Kinetic profiles for shock waves of scalar conservation laws

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Abstract

The subject of this work are one-dimensional kinetic BGK models, regarded as relaxation models for genuinely non-linear scalar conservation laws. Kinetic profiles of shock waves in the form of travelling wave solutions of the kinetic model are studied. In particular, recent results on the existence and dynamic stability of small amplitude travelling waves are reviewed, and a new result on the existence of big travelling waves is proven.

1 Introduction

We consider the following one-dimensional BGK type equation

$$\partial_t f + v\partial_x f = M(\rho_f, v) - f \quad \text{for all } (t, x) \in [0, T] \times \mathbb{R} =: Q_T, \ v \in \Omega \qquad (1.1)$$

here $f(t, x, v)$ can be interpreted (in analogy with the Boltzmann equation) as a probability density of particles that move with velocity $v \in \Omega$ at the time-space position $(t, x) \in Q_T$. We shall assume that (Ω, dv) is a measure space with $\Omega \subset \mathbb{R}$. In particular, discrete velocity models, where (1.1) is a hyperbolic system, are included in our assumptions.

The function $\rho_f(t, x)$ in (1.1) is the macroscopic density corresponding to the distribution f, i.e., the zeroth order velocity moment

$$\rho_f(t, x) = \int f(t, x, v)dv \quad \text{for all } (t, x) \in Q_T \qquad (1.2)$$

where here and in the following we omit Ω under the integral sign in all integrations with respect to v. The 'Maxwellian' $M(\rho, v)$ is an equilibrium distribution satisfying the moment conditions

$$\int M(\rho, v)dv = \rho, \quad \text{and} \quad \int v\, M(\rho, v)dv = a(\rho), \qquad (1.3)$$

for a macroscopic flux function $a(\rho)$ that will be assumed smooth and genuinely non-linear, actually (without loss of generality) concave: $a''(\rho) < 0$. The properties (1.3) ensure, at least formally, that the macroscopic limit equation (scaling with $(t, x) \to (t/\varepsilon, x/\varepsilon)$ and taking $\varepsilon \to 0$) of (1.1) is the scalar conservation law

$$\partial_t \rho + \partial_x a(\rho) = 0. \qquad (1.4)$$

Weak solutions of initial value problems for (1.4) can be made unique by an entropy condition. Thus, it would be desirable to introduce an entropy already on the kinetic

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level. This is possible under an additional structure condition on the Maxwellian: We shall assume that the Maxwellian is a smooth and strictly increasing function of ρ:

$$\partial_\rho M(\rho, v) > 0 \,. \tag{1.5}$$

Then there exists a function $\phi(f, v)$ such that $f = M(\rho, v)$ is equivalent to $\rho = \phi(f, v)$. With the primitive $\Phi(f, v)$ ($\partial_f \Phi = \phi$), solutions of (1.1) formally satisfy the entropy inequality

$$\frac{d}{dt} \int_{\mathbb{R}} \int \Phi(f, v) \, dv \, dx = \int_{\mathbb{R}} \int (M(\rho_f, v) - f)(\phi(f, v) - \rho_f) dv \, dx \leq 0 \,.$$

The condition (1.5) can be seen as a subcharacteristic condition. It can be used for proving stability results such as a TVD property corresponding to entropy solutions of the macroscopic equation (1.4), see [1], [6].

Examples of Maxwellians $M(\rho, v)$ that satisfy the moment conditions (1.3) for a given flux function a (satisfying $a(0) = 0$) as well as (1.5), are given by

$$M(\rho, v) = \int_0^\rho \psi(v - a'(r)) \, dr \,, \qquad v \in \Omega = \mathbb{R} \,, \tag{1.6}$$

where ψ is a positive even function such that $\int \psi \, dv = 1$. For example, for the Burgers' flux $a(\rho) = \rho^2/2$, choosing $\psi(v) = \pi^{-1/2} \exp(-v^2)$, gives $M(\rho, v) = \frac{1}{2}(\mathrm{erf}(v) - \mathrm{erf}(v - \rho))$. A well known kinetic model for scalar conservation laws is the Perthame-Tadmor model, see [10] reading

$$\partial_t f + a'(v) \partial_x f = \chi(\rho_f, v) - f, \tag{1.7}$$

with

$$\chi(\rho, v) = \begin{cases} 1 & \text{if } 0 < v < \rho, \\ -1 & \text{if } \rho < v < 0, \\ 0 & \text{otherwise.} \end{cases}$$

In this case the Maxwellian is a discontinuous function. This lack of smoothness is an obstacle for the study of small waves by perturbation arguments as carried out here for the BGK model (1.1). Existence of big travelling waves has been studied by Golse [4].

