Convergence to global equilibrium for spatially inhomogeneous kinetic models of non-micro-reversible processes

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Abstract

We study the long-time asymptotics of linear kinetic models with periodic boundary conditions or in a rectangular box with specular reflection boundary conditions. An entropy dissipation approach is used to prove decay to the global equilibrium under some additional assumptions on the equilibrium distribution of the mass preserving scattering operator. We prove convergence at an algebraic rate depending on the smoothness of the solution. This result is compared to the optimal result derived by spectral methods in a simple one dimensional example.

Key words: kinetic transport equations, relative entropy, entropy dissipation, long-time asymptotics

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1 Introduction

We investigate initial value problems of the form

$$\partial_t f + v(k) \cdot \nabla_x f = Q(f) \tag{1}$$

where $f = f(t,x,k) \geq 0$ denotes the particle distribution function, depending on time $t \geq 0$, position $x \in \mathbb{T}^d$ (a d-dimensional torus, $d \geq 1$) and momentum $k \in B \subset \mathbb{R}^d$. For the velocity-momentum relation $v = v(k) \in \mathbb{R}^d$ we assume continuity, oddness $(v(-k) = -v(k))$ and that flow in any direction is possible:

$$\forall z \in S^{d-1}\ \exists k \in B: \quad v(k) \cdot z \neq 0\,. \tag{2}$$

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The set of admissible momentum vectors B is symmetric about the origin. The simplest example are free classical particles with $v(k) = k$ and B rotationally invariant, e.g. a ball, a sphere, or $B = \mathbb{R}^3$. Other examples are relativistic particles with $v(k) = k(1 + k^2/c^2)^{-1/2}$ and $B = \mathbb{R}^3$, or semi-classical particles moving in a periodic potential where $v(k) = \nabla_k \epsilon(k)$, $\epsilon(k)$ is the band diagram, and B the first Brillouin zone. In this case all functions of k satisfy periodic boundary conditions on ∂B. In all cases we assume B to be equipped with a measure, which we denote by $\mathrm{d}k$ for simplicity.

The scattering operator Q acts only on the variable k and is assumed linear and of the form

$$Q(f)(k) = \int_B [S(k', k) f' - S(k, k') f] \mathrm{d}k',$$

where $f' := f(k')$, and $S(k, k') \geq 0$ is the scattering rate.

We assume the existence of a normalized equilibrium distribution $M(k)$ with vanishing mean velocity and bounded velocity moments up to fourth order:

$$(a) \quad M \in L^1(B, \mathrm{d}k),\ M \geq 0\,, \qquad (b) \quad Q(M) = 0\,,$$

$$(c) \quad \int_B M(k)\, \mathrm{d}k = 1\,, \qquad (d) \quad \int_B v(k) M(k)\, \mathrm{d}k = 0\,,$$

$$(e) \quad m_j := \int_B |v(k)|^j M(k)\, \mathrm{d}k < \infty\,,\ j \leq 4\,.$$

A sufficient condition for $Q(M) = 0$ is the detailed balance or micro-reversibility condition $S(k', k) M' = S(k, k') M$, $\forall k, k' \in B$, which we shall *not* assume here.

It has been shown by Degond, Goudon, and Poupaud [4] (see also [11]) that even without micro-reversibility there is an entropy equation:

$$\int_B \frac{Q(f) f}{M} \mathrm{d}k = -\frac{1}{4} \int_B \int_B [S(k', k) M' + S(k, k') M] \left(\frac{f}{M} - \frac{f'}{M'} \right)^2 \mathrm{d}k' \mathrm{d}k\,, \qquad (3)$$

showing that the bilinear form $\int_B \frac{Q(f) g}{M} \mathrm{d}k$ is non-positive (however it is non symmetric without detailed balance).

