in the weak formulation of the SHE equation to see that F is indeed its unique solution corresponding to Φ. Then it is trivial to conclude that $\Psi_\gamma(\Phi_n)\to\Psi_\gamma(\Phi)$ in $L^2_{t,loc}(W^{1,\infty}_x)$.

Finally, we will prove that $\Psi_\gamma(\mathcal{B}_\gamma)$ is a relatively compact subset of $\mathcal{B}_\gamma$. Due to Lemma 2.5, $\varrho_F \in L^\infty_{t,loc}(L^{5/3}_x)$ and $j_F \in L^2_{t,loc}(L^{p_\alpha}_x)$. Since

$$(1-\gamma\Delta)^2\Delta(\partial_t\Psi_\gamma(\Phi)) = \nabla_x\cdot j_F\,,$$

elliptic regularity implies that $\Psi_\gamma(\mathcal{B}_\gamma)$ lies within a bounded set of $L^2_{t,loc}(W^{6,5/3}_x)$ and $\partial_t\Psi_\gamma(\mathcal{B}_\gamma)$ in a bounded subset of $L^2_{t,loc}(W^{1,p_\alpha}_x)$. Moreover, in

$$W^{6,5/3}_x \hookrightarrow W^{1,\infty}_x \hookrightarrow W^{1,p_\alpha}_x\,,$$

the first embedding is continuous and compact ($d\le 3$), the second continuous, and the left and right spaces are reflexive. Consequently, by the classical Aubin-Lions lemma [6], the set $\Psi_\gamma(\mathcal{B}_\gamma)$ is relatively compact in $L^2_{t,loc}(W^{1,\infty}_x)$. Therefore, the Schauder theorem implies the existence of a fixed point $\Phi_\gamma\in\mathcal{B}_\gamma$ of the mapping Ψ_γ. □

Finally, we remove the regularization by passing to the limit $\gamma\to 0$. For this, we will need a bound on the kinetic energy $\mathcal{E}[F_\gamma]$ that is independent of the $W^{1,\infty}_x$-norm of Φ_γ:

Lemma 2.7. *Let (F_γ,Φ_γ) be a solution of the regularized SHE-Poisson system 7, 14 as found in Theorem 2.6. Then the total energy defined as*

$$\mathcal{E}_{tot}[F_\gamma](t) := \mathcal{E}[F_\gamma](t) + \frac{1}{2}\int_{\mathbb{T}^d}|\nabla_x(1-\gamma\Delta_x)\Phi_\gamma|^2dx\,,$$

satisfies

$$\mathcal{E}_{tot}[F_\gamma](t) = \mathcal{E}_{tot}[F^0] < \infty\,,$$

for all $t\ge 0$.

Proof. Let us observe that $\tilde\nabla(\varepsilon+\Phi_\gamma)=0$. Therefore, using $\varepsilon+\Phi_\gamma$ as a test function in 7 yields (after an approximation procedure)

$$\int_{\mathbb{T}^d}\int_0^\infty(\varepsilon+\Phi_\gamma)\partial_t F_\gamma N(\varepsilon)d\varepsilon\,dx = 0\,.$$

On the other hand,

$$\int_{\mathbb{T}^d}\int_0^\infty \Phi_\gamma \partial_t F_\gamma N(\varepsilon) d\varepsilon\, dx = \int_{\mathbb{T}^d} \Phi_\gamma \partial_t \varrho_{F_\gamma}\, dx$$
$$= -\int_{\mathbb{T}^d} \Phi_\gamma (1-\gamma\Delta_x)^2 \Delta_x \partial_t \Phi_\gamma\, dx = \frac{d}{dt}\frac{1}{2}\int_{\mathbb{T}^d} |\nabla_x (1-\gamma\Delta_x)\Phi_\gamma|^2 dx\,,$$

which proves $\frac{d}{dt}\mathcal{E}_{tot}[F_\gamma] = 0$.