It is well-known that equation (1.4) exhibits shock wave solutions, i.e. solutions of the form

$$\rho(t, x) = \begin{cases} \rho_- & \text{if } x - st < x_0, \\ \rho_+ & \text{if } x - st > x_0, \end{cases}$$

where the constants $\rho_\pm$ and the wave speed s are related by the Rankine-Hugoniot condition

$$s = \frac{a(\rho_+) - a(\rho_-)}{\rho_+ - \rho_-} \,. \tag{1.8}$$

The admissibility condition

$$\frac{a(\rho) - a(\rho_-)}{\rho - \rho_-} - s > 0 \quad \text{for all } \rho \in (\min(\rho_+, \rho_-), \max(\rho_+, \rho_-)), \tag{1.9}$$

can be derived by a vanishing diffusion argument (cf. [11]), for example by constructing viscous profiles, i.e. travelling wave solutions of the regularized parabolic conservation law

$$\partial_t \rho + \partial_x a(\rho) = \mu \partial_x^2 \rho. \tag{1.10}$$

In this framework (1.9) gives a necessary and sufficient condition for existence of travelling wave solutions connecting the values $\rho_\pm$ at $(x - st)/\mu = \pm\infty$. For the concave flux functions $a(\rho)$ considered here, (1.9) reduces to the condition $\rho_- < \rho_+$.

In this work, instead of (1.10), the kinetic regularization (1.1) is studied, in particular the questions, whether (1.1) permits travelling waves, and whether these are selected in the macroscopic limit. Stability of these solutions is therefore an issue.

The travelling wave problem follows by changing to the travelling wave variable $\xi = x - st$, with s being the wave speed:

$$(v - s)\partial_\xi f = M(\rho_f, v) - f, \qquad \xi \in \mathbb{R}, \ v \in \Omega, \tag{1.11}$$

subject to the far-field conditions

$$f(\pm\infty, v) = M(\rho^\pm, v), \qquad v \in \Omega. \tag{1.12}$$

The Rankine-Hugoniot condition (1.8) is derived as a necessary condition for existence by integrating equation (1.11) with respect to v and by (1.3).

In Section 2, we review our recent results [3] on the existence and stability of small amplitude travelling waves assuming

$$\rho^+ - \rho^- = \varepsilon \qquad \text{with } 0 < \varepsilon << 1. \tag{1.13}$$

It turns out that it is appropriate to rescale the travelling wave variable by $\xi \to \xi/\varepsilon$, to get

$$\varepsilon(v - s)\partial_\xi f = M(\rho_f, v) - f \quad \xi \in \mathbb{R}, \ v \in \Omega. \tag{1.14}$$

To prove existence, f is constructed as a small perturbation of the Maxwellian $M(\rho_-, v)$. Partially, our existence proof adapts ideas from Caflisch and Nicolaenko [2], where existence of weak shocks for the Boltzmann equation of gas dynamics has been proven. We shall only outline the proof here. Details can be found in [3].

In Section 2 we also sketch the proof of stability of small amplitude travelling waves. The idea is to decouple the equation into a macroscopic part and a small microscopic part. Then we use L^2-type energy (actually entropy) methods for the macroscopic equation, which can be extended to also control the microscopic part. Similar techniques have been used by Liu and Yu [8] for the Boltzmann equation. For the Broadwell model, a discrete velocity model for the Boltzmann equation, energy estimates have also been used in [7].

In Section 3 we prove existence of what we call *big* travelling waves, i.e. the smallness assumption on the amplitude of the wave is removed. Here we shall only

consider the continuous velocity case $v \in \mathbb{R}$. The existence proof is non-constructive and, consequently, the result is somewhat weaker than for small waves, since the far-field conditions (1.12) are only satisfied in a weak sense. The main ideas of the proof are similar to Golse's work [4] on the Perthame-Tadmor model (1.7).