Motivated by the entropy equation we shall consider the weighted L^2-scalar product

$$\langle f, g \rangle_M = \int_B \frac{fg}{M}\, \mathrm{d}k\,.$$

For the scattering rate we assume the existence of positive constants γ and Γ such that

$$\gamma \leq \frac{S(k', k)}{M(k)} \leq \Gamma \qquad \forall k, k' \in B\,.$$

This implies the boundedness of $Q : L^2(B, \frac{\mathrm{d}k}{M}) \to L^2(B, \frac{\mathrm{d}k}{M})$ (actually $\|Q(f)\|_M \leq 2\Gamma \|f\|_M$) and

$$\int_B f\, \mathrm{d}k = \langle f, M \rangle_M = 0 \quad \Longrightarrow \quad \gamma \|f\|_M^2 \leq -\langle Q(f), f \rangle_M\,, \qquad (4)$$

i.e. coercivity on the orthogonal complement of the kernel of Q, spanned by M. As a consequence, for $g \in L^2(B, \frac{\mathrm{d}k}{M})$ with $\int_B g \mathrm{d}k = 0$ the problem

$$Q(f) = g\,, \qquad \int_B f\,\mathrm{d}k = 0\,,$$

has a unique solution f in $L^2(B, \frac{\mathrm{d}k}{M})$ (see [4]). The necessity of the solvability criterion $\int_B g\,\mathrm{d}k = 0$ follows from the mass conservation property of Q, also implying (by integration of (1) with respect to k) the macroscopic conservation law

$$\partial_t \rho + \nabla_x \cdot J = 0$$

for the macroscopic mass density $\rho(t,x) = \int_B f(t,x,k)\mathrm{d}k$, where J is the flux density $J(t,x) = \int_B v(k) f(t,x,k)\mathrm{d}k$.

Existence of a unique solution of (1) subject to an initial condition

$$f(t=0) = f_0 \in L^2\left(\mathbb{T}^d; L^1(B) \cap L^2\left(B, \frac{\mathrm{d}k}{M}\right)\right) \tag{5}$$

follows by standard arguments.

We shall be concerned with the convergence as $t \to \infty$ of the solution of (1) to the global equilibrium $\rho_\infty M(k)$, where ρ_∞ is determined by conservation of total mass,

$$\rho_\infty = \frac{1}{\mu(\mathbb{T}^d)} \int_{\mathbb{T}^d} \rho(t,x)\,\mathrm{d}x = \frac{1}{\mu(\mathbb{T}^d)} \int_{\mathbb{T}^d} \int_B f_0(x,k)\,\mathrm{d}k\,\mathrm{d}x\,.$$

With the relative entropy

$$H(f|g) := \int_{\mathbb{T}^d} \|f - g\|_M^2 \mathrm{d}x,$$

the entropy equation (3) and the coercivity (4) imply

$$\frac{\mathrm{d}}{\mathrm{d}t} H(f|\rho_\infty M) \le -\gamma H(f|\rho M)\,. \tag{6}$$

Arguments, which are standard by now (see [5]) use this inequality for proving weak convergence of f to the global equilibrium $\rho_\infty M$.

In this work we shall prove *strong convergence* of smooth solutions *at an algebraic rate* depending on the smoothness. We shall use the entropy-entropy dissipation approach developed by Desvillettes and Villani [6]. In [6] it has been applied to a linear Fokker-Planck equation with a confining potential. There the necessary smoothness is produced by a hypo-ellipticity property. In the case considered here, regularization effects cannot be expected since the scattering operator is bounded. We will use smoothness assumptions on the initial data and propagation of regularity instead. A recent comparable study is [2], where the special (micro-reversible) case

$$v = k, \quad B = \mathbb{R}^3, \quad S(k',k) = M(k)$$

with a Maxwellian M of vanishing mean velocity and given constant temperature is considered. Similarly to [6], the whole space problem with confining potential is treated there.

It is important to point out that the approach of Desvillettes and Villani has the potential to deal with nonlinear problems. Recently it has been applied to the (gas dynamics) Boltzmann equation by the same authors.

Shortcomings of the approach are the required smoothness and the fact that exponential rates of convergence cannot be provided in general. This will be illustrated by a simple one-dimensional example at the end of this work, which can be solved by spectral analysis. As a result, exponential convergence to global equilibrium, even for unsmooth solutions, is shown.