It remains to show that $\mathcal{E}_{tot}[F^0]$ is well defined. As in the proof of Lemma 2.5, we prove

$$\|\varrho_F\|_{L_x^{5/3}} \le \frac{5}{3}\|F\|_{L^\infty_{x,\varepsilon}}^{2/5}\mathcal{E}[F]^{3/5}$$

The inequality $\|u\|_{L^p_x} \le \|(1-\gamma\Delta_x)u\|_{L^p_x}$, $p\ge 1$, together with elliptic regularity, implies a (uniform in γ) bound in $W_x^{2,5/3}$ for the solution of 14. Sobolev imbedding ($d\le 3$) implies

$$\|\Phi_\gamma\|_{L_x^\infty} \le c_1 \|F_\gamma\|_{L^\infty_{x,\varepsilon}}^{2/5}\mathcal{E}[F_\gamma]^{3/5} + c_2\,.$$

Integration by parts gives

$$\int_{\mathbb{T}^d} |\nabla_x(1-\gamma\Delta_x)\Phi_\gamma|^2 dx = \int_{\mathbb{T}^d} \Phi_\gamma(\varrho_{F_\gamma} - C) dx \le \|\Phi_\gamma\|_{L_x^\infty}(M + C|\mathbb{T}^d|)\,,$$

completing the proof of the boundedness of the potential energy and, therefore, the total energy at $t=0$. □

We are now ready to carry out the limit $\gamma\to 0$, concluding the proof of Theorem 2.1. Due to the boundedness of $\mathcal{E}[F_\gamma]$ established in the previous lemma, we obtain by the same estimates as in the proof of Lemma 2.5 the uniform-in-γ boundedness of

$$\varrho_{F_\gamma} \in L_t^\infty(L_x^{5/3}) \quad \text{and} \quad j_{F_\gamma} \in L_t^2(L_x^{p_\alpha})\,,$$

and therefore also of

$$\Phi_\gamma \in L_t^\infty(W_x^{2,5/3}) \quad \text{and} \quad \partial_t\Phi_\gamma \in L_t^2(W_x^{1,p_\alpha})\,.$$

An application of the Aubin-Lions lemma [6] implies the strong convergence (restricted to a subsequence) of $\Phi_\gamma \to \Phi$ in $L_t^2(W_x^{1,2})$ as $\gamma\to 0$. We also have $F_\gamma \to F$ in $L^\infty_{t,x,\varepsilon}$-weak$*$. Therefore, for any test function φ we have $\nabla_x\varphi - \nabla_x\Phi_\gamma\partial_\varepsilon\varphi \to \tilde\nabla\varphi$ strongly in $L^2_{t,x,\varepsilon}(D(\varepsilon))$. As in the proof of Lemma 2.4 (3), the uniform boundedness of $\nabla_x F_\gamma - \partial_\varepsilon(F_\gamma\nabla_x\Phi_\gamma)$ in $L^2_{t,x,\varepsilon}(D(\varepsilon))$ follows. By the weak$*$ convergence of F_γ and the strong convergence of $\nabla_x\Phi_\gamma$, its weak limit is $\tilde\nabla F$. These arguments justify passing to the limit in the second line of 7, completing the proof of Theorem 2.1, since all other terms in 7 and 14 are linear.

3. **Long time behavior.** In this section we study the long time behavior of the SHE-Poisson system 1–4, posed on an interval $\Omega\subset\mathbb{R}$, subject to the initial condition 5 and with well prepared boundary data

$$\Phi(t,x) = \Phi_b(x)\,, \qquad x\in\partial\Omega\,,\ t\ge 0\,, \tag{17}$$
$$F(t,x,\varepsilon) = \Psi(\varepsilon + \Phi_b(x))\,, \qquad x\in\partial\Omega\,,\ t\ge 0\,, \tag{18}$$

We adopt the following assumptions on the boundary data:

A4. The boundary datum Φ_b is nonnegative.