2 Small amplitude travelling waves: existence and stability

In this section we shall be concerned with the existence of solutions of (1.14) subject to (1.12). We start with a formal derivation of an approximation for travelling waves. We set

$$f = M(\rho, v) + \varepsilon^2 f^{\perp}, \qquad \int f^{\perp} dv = 0,$$

with

$$\rho = \rho_- + \varepsilon u, \qquad s = a'(\rho_-) + \varepsilon \sigma. \tag{2.1}$$

Then integration of (1.14) with respect to v and ξ gives

$$a(\rho) - a(\rho_-) + \varepsilon^2 \int v f^{\perp} dv = s(\rho - \rho_-).$$

Comparing $O(\varepsilon^2)$-terms in this equation, we derive

$$\frac{a''(\rho_-)}{2} u^2 - \sigma u = -\int (v - a'(\rho_-)) f^{\perp} dv.$$

On the other hand, from the $O(\varepsilon^2)$-terms in (1.14) we get

$$f^{\perp} = -(v - a'(\rho_-)) \partial_{\rho} M(\rho_-, v) \partial_{\xi} u,$$

leading to

$$\frac{a''(\rho_-)}{2} u^2 - \sigma u = A \, \partial_{\xi} u \tag{2.2}$$

with the *diffusion* coefficient

$$A := \int (v - a'(\rho_-))^2 \partial_{\rho} M(\rho_-, v) \, dv > 0, \tag{2.3}$$

which is positive by (1.5). Observe that σ must be negative, since $a''(\rho_-) < 0$, this way u connects 0 at $-\infty$ to a larger value at $+\infty$. Then, formally, the perturbation u, that carries the profile of the small wave, is a travelling wave solution of the viscous Burgers' equation corresponding to (2.2). In the section we sketch how to make this argument rigorous, we also explain stability in the subsequent section.

We shall work in the weighted Hilbert space H_v, in the v -direction, defined by the scalar product

$$\langle f, g \rangle_v = \int \frac{fg}{F} dv \quad \text{and norm } \|f\|_v = \langle f, f \rangle_v^{\frac{1}{2}}.$$

The inverse of the weight being $F := \partial_\rho M(\rho_-, v)$. We also consider the L^2-norm and H^1-norm in the x direction, and we denote their norms by $\| \cdot \|_x$ and $\| \cdot \|_{H^1}$, respectively. Finally we define the Hilbert space $\mathcal{H}_v$ by the scalar product

$$\langle f, g \rangle_{x,v} = \iint \frac{fg}{F} dv \, dx,$$

and denote the corresponding norm by $\| \cdot \|_{x,v}$.

Finally, we shall denote by $\mathcal{H}_v^k$ the space of functions which derivatives in the x-direction up to order k are in $\mathcal{H}_v$, and the corresponding norm by

$$\|f\|_{\mathcal{H}_v^k} = \left(\|f\|_{x,v}^2 + \ldots + \|\partial_x^k f\|_{x,v}^2 \right)^{\frac{1}{2}}.$$

Existence and stability of small waves relay on the following assumptions on M. For ρ in $\mathbb{R}$, we assume that M is a smooth, at least C^2, function of ρ. Moreover, for a given ρ, the moments of $\partial_\rho M(\rho, v)$ are bounded up to the third moment, i.e.

$$\int (v - s)^k \partial_\rho M(\rho, v) \, dv < \infty \quad \text{with } k = 0, 1, 2, 3. \tag{2.4}$$

And finally, for given ρ_1 and ρ_2 the following holds

$$\int \frac{(\partial_\rho^2 M(\rho_1, v))^2}{\partial_\rho M(\rho_2, v)} \, dv < \infty. \tag{2.5}$$

2.1 Existence

As for the formal argument we want to expand f in ε-powers. Instead of taking the leading order to be $M(\rho_f)$, we take f_{as}, which is constructed as follows

$$f_{as} := M(\rho, v) + \varepsilon^2 f^\perp. \tag{2.6}$$

The argument ρ of M satisfies an ODE that can be regarded as an approximation of (2.2). In particular ρ satisfies the conditions

$$\rho(-\infty) = \rho_- \quad \text{and} \quad \rho(+\infty) = \rho_+. \tag{2.7}$$

The function $f^\perp$ is chosen in such a way that

$$\rho^\perp = O(\varepsilon) \quad \text{and that} \quad \varepsilon(v - s)\partial_\xi f_{as} = M(\rho_{f_{as}}) - f_{as} + O(\varepsilon^3).$$

The conditions (2.7) satisfied by ρ, also imply that f_{as} approaches $M(\rho_\pm, v)$ as $\xi \to \pm\infty$. We have the following existence result.