A further comment is concerned with boundary conditions. Let $\mathbb{T}^d$ be represented by an interval in $\mathbb{R}^d$, centered around the origin, with periodic boundary conditions. Assume further that the initial datum f_0 is invariant under the transformations $(x_i, k_i) \mapsto (-x_i, -k_i), i = 1, \dots, d$. Then this symmetry will be propagated by (1), if $v(k)$ and $S(k, k')$ have the according symmetries. The crucial observation is that on the subinterval Ω of $\mathbb{T}^d$ defined by $x_i \geq 0, i = 1, \dots d$, distribution functions with the symmetry of f_0 satisfy specular reflection boundary conditions on $\partial\Omega$ [9]. This shows that our analysis applies to the case where the spatial domain is an *interval* with specular reflection boundary conditions.
Smoothness with respect to x of f_0 as a function defined on $\mathbb{T}^d$ requires compatibility conditions along $\partial\Omega$. An example with $d = 1$ will be discussed in the last section. For general domains Ω with specular reflection boundary conditions, smoothness of the solution is a delicate question (see [10]).

2 The convergence result

In this section the following result will be proved:

Theorem. *Let $n \geq 2$ and $f_0 \in L^1(\mathbb{T}^d \times B) \cap L^2\left(B, \frac{dk}{M}; H^n(\mathbb{T}^d)\right)$. Then there exists $C > 0$, such that the solution f of the initial value problem* (1), (5) *satisfies*

$$\|f - \rho_\infty M\|_{L^2(\mathbb{T}^d \times B, dx dk/M)} \leq C t^{(1-n)/2} .$$

As a preliminary step, we prove a smoothness result

Lemma. *Under the assumptions of the above theorem, f satisfies*

$$\|f(t, \cdot, \cdot)\|_{L^2\left(B, dk/M;\ H^n(\mathbb{T}^d)\right)} \leq \|f_0\|_{L^2\left(B, dk/M;\ H^n(\mathbb{T}^d)\right)} . \tag{7}$$

Proof. For $n = 0$ the result is a consequence of the entropy equation (3) which implies that the above norm of f is non increasing with time. Since the coefficients in the transport equation (1) are independent of x, the same equation holds for partial derivatives of f with respect to x. Therefore the left hand side of (7) also decays for positive n. □

The simplicity of the proof of this result strongly relies on the periodic boundary conditions, the linearity of the transport equation, and on the fact that the transport equation does not contain position dependent coefficients. In [2] and [6], where the whole space problem with a confining potential is treated, the proofs of results comparable to the Lemma are a major part of the analysis.

Now we proceed with the proof of the Theorem:

Proof. The main argument starts with the inequality (6). The problem is that it allows the decay to global equilibrium to stop as soon as a local equilibrium $f(t,x,k) = \rho(t,x)M(k)$ is reached. Therefore we have to prove that in such a situation f moves out of the local equilibrium as long as it is still away from the global equilibrium. For this purpose, we compute derivatives with respect to time of the relative entropy of f with respect to the local equilibrium:

$$\frac{\mathrm{d}}{\mathrm{d}t}H(f|\rho M) = 2\int_{\mathbb{T}^d} \langle Q(f), f\rangle_M \,\mathrm{d}x + 2\int_{\mathbb{T}^d} \rho\nabla_x \cdot J\,\mathrm{d}x$$

Since $H(f|\rho M)$ is nonnegative, it is no surprise that the right hand side vanishes for $f = \rho M$. In the computation of the second order time derivative of $H(f|\rho M)$, we shall use the momentum balance equation

$$\partial_t J + \nabla_x \cdot P = \int_B vQ(f)\mathrm{d}k$$

with the pressure tensor $P := \int_B v\otimes vf\,\mathrm{d}k$. The right hand side can be interpreted as momentum relaxation term. Now we rewrite the pressure tensor as

$$P = \int_B v\otimes v(f-\rho M)\mathrm{d}k + \rho T \quad ,$$

where the temperature tensor $T := \int_B v\otimes vM\,\mathrm{d}k$ is positive definite as a consequence of (2) and of the continuity of $v(k)$.