A5. The profile $\Psi : \mathbb{R} \to (0, +\infty)$ is smooth and strictly decreasing, with $\Psi'(\varepsilon)$ bounded for $\varepsilon \in (0, \infty)$ and with finite energy:

$$\int_0^\infty \varepsilon \Psi(\varepsilon) N(\varepsilon)\, d\varepsilon < \infty\,.$$

Furthermore, there exists a constant $C_0 > 0$, such that

$$\sup\left\{(\Psi^{-1})'(s) : s \in \left(0, \|F^0\|_{L^\infty_{x,\varepsilon}}\right)\right\} \le -C_0 < 0\,. \tag{19}$$

In the following it will occasionally be useful to use the microscopic total energy $z = \varepsilon + \Phi(t, x)$ as an independent variable instead of the kinetic energy ε. In terms of the variables (x, z), the operator $\tilde{\nabla}$ becomes ∇_x.

We say that the couple (F, Φ) is a weak solution of the SHE-Poisson system 1–4 with the Dirichlet boundary conditions 17, 18, if the boundary conditions are satisfied in the sense that

$$\Phi(t, \cdot) - \Phi_b \in H^1_0(\Omega)\,, \qquad F(t, \cdot, z - \Phi(t, \cdot)) - \Psi(z) \in H^1_0(\Omega) \tag{20}$$

for every fixed t and z, where Φ_b now has to be understood as the linear interpolation of the boundary data in (17). The weak formulation 7 of the SHE equation now has to be satisfied for all test fuctions $\varphi \in C^\infty_0([0, \infty) \times \Omega \times [0, \infty))$, and Φ is a weak solution of the Dirichlet problem 4, 20.

Remark 1. Because of the lack of energy conservation, the existence analysis performed in Section 2 for the spatially periodic case cannot be straightforwardly modified for the case of the Dirichlet problem, considered in this section. In particular, the estimates for the decoupled problem pose a difficulty. However, it will be shown in the following that the main a priori estimates for the coupled problem can still be carried out. For example, boundedness of the kinetic energy will be a consequence of entropy dissipation. Existence of a weak solution satisfying the maximum principle estimates $F \ge 0$ and $\|F(t, \cdot, \cdot)\|_{L^\infty_{x,\varepsilon}} \le \|F^0\|_{L^\infty_{x,\varepsilon}}$, as well as the mass conservation property $\|F(t, \cdot, \cdot)\|_{L^1_{x,\varepsilon}(N)} = \|F^0\|_{L^1_{x,\varepsilon}(N)}$ will be assumed. Moreover, the result of Theorem 3.1 below holds under the assumption of sufficient regularity of F. To avoid technicalities, we merely pose the assumption that F is regular enough to justify the proof of Lemma 3.2.

We define the equilibrium profile (F_∞, Φ_∞) corresponding to the time-independent boundary data 17–18 by

$$\begin{aligned}
F_\infty(x, \varepsilon) &= \Psi(\varepsilon + \Phi_\infty(x))\,, \qquad \varepsilon > 0,\ x \in \Omega\,,\\
-\Delta_x \Phi_\infty &= \varrho_\infty := \int_0^\infty \Psi(\varepsilon + \Phi_\infty) N(\varepsilon) d\varepsilon\,, \qquad \text{in } \Omega\,,\\
\Phi_\infty(x) &= \Phi_b(x)\,, \qquad x \in \partial\Omega\,.
\end{aligned}$$

Since the right hand side of the semilinear elliptic equation for Φ_∞ is strictly decreasing as a function of Φ_∞, the solvability of the Dirichlet problem is a standard result. The solution is smooth and nonnegative and, consequently, the same is true for the equilibrium density ϱ_∞.