Theorem 2.1 *Under the assumptions (2.4) and (2.5), and for ε small enough, there exists a solution f of (1.14) unique in a ball in $\mathcal{H}_v^1$ with center f_{as}.*

Next we review the main ideas of the proof of Theorem 2.1. We introduce g such that $\varepsilon^{3/2} g = f - f_{as}$. The power $3/2$ turns out to be the right scaling in order to get small non-linear and residual terms, since f_{as} solves (1.14) up to order ε^3. We then seek a g that solves

$$\varepsilon(v - s)\partial_\xi g + g - F\rho_g = \frac{1}{\varepsilon^{\frac{3}{2}}} \{ M(\rho_{as} + \varepsilon^{\frac{3}{2}}\rho_g) - M(\rho_{as}) - \varepsilon^{\frac{3}{2}} F \rho_g \} + \varepsilon^{\frac{3}{2}} h, \tag{2.8}$$

and satisfies

$$g(\pm\infty, v) \equiv 0, \quad \text{for all } v \in \Omega. \tag{2.9}$$

Here

$$h := \frac{1}{\varepsilon^3}\{M(\rho_{as}) - f_{as} - \varepsilon(v - s)\partial_\xi f_{as}\} = O(1). \tag{2.10}$$

We also observe that the condition (2.9) implies the orthogonality condition

$$\int (v - s)g\, dv = 0. \tag{2.11}$$

The next step to get existence of equation (2.8) is to decompose g into a macroscopic part and a microscopic part, this reads

$$g(\xi, v) = z(\xi)\Phi(v) + \varepsilon w(\xi, v), \tag{2.12}$$

where $\Phi := F\left(1 + \varepsilon\frac{\sigma}{A}(v - s)\right)$, and z is defined by the condition $\int (v-s)\Phi w/F\, dv = 0$. A similar projection was introduced in [2]. This way, we linearise g around the limiting state $M(\rho_-, v)$ with a small correction. The choice of the coefficient σ/A, in front of the correction term, guarantees that $\int (v - s)\Phi\, dv = 0$. This, (2.9) and (2.11) imply that

$$w(\pm\infty, v) \equiv 0, \; z(\pm\infty) = 0, \tag{2.13}$$

and consequently also that

$$\int (v - s)w\, dv = 0, \quad \int (v - s)^2 w\, dv = 0. \tag{2.14}$$

Next we write down an equation for z (macroscopic equation) and an equation for w (microscopic equation). First, substituting (2.12) into (2.8), dividing by ε, and using $\rho_g = z\rho_\Phi + \varepsilon\rho_w$. The resulting equation is multiplied by $(v - s)$ and integrated with respect to v, this giving the macroscopic equation

$$\tilde{A}\partial_\xi z = \alpha z + \varepsilon^{\frac{1}{2}}\beta + \varepsilon\gamma \tag{2.15}$$

which is, in a sense, a linearised version of (2.2); here $\alpha = \rho_\Phi(a'(\rho_{as}) - s)/\varepsilon = O(1)$, and $\tilde{A} := \int (v - s)^2\Phi\, dv = A + O(\varepsilon) \neq 0$. We do not write the terms β and γ explicitly, instead we observe that $\alpha = O(1)$, $\beta = O(\rho_g^2 + \varepsilon)$ and $\gamma = O(\rho_w)$, or more precisely

$$\|\beta\|_x \leq C\|\rho_g\|_\infty\|\rho_g\|_x + O(\varepsilon), \; \|\gamma\|_x \leq C'\|\rho_w\|_x, \tag{2.16}$$

for some constants C and C'. Equation (2.15) gives an expression for $\partial_\xi z$, which substituted into the expanded version of (2.8) gives the microscopic equation

$$\varepsilon(v - s)\partial_\xi w = F\rho_w - w + \tilde{\alpha}z + \varepsilon^{\frac{1}{2}}\tilde{\beta} + \varepsilon\tilde{\gamma}. \tag{2.17}$$

here the coefficients formally satisfy $\tilde{\alpha} = O(1)$, $\tilde{\beta} = O(\rho_g^2 + h)$ and $\tilde{\gamma} = O(\rho_w)$, and with the aid of assumptions (2.4) and (2.5), they also satisfy

$$\|\tilde{\alpha}\|_{x,v} < \infty, \; \|\tilde{\beta}\|_{x,v} \leq C\|\rho_g\|_\infty\|\rho_g\|_x + \|h\|_{x,v}, \; \|\tilde{\gamma}\|_{x,v} \leq C'\|\rho_w\|_x, \tag{2.18}$$

for some constants C, C'.