A symmetrized version of the scattering operator is given by

$$Q^s(f) = \int_B \phi(k,k')\left(\frac{f'}{M'} - \frac{f}{M}\right)\mathrm{d}k' \,, \qquad \phi(k,k') = \frac{S(k',k)M' + S(k,k')M}{2} \,.$$

The maps Q^s and Q produce the same quadratic form $\langle Q(f), f\rangle_M = \langle Q^s(f), f\rangle_M$, where the latter is derived from a symmetric bilinear form. Now it is straightforward to compute the second order time derivative of the relative entropy with respect to the local equilibrium:

$$\begin{aligned}\frac{\mathrm{d}^2}{\mathrm{d}t^2}H(f|\rho M) &= 2\int_{\mathbb{T}^d}(\nabla_x\rho)^T T(\nabla_x\rho)\mathrm{d}x - 2\int_{\mathbb{T}^d}(\nabla_x\cdot J)^2\,\mathrm{d}x \\ &+2\int_{\mathbb{T}^d}\nabla_x\rho\cdot\int_B v\otimes v\nabla_x(f-\rho M)\mathrm{d}k\mathrm{d}x \\ &-2\int_{\mathbb{T}^d}\nabla_x\rho\cdot\int_B vQ(f)\mathrm{d}k\mathrm{d}x + 4\int_{\mathbb{T}^d}\langle Q^s(f),Q(f)\rangle_M\,\mathrm{d}x \\ &-4\int_{\mathbb{T}^d}\langle Q^s(f), v\cdot\nabla_x f\rangle_M\,\mathrm{d}x\,. \end{aligned} \tag{8}$$

Note that when $f=\rho M$ all terms except the first one on the right hand side vanish. Exactly this term leads to the desired result that $\frac{\mathrm{d}^2}{\mathrm{d}t^2}H(f|\rho M)$ is positive whenever $f=\rho M$, but f has not reached the global equilibrium $\rho_\infty M$. We estimate it from below by

$$\int_{\mathbb{T}^d}(\nabla_x\rho)^T T(\nabla_x\rho)\mathrm{d}x \ge K I(\rho M|\rho_\infty M),$$

where K is a positive constant coming from the positive definiteness of T, and

$$I(f|g) := H(\nabla_x f|\nabla_x g)$$

is the Fisher information. In estimating the remaining terms in (8), it is important that the bounds vanish for $f=\rho M$. For the first term we derive

$$\int_{\mathbb{T}^d}(\nabla_x\cdot J)^2\mathrm{d}x = \int_{\mathbb{T}^d}\left(\nabla_x\cdot\int_B\sqrt{M(k)}v(k)\frac{(f-\rho M)}{\sqrt{M(k)}}\,\mathrm{d}k\right)^2\mathrm{d}x \le m_2 I(f|\rho M),$$

where we have used that M has zero mean velocity. The other terms are estimated similarly by further applications of the Cauchy-Schwarz inequality:

$$\begin{gathered}\left|\int_{\mathbb{T}^d}\nabla_x\rho\cdot\int_B v\otimes v\nabla_x(f-\rho M)\mathrm{d}k\,\mathrm{d}x\right| \le \sqrt{m_4 I(\rho M|\rho_\infty M)I(f|\rho M)}\,, \\ \left|\int_{\mathbb{T}^d}\nabla_x\rho\cdot\int_B vQ(f)\mathrm{d}k\,\mathrm{d}x\right| \le 2\Gamma\sqrt{m_2 I(\rho M|\rho_\infty M)H(f|\rho M)}\,, \\ \left|\int_{\mathbb{T}^d}\langle Q^s(f),Q(f)\rangle_M\,\mathrm{d}x\right| = \left|\int_{\mathbb{T}^d}\langle Q^s(f-\rho M),Q(f-\rho M)\rangle_M\,\mathrm{d}x\right| \le 4\Gamma^2 H(f|\rho M)\,.\end{gathered} \tag{9}$$

And finally, the most complicated term

$$\begin{aligned}&\int_{\mathbb{T}^d}\langle Q^s(f), v\cdot\nabla_x f\rangle_M\,\mathrm{d}x \\ &= \int_{\mathbb{T}^d}\int_B\frac{1}{M}Q^s(f-\rho M)v\cdot\nabla_x(f-\rho M)\mathrm{d}k\,\mathrm{d}x + \int_{\mathbb{T}^d}\int_B Q^s(f)v\cdot\nabla_x\rho\,\mathrm{d}k\,\mathrm{d}x\end{aligned} \tag{10}$$