For (F, Φ) a weak solution of the SHE-Poisson system and (F_∞, Φ_∞) the corresponding equilibrium profile, we define the relative entropy (see [3])

$$\mathcal{H}[F|F_\infty](t) = \int_\Omega \int_0^\infty \mathcal{R}_\Psi[F(t)|F_\infty] N\, d\varepsilon\, dx + \frac{1}{2}\int_\Omega |\nabla_x(\Phi(t) - \Phi_\infty)|^2 dx, \tag{21}$$

with

$$\mathcal{R}_\Psi[F|G] = \int_F^G [\Psi^{-1}(s) - \Psi^{-1}(G)]\,\mathrm{d}s = \beta_\Psi(F) - \beta_\Psi(G) + (F-G)\Psi^{-1}(G),$$

and $\beta_\Psi(F) = -\int_0^F \Psi^{-1}(s)\,\mathrm{d}s$. Before stating the main result of this section, we adopt two more assumptions:

A6. The initial datum F^0 has bounded relative entropy,

$$\int_\Omega \int_0^\infty \mathcal{R}_\Psi[F^0|F_\infty] N\, d\varepsilon\, dx < \infty\,.$$

A7. There exists a positive constant c_Ψ, such that for every $\underline{F} > 0$,

$$\beta_\Psi(F) \ge -c_\Psi F \Psi^{-1}(\underline{F})\,, \quad \text{for } F \ge \underline{F}\,.$$

The assumption **A7** looks rather mysterious, but it is satisfied for typical choices of the equilibrium profile Ψ, in particular, it has been used already for the Maxwellian $\Psi(z) = e^{-z}$ (as is standard in the corresponding problem for the Boltzmann equation [8, 12, 11]) in the proof of Lemma 2.4.

Theorem 3.1. *Let $d = 1$, $1/2 \le \alpha < 1$, and the assumptions* **A1**–**A7** *be satisfied. Then the weak solution (F, Φ) of the SHE-Poisson system 1–4 with the Dirichlet boundary conditions 17–18 converges to the equilibrium (F_∞, Φ_∞) in $L^2_{x,\varepsilon}(N) \times L^2_x$, and the decay of the distance to the partial equilibrium $F_\Phi(t,x,\varepsilon) = \Psi(\varepsilon + \Phi(t,x))$ can be quantified by*

$$\|F - F_\Phi\|_{L^2_{x,\varepsilon}(N)} \le \frac{c}{(1 + (1-\alpha)t)^{1/(1+3\alpha)}}\,.$$

The proof of this theorem will be performed in two steps. First, in Lemma 3.2 we derive an entropy-entropy production inequality. Then, by introducing a coordinate transformation, we bound the entropy production by the entropy in Lemma 3.4, using a special Poincaré-type estimate, which we are only able to prove in the one-dimensional case and for $\alpha < 1$.

Lemma 3.2. *Let (F, Φ) be a weak solution of the SHE-Poisson system 1–4 with the Dirichlet boundary conditions 17–18, and (F_∞, Φ_∞) the corresponding equilibrium profile. Then we have*

$$c_1 \int_\Omega \int_0^\infty (F(t) - F_\Phi)^2 N\, d\varepsilon\, dx + \frac{1}{2}\int_\Omega |\nabla_x \Phi(t) - \nabla_x \Phi_\infty|^2 dx$$
$$\le \mathcal{H}(0) - c_2 \int_0^t \int_\Omega \int_0^\infty D|\tilde\nabla F(\tau)|^2 d\varepsilon\, dx\, d\tau\,,$$

with positive constants c_1, c_2, and $F_\Phi(t,x,\varepsilon) = \Psi(\varepsilon + \Phi(t,x))$.

Proof. With the assumption of sufficient regularity of F, as mentioned in Remark 1, we first evaluate the time derivative of $\mathcal{H}$:

$$\frac{d}{dt}\int_\Omega \int_0^\infty \mathcal{R}_\Psi[F(t)|F_\infty] N\, d\varepsilon\, dx = \int_\Omega \int_0^\infty \partial_t F \left(\Psi^{-1}(F_\infty) - \Psi^{-1}(F)\right) N\, d\varepsilon\, dx$$
$$= -\int_\Omega \int_0^\infty D\tilde\nabla F \cdot \tilde\nabla\left(\varepsilon + \Phi_\infty - \Psi^{-1}(F)\right) d\varepsilon\, dx$$
$$= \int_\Omega \int_0^\infty (\Psi^{-1})'(F) D|\tilde\nabla F|^2 d\varepsilon\, dx - \int_\Omega \int_0^\infty D\tilde\nabla F \cdot \nabla_x(\Phi_\infty - \Phi) d\varepsilon\, dx\,.$$

The last term can be rewritten in terms of the flux j_F:

$$-\int_\Omega \int_0^\infty D\tilde{\nabla} F \cdot \nabla_x(\Phi_\infty - \Phi) d\varepsilon\, dx = -\int_\Omega (\Phi_\infty - \Phi)\nabla_x \cdot j_F\, dx\,,$$

and using the continuity equation 8,

$$\begin{aligned}
-\int_\Omega (\Phi_\infty - \Phi)\nabla_x \cdot j\, dx &= -\int_\Omega (\Phi_\infty - \Phi)\partial_t(\varrho_\infty - \varrho) dx \\
&= \int_\Omega (\Phi_\infty - \Phi)\Delta_x[\partial_t(\Phi_\infty - \Phi)] dx = -\frac{1}{2}\frac{d}{dt}\left\|\nabla_x(\Phi - \Phi_\infty)\right\|^2_{L^2_x}\,.
\end{aligned}$$

Consequently, we arrive at

$$\begin{aligned}
\frac{d}{dt}\mathcal{H}[F(t)|F_\infty](t) &= \int_\Omega \int_0^\infty (\Psi^{-1})'(F(t)) D|\tilde{\nabla} F(t)|^2 d\varepsilon\, dx \\
&\le -C_0 \int_\Omega \int_0^\infty D|\tilde{\nabla} F|^2 d\varepsilon\, dx\,, \qquad (22)
\end{aligned}$$

where the inequality is due to **A5**. Moreover, we estimate the integrand $\mathcal{R}_\Psi[F|F_\infty]$ in the relative entropy by

$$\begin{aligned}
\mathcal{R}_\Psi[F(t)|F_\infty] &= \frac{1}{2}\beta_\Psi{}''(\xi)(F(t) - F_\infty)^2 \\
&= \frac{1}{2}(-\Psi^{-1})'(\xi)(F(t) - F_\infty)^2 \\
&\ge \frac{C_0}{2}(F(t) - F_\infty)^2\,,
\end{aligned}$$

with some $\xi \in \left(0, \left\|F^0\right\|_{L^\infty_{x,\varepsilon}}\right)$. On the other hand

$$\begin{aligned}
\|F - F_\Phi\|_{L^2_{x,\varepsilon}(N)} &\le \|F - F_\infty\|_{L^2_{x,\varepsilon}(N)} + \|F_\Phi - F_\infty\|_{L^2_{x,\varepsilon}(N)} \\
&\le \|F - F_\infty\|_{L^2_{x,\varepsilon}(N)} + c\|\Phi - \Phi_\infty\|_{L^2_x} \\
&\le \|F - F_\infty\|_{L^2_{x,\varepsilon}(N)} + c\|\nabla_x\Phi - \nabla_x\Phi_\infty\|_{L^2_x}\,,
\end{aligned}$$

where the second inequality is due to the Lipschitz continuity of Ψ, and the third is due to the Poincaré inequality on $H^1_0(\Omega)$. Using this estimate in the time-integrated version of 22, we conclude. □

Lemma 3.3. *The total kinetic energy*

$$\mathcal{E}[F] = \int_\Omega e[F] dx\,, \qquad e[F] = \int_0^\infty \varepsilon F N\, d\varepsilon\,,$$

is bounded uniformly in time.