The first term on the right hand side is again split into two parts according to the gain and loss terms in Q^s. The second part originating from the loss term can be written as the integral of a divergence and, thus, vanishes by the divergence theorem due to the periodic boundary conditions. Therefore we have

$$\begin{aligned}&\int_{\mathbb{T}^d}\int_B \frac{1}{M} Q^s(f-\rho M) v\cdot\nabla_x (f-\rho M)\mathrm{d}k\,\mathrm{d}x \\ &= \int_{\mathbb{T}^d}\int_B\int_B \frac{\phi(k',k)}{MM'}(f'-\rho M')v\cdot\nabla_x(f-\rho M)\mathrm{d}k'\,\mathrm{d}k\,\mathrm{d}x\,.\end{aligned}$$

With the boundedness assumption on the scattering kernel and the Cauchy-Schwarz inequality the modulus of this term can be estimated by

$$\Gamma\sqrt{m_2 I(f|\rho M)H(f|\rho M)} \le \frac{\Gamma\sqrt{m_2}}{2}\big(I(f|\rho M)+H(f|\rho M)\big)\,.$$

The last term in (10) is estimated analogously to (9):

$$\left|\int_{\mathbb{T}^d}\nabla_x\rho\cdot\int_B vQ^s(f)\,\mathrm{d}k\,\mathrm{d}x\right| \le 2\Gamma\sqrt{m_2 I(\rho M|\rho_\infty M)H(f|\rho M)}\,.$$

Combining all these estimates, we have

$$\begin{aligned}\frac{\mathrm{d}^2}{\mathrm{d}t^2}H(f|\rho M) \ge KI(\rho M|\rho_\infty M) &- C_1\sqrt{I(\rho M|\rho_\infty M)\big(H(f|\rho M)+I(f|\rho M)\big)} \\ &- C_2 H(f|\rho M) - C_3 I(f|\rho M)\,,\end{aligned} \tag{11}$$

with positive constants K, C_1, C_2, C_3. We simplify this inequality using the fact that $\forall\,\delta>0\;\exists\, C_\delta>0$, such that

$$\sqrt{I(\rho M|\rho_\infty M)\big(H(f|\rho M)+I(f|\rho M)\big)} \le \delta I(\rho M|\rho_\infty M) + C_\delta H(f|\rho M) + C_\delta I(f|\rho M)\,,$$

and the Poincaré inequality [7]

$$I(\rho M|\rho_\infty M) = \int_{\mathbb{T}^d}|\nabla_x\rho|^2\mathrm{d}x \ge C\int_{\mathbb{T}^d}(\rho-\rho_\infty)^2\,\mathrm{d}x = CH(\rho M|\rho_\infty M)\,.$$

Choosing δ small enough, (11) takes the form

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}H(f|\rho M) \ge KH(\rho M|\rho_\infty M) - C_1 H(f|\rho M) - C_2 I(f|\rho M)\,, \tag{12}$$

with different, but still positive constants.

In the next step we use the additivity of the relative entropy, i.e.,

$$H(f|\rho M) + H(\rho M|\rho_\infty M) = H(f|\rho_\infty M)\,,$$

to derive

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} H(f|\rho M) \geq K H(f|\rho_\infty M) - \tilde{C}_1 H(f|\rho M) - C_2 I(f|\rho M)\,. \qquad (13)$$

Now we need an estimate for the term $I(f|\rho M)$. This is done by standard interpolation [12] in the position variable:

$$\|\nabla_x u\|_{L^2(\mathbb{T}^d)} \leq C \|u\|_{L^2(\mathbb{T}^d)}^{1-1/n} \|u\|_{H^n(\mathbb{T}^d)}^{1/n}\,, \qquad \forall\, u \in H^n(\mathbb{T}^d)\,,\ n \geq 1\,.$$

We derive

$$\begin{aligned} I(f|\rho M) &= \int_B \frac{1}{M} \|\nabla_x (f - \rho M)\|_{L^2(\mathbb{T}^d)}^2 \mathrm{d}k \\ &\leq C \int_B \left(\frac{1}{M} \|f - \rho M\|_{L^2(\mathbb{T}^d)}^2 \right)^{1-1/n} \left(\frac{1}{M} \|f - \rho M\|_{H^n(\mathbb{T}^d)}^2 \right)^{1/n} \mathrm{d}k \\ &\leq C H(f|\rho M)^{1-1/n} \left(\int_B \frac{1}{M} \|f - \rho M\|_{H^n(\mathbb{T}^d)}^2 \mathrm{d}k \right)^{1/n}\,, \end{aligned} \qquad (14)$$