Proof. This result is the consequence of the fact that the kinetic energy is controlled by the relative entropy. The proof of the previous lemma implies $\mathcal{H}[F(t)|F_\infty] \le \mathcal{H}[F^0|F_\infty]$, and in the following, we shall prove the existence of a constant c, such that

$$\mathcal{H}[F|F_\infty] \ge \frac{1}{2}\mathcal{E}[F] - c\,, \qquad (23)$$

which implies the result of the lemma.

A straightforward computation shows

$$\begin{aligned}\int_0^\infty \mathcal{R}[F|F_\infty]N\,d\varepsilon &= \int_0^\infty \beta_\Psi(F)N\,d\varepsilon + e[F] + e[F_\infty] + \Phi_\infty(\varrho_F + \varrho_\infty) \\ &\geq \int_0^\infty \beta_\Psi(F)N\,d\varepsilon + e[F]\,. \end{aligned} \tag{24}$$

By its definition, β_Ψ is strictly convex and satisfies $\beta_\Psi(0) = \beta_\Psi(\hat{F}) = 0$, $\beta'_\Psi(\Psi(0)) = 0$, with $0 < \Psi(0) < \hat{F}$. This implies

$$\int_0^\infty \beta_\Psi(F)N\,d\varepsilon \geq \int_{F<\Psi(\varepsilon/(2c_\Psi))} \beta_\Psi(F)N\,d\varepsilon + \int_{\Psi(\varepsilon/(2c_\Psi))\leq F<\hat{F}} \beta_\Psi(F)N\,d\varepsilon$$
$$\geq \int_0^\infty \beta_\Psi(\Psi(\varepsilon/2c_\Psi))N\,d\varepsilon - \frac{1}{2}e[F] = -2(2c_\Psi)^{3/2}e[\Psi] - \frac{1}{2}e[F]\,.$$

In the estimate of the second term assumption **A7** has been used, and in the final computation of the first term, the explicit form of the density of states $N(\varepsilon) = \sqrt{\varepsilon}$. Combining this last estimate with 24 and integration with respect to x shows 23, completing the proof. □

Lemma 3.4. *The entropy dissipation rate satisfies*

$$\int_\Omega \int_0^\infty D|\tilde{\nabla}F|^2 d\varepsilon\,dx \geq c(1-\alpha)\left(\int_\Omega \int_0^\infty (F - F_\Phi)^2 N\,d\varepsilon\,dx\right)^r\,, \tag{25}$$

where c is a time-independent constant and $r = 3(1+\alpha)/2$.

Proof. Setting $G = F - F_\Phi$, the Hölder inequality with exponent 3 and the identity $N(\varepsilon)^3 = \varepsilon N(\varepsilon)$ imply

$$\int_\Omega \int_0^\infty G^2 N\,d\varepsilon\,dx \leq \mathcal{E}[G]^{1/3}\|G\|^{1/3}_{L^\infty_{x,\varepsilon}}\|G\|^{4/3}_{L^2_{x,\varepsilon}}$$

We introduce the coordinate transformation $(x,\varepsilon) \to (x,z)$ with the total energy $z := \varepsilon + \Phi$, such that the operator $\tilde{\nabla} = \partial_x - \partial_x\Phi\,\partial_\varepsilon$ transforms to ∂_x. Due to the nonnegativity of the boundary data Φ_b (Assumption **A4**), we have $z \geq 0$. Note that $F_\Phi(t,x,z-\Phi(t,x)) = \Psi(z)$, implying $\partial_x F = \partial_x G$. Therefore, we have to prove

$$\int_0^\infty \int_{\Omega_{\Phi,z}} G^2 dx\,dz \leq c\left(\int_0^\infty \int_{\Omega_{\Phi,z}} D(z-\Phi)(\partial_x G)^2 dx\,dz\right)^{\frac{3}{2r}}\,, \tag{26}$$

for G satisfying $G(x,z) = 0$ for $x \in \partial\Omega$, where $\Omega_{\Phi,z} = \{x \in \Omega :\ \Phi(x) < z\}$.