where the last estimate is due to the Hölder inequality. The boundedness of the last factor on the right hand side is ensured by the Lemma:

$$\int_B \frac{1}{M} \|f - \rho M\|_{H^n(\mathbb{T}^d)}^2 \mathrm{d}k \leq \|f(t,\cdot,\cdot)\|_{L^2\left(B, \frac{\mathrm{d}k}{M}; H^n(\mathbb{T}^d)\right)}^2 \leq \|f_0\|_{L^2\left(B, \frac{\mathrm{d}k}{M}; H^n(\mathbb{T}^d)\right)}^2\,.$$

Recalling (6), we have derived the following system of differential inequalities for the relative entropies with respect to local and global equilibrium:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} H(f|\rho_\infty M) &\leq -\gamma H(f|\rho M)\,, \\ \frac{d^2}{\mathrm{d}t^2} H(f|\rho M) &\geq K H(f|\rho_\infty M) - C_1 H(f|\rho M) - C_2 H(f|\rho M)^{1-1/n}\,. \end{aligned} \qquad (15)$$

Since $H(f|\rho M)$ is bounded, the term $C_1 H(f|\rho M)$ can be dropped by increasing C_2.

Theorem 6.2 from [6] now ensures

$$H(f|\rho_\infty M) \leq C t^{-n+1}\,,$$

completing the proof. Note that for $n = \infty$ in (15), we would have exponential convergence by Theorem 6.2 of [6]. □

The Csiscár-Kullback inequality (compare [1]),

$$\|f\|_{L^1(\mathbb{T}^d \times B)} \leq C \|f\|_{L^2(\mathbb{T}^d \times B, \mathrm{d}x\mathrm{d}k/M)}\,,$$

a simple consequence of the normalization of M and of the boundedness of $\mathbb{T}^d$, can be used to estimate decay of the L^1-norm rather than of the weighted L^2-norm:

Corollary. *Under the assumptions of the Theorem, the following estimate holds:*

$$\|f - \rho_\infty M\|_{L^1(\mathbb{T}^d \times B)} \leq C t^{(1-n)/2}\,.$$

3 A one-dimensional example

In this section we shall present a simple one dimensional problem that can be solved explicitly by spectral methods. We will show the relation between specular reflection and periodic boundary conditions and compare the decay estimate of the preceding section to the optimal result.

One advantage of the spectral theory approach is that it is not relying on smoothness of the solutions. While the assumption $f_0 \in L^2(B, H^1(\mathbb{T}^d))$ in Chapter 2 is necessary to give $\frac{d^2}{d^2t}H(f|\rho M)$ a meaning and even more regularity is needed to guarantee fast convergence, spectral theory shows *exponential* convergence for *every* L^2 solution.

We treat the following problem:

$$\partial_t f + k\partial_x f = \langle f \rangle - f \tag{16}$$

with $k \in \{+1, -1\}$, initial condition $f(t=0) = f_0$, and periodic boundary conditions in x with period $2L$, i.e., $T^1 = (-L, L)$ is a possible choice. The expression $\langle f \rangle$ is the mean value

$$\langle f \rangle (t,x) := \frac{f(t,x,1) + f(t,x,-1)}{2} .$$

This can be considered as a one dimensional neutron transport or radiative transfer equation and falls into the class of equations treated in this paper. We chose a simple one dimensional example because eigenfunctions and eigenvalues can be calculated explicitly, however our convergence result from Section 2 also applies to similar problems, i.e. for example mass preserving neutron transport or radiative transfer in higher dimensions. The entropy method was used to show exponential convergence for the homogeneous radiative transfer equation in [8]. For a review of neutron transport and spectral considerations leading to convergence results, see [3].

Equation (16) will be solved by spectral methods. The spectral problem

$$\lambda f + k\partial_x f = \langle f \rangle - f$$

can be written as the system of ordinary differential equations

$$\begin{aligned} \lambda \rho_\lambda + \partial_x j_\lambda &= 0 \,, \\ \lambda j_\lambda + \partial_x \rho_\lambda &= -j_\lambda \,, \end{aligned} \tag{17}$$

by using the macroscopic density and flux

$$\rho(x) = f(x,+1) + f(x,-1) \,, \qquad J(x) := f(x,+1) - f(x,-1) \,.$$

The eigenvalues are solutions of

$$\lambda_l (\lambda_l + 1) = -\left(\frac{l\pi}{L}\right)^2 , \qquad \text{i.e., } \lambda_{l,\pm} = -\frac{1}{2} \pm \sqrt{\frac{1}{4} - \left(\frac{l\pi}{L}\right)^2} , \qquad l \geq 0 \,.$$

The eigenvalues $\lambda_{0,+} = 0$ and $\lambda_{0,-} = -1$ are simple with the eigenspaces $E_{0,+}$ =span$\{(1,0)\}$ and $E_{0,-}$ =span$\{(0,1)\}$. All the other eigenvalues have multiplicity two with the eigenspaces

$$E_{l,\pm} = \text{span}\left\{ \left(-\frac{l\pi}{\lambda_{l,\pm} L} \cos \frac{l\pi x}{L},\ \sin \frac{l\pi x}{L} \right),\ \left(\frac{l\pi}{\lambda_{l,\pm} L} \sin \frac{l\pi x}{L},\ \cos \frac{l\pi x}{L} \right) \right\}.$$

It is easily seen that the eigenfunctions form a basis of $L^2\big((-L,L)\big)^2$. Therefore, the initial value problem is solved completely for initial data $f_0(\cdot,\pm 1) \in L^2\big((-L,L)\big)$.

Observe that all eigenvalues have their real part in the interval $[-1,0]$. There is a spectral gap g of positive length between $\lambda_{0,+} = 0$ and the nearest eigenvalue $\lambda_{1,+}$:

$$g = \begin{cases} \frac{1}{2} & \text{if } L < 2\pi, \\ \frac{1}{2} - \sqrt{\frac{1}{4} - \left(\frac{\pi}{L}\right)^2} & \text{else.} \end{cases}$$

So we derived convergence of f as $t \to \infty$ (with respect to the L^2-norm) to

$$f_\infty = \frac{1}{4L} \int_{-L}^{L} (f_0(x,1) + f_0(x,-1))dx$$

at the exponential rate e^{-tg}.

As noted before, exponential convergence could be recovered in the entropy approach by showing that

$$I(f|\rho M) \le C H(f|\rho M) \tag{18}$$

holds and, thus, (15) is valid with $n = \infty$. However straightforward calculations show that this is not even possible in this simple case. It is well known [1] that validity of a Sobolev inequality of type (18) would imply the existence of a spectral gap of positive length. Our example, as many others, demonstrates that the converse is not true.

We conclude our work by commenting on specular reflection boundary conditions. Let us consider equation (16) on $\Omega = (0,L)$ with specular reflection boundary conditions

$$f(t,0,+1) = f(t,0,-1)\,, \qquad f(t,L,+1) = f(t,L,-1)\,.$$

Proceeding as proposed in the introduction, the initial data have to be continued to $(-L,L)$ by

$$f_0(x,k) = f_0(-x,-k)\,, \tag{19}$$

and the problem with periodic boundary conditions is solved. However, this periodic continuation has to satisfy regularity assumptions when the entropy approach is applied. It is of course *not* sufficient that $f_0(\cdot,\pm 1) \in H^n\big((0,L)\big)$. Additionally the initial data have to satisfy the following compatibility conditions:

$$\partial_x^m f_0(0,k) = (-1)^m \partial_x^m f_0(0,-k)\,, \qquad 0 \le m \le n-1\,. \tag{20}$$

Conversely, for smooth periodic initial data satisfying (19), this symmetry (and, thus, also (20)) is propagated by the transport equation (being invariant under the map $(x,k) \to (-x,-k)$). Thus, if the solution is reduced to the interval $(0,L)$, it satisfies specular reflection boundary conditions.

Similar compatibility conditions arise in higher dimensional rectangular domains. For domains with curved boundaries and specular reflection boundary conditions, appropriate compatibility conditions are hard to formulate and are only expected to exist under convexity assumptions on the domain. Related results can be found in the recent work [10] on classical solutions of Vlasov-Poisson with specular reflection. There, only first order derivatives have to be controlled, which turns out to be difficult enough.

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