Since $-\partial_x^2\Phi = \varrho_F \geq 0$, the potential Φ is concave and bounded over Ω. Therefore, for any fixed time $t \geq 0$, we may define

$$\begin{aligned}\underline{\Phi} &:= \inf_{x\in\Omega}\Phi(x) \geq \inf_{x\in\partial\Omega}\Phi_b(x) \geq 0\,, \\ \overline{\Phi} &:= \sup_{x\in\Omega}\Phi(x) < \infty\,.\end{aligned}$$

We fix a constant $\delta > 0$ and define the disjoint intervals

$$I^1_\delta := (\underline{\Phi}, \overline{\Phi} - \delta)\,, \quad I^2_\delta := (\overline{\Phi} - \delta, \overline{\Phi} + \delta) \quad \text{and} \quad I^3_\delta := (\overline{\Phi} + \delta, \infty)\,.$$

We split the domain of integration in 26 accordingly and prove the inequality in each of the three parts:

- We start with I_δ^3. For a fixed $z \in I_\delta^3$, we have $\Omega_{\Phi,z} = \Omega$ and
$$z - \Phi(x) \geq \delta \qquad \text{for all} \quad x \in \Omega\,.$$
Therefore,
$$\begin{aligned}\int_{I_\delta^3}\int_\Omega G^2 dx\,dz &\leq c\int_{I_\delta^3}\int_\Omega (\partial_x G)^2 dx\,dz \\ &\leq \frac{c}{\delta^\alpha}\int_{I_\delta^3}\int_\Omega D(z-\Phi)(\partial_x G)^2 dx\,dz\,,\end{aligned}$$
where c is the constant of the Poincaré inequality and depends only on Ω.
- For I_δ^2, we obtain
$$\int_{I_\delta^2}\int_{\Omega_{\Phi,z}} G^2 dx\,dz \leq 2\delta|\Omega|\,\|G\|^2_{L^\infty_{x,\varepsilon}}\ .$$
- Finally, the estimate for $z \in I_\delta^1$ is a consequence of the (possibly not strict) concavity of Φ. At the points $x \in \Phi^{-1}(\overline{\Phi}-\delta)$, the slope of Φ is at least $\delta/|\Omega|$. Therefore, for any fixed $z \in I_\delta^1$, we have $|\Phi'(x)| \geq \delta/|\Omega|$ for all $x \in \Omega_{\Phi,z}$. Consequently,
$$z - \Phi(x) \geq \frac{\delta}{|\Omega|}\operatorname{dist}(x, \partial(\Omega\backslash\Omega_{\Phi,z}))\,,$$
and
$$\int_{\Omega_{\Phi,z}} \frac{dx}{(z-\Phi)^\alpha} \leq \left(\frac{|\Omega|}{\delta}\right)^\alpha \int_{\Omega_{\Phi,z}} \frac{dx}{\operatorname{dist}(x,\partial(\Omega\setminus\Omega_{\Phi,z}))^\alpha} \leq \frac{|\Omega|}{(1-\alpha)\delta^\alpha}\,.$$
Noting that the boundary of each connected component of $\Omega_{\Phi,z}$ contains at least one point of $\partial\Omega$, we use the weighted Poincaré inequality 27 with $w(x) = D(z-\Phi(x))$. This yields
$$\begin{aligned}\int_{\Omega_{\Phi,z}} G^2 dx &\leq \int_{\Omega_{\Phi,z}} \frac{dx}{D(z-\Phi)}\int_{\Omega_{\Phi,z}} D(z-\Phi)(\partial_x G)^2 dx \\ &\leq \frac{|\Omega|}{(1-\alpha)\delta^\alpha}\int_{\Omega_{\Phi,z}} D(z-\Phi)(\partial_x G)^2 dx\,.\end{aligned}$$

Combining the three partial estimates, we have
$$\int_0^\infty\int_{\Omega_{\Phi,z}} G^2 dx\,dz \leq c_1\delta + \frac{c_2}{(1-\alpha)\delta^\alpha}\int_0^\infty\int_{\Omega_{\Phi,z}} D(z-\Phi)(\partial_x G)^2 dx\,dz\,,$$
where c_1 and c_2 are time-independent constants. Optimizing in δ gives
$$\int_0^\infty\int_{\Omega_{\Phi,z}} G^2 dx\,dz \leq c\left(\frac{1}{1-\alpha}\int_0^\infty\int_{\Omega_{\Phi,z}} D(z-\Phi)(\partial_x G)^2 dx\,dz\right)^{\frac{1}{1+\alpha}}\,,$$
and a transformation back to the original coordinates (x,ε) concludes the proof of 25. ∎

We now conclude the proof of Theorem 3.1. Denoting
$$u(t) = \int_\Omega\int_0^\infty (F-F_\Phi)^2 N\,d\varepsilon\,dx\,,$$

Lemmata 3.2 and 3.4 yield the estimate

$$u(t) \le c_1 - c_2(1-\alpha)\int_0^t u(s)^r\,\mathrm{d}s\,,$$

with time-independent constants c_1 and c_2. An application of the Gronwall lemma gives the result

$$u(t) \le \frac{c}{(1+(1-\alpha)t)^{1/(r-1)}} \qquad \text{for all } t \ge 0\,.$$

Lemma 3.2 provides a uniform-in-time bound for $\nabla_x\Phi(t,\cdot)$ in L^2_x. As a consequence, restricting to subsequences, $\Phi(t,\cdot) \to \hat\Phi$ in L^2_x as $t\to\infty$, implying $F_\Phi \to F_{\hat\Phi}$ and therefore also $F \to F_{\hat\Phi}$ in $L^2_{x,\varepsilon}(N)$. Passing to the limit in the Poisson equation $-\Delta_x\Phi = \varrho_F$ implies $\hat\Phi = \Phi_\infty$, leading to the convergence $F \to F_\infty$ in $L^2_{x,\varepsilon}(N)$ (without restricting to a subsequence).

4. **Appendix: The weighted Poincaré inequality.** For completeness, we include a proof of the inequality 27 below. Let Ω be a bounded domain in $\mathbb{R}^d$, $u \in C_0^\infty(\Omega)$ and w a nonnegative weight. For a fixed $\tilde x := (x_2,\dots,x_d) \in \mathbb{R}^{d-1}$, let us denote

$$\Omega^{-1}(\tilde x) = \{x_1 \in \mathbb{R};\, (x_1,\tilde x) \in \Omega\}\,.$$

Then, we have

$$\begin{aligned} u(x_1,\tilde x)^2 &\le \left(\int_{\Omega^{-1}(\tilde x)} |\partial_{x_1} u(y,\tilde x)|\,\mathrm{d}y\right)^2 \\ &\le \left(\int_{\Omega^{-1}(\tilde x)} \frac{\mathrm{d}y}{w(y,\tilde x)}\right)\left(\int_{\Omega^{-1}(\tilde x)} w(y,\tilde x)|\partial_{x_1} u(y,\tilde x)|^2\,\mathrm{d}y\right). \end{aligned}$$

Integrating over Ω, we obtain

$$\int_\Omega u(x)^2\,\mathrm{d}x \le \mathrm{diam}(\Omega)\left(\sup_{\tilde x\in\mathbb{R}^{d-1}} \int_{\Omega^{-1}(\tilde x)} \frac{\mathrm{d}y}{w(y,\tilde x)}\right)\left(\int_\Omega w(x)|\partial_{x_1}u(x)|^2\,\mathrm{d}x\right), \quad (27)$$

which we call the weighted Poincaré inequality.

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