We make one more manipulation to get a final equation for w. We introduce the following linear operator $\mathcal{M}$,

$$\mathcal{M}w := F\rho_w - w - (v-s)^2 F \int (v-s)^2 w \, dv.$$

The operator $\mathcal{M}$ is symmetric and strictly negative definite in $\mathcal{H}_v$, in contrast to the linear operator $\mathcal{L}w = F\rho_w - w$ which is negative semidefinite.

One can now prove that if w is a solution of

$$\varepsilon(v-s)\partial_\xi w = \mathcal{M}w + \tilde{\alpha}z + \varepsilon^{\frac{1}{2}}\tilde{\beta} + \varepsilon\tilde{\gamma}, \qquad (2.19)$$

subject to (2.14) at $\xi = \pm\infty$ then w is a solution (2.17) and satisfies (2.14) for all $\xi \in \mathbb{R}$, and vice versa. A similar result was proved in [2]. Therefore we shall solve (2.15) coupled with (2.19) subject to (2.13), with the advantage that the linear operator $\mathcal{M}$ is coercive.

The second step consists of solving the associated linear equations

$$\varepsilon(v-s)\partial_\xi w = \mathcal{M}w + f_w \quad \text{with } f_w \in \mathcal{H}_v. \qquad (2.20)$$

subject to (2.14) at $\xi = \pm\infty$, and the linear ODE

$$\tilde{A}\partial_\xi z = \alpha z + f_z, \quad \text{with } f_z \in L^2(\mathbb{R}), \qquad (2.21)$$

subject to the initial condition $z(0) = 0$.

We have the following estimates. If z solves (2.21), then there exist positive constants, say C, such that

$$\|z\|_x \leq C\|f_z\|_x \quad \text{and} \quad \|\partial_\xi z\|_x \leq C\|f_z\|_x.$$

For the linear equation (2.20), by the coercivity of $\mathcal{M}$, we get

$$\|w\|_{x,v} \leq \frac{1}{\kappa}\|f_w\|_{x,v}.$$

Existence of the linear w-equation is achieved by discretisation of the velocity space, and using the coercivity of the operator $\mathcal{M}$, so that one can diagonalise the discrete $\mathcal{M}$ and get a system of decoupled ODEs to be solved. In the case of continous velocity space, the coercivity estimate is then applied to get uniform estimates that let us take the limit to the continuous velocity space.

The existence proof of the non-linear problem uses a fixed-point map argument and the smallness of ε. We define the fixed-point map, $\mathcal{F}$, for a given pair $(\bar{z}, \bar{w})$ corresponding to the projections of a $\bar{g}$. We then solve for (z, w) the equations (2.15) and (2.17), with the coefficents α, β, γ, $\tilde{\alpha}$, $\tilde{\beta}$ and $\tilde{\gamma}$ evaluated using $\bar{g}$, and by imposing the initial data $z(0) = 0$. The solution pair (z, w) of this linear sytem gives $g = \mathcal{F}(\bar{g})$. To prove that the operator has a fixed point, one uses the estimates on the linear equations, (2.16) and (2.18). To estimate the L^2-norms of the quadratic terms on $\rho_{\bar{g}}$, we need to control $\|\rho_{\bar{g}}\|_\infty$. This can be done by observing that, in one dimension, the H^1-norm controls the L^∞-norm. Thus, it is also necessary to get estimates on the first derivatives of z and w.

2.2 Stability

To study stability we shall first write equation (1.1) in the travelling wave variable $\xi = (x - st)\varepsilon$, and introduce the *parabolic* scaling $t \to t/\varepsilon^2$, where ε is the amplitude of the wave. Then equation (1.1) reads

$$\varepsilon^2 \partial_t f + \varepsilon(v - s)\partial_\xi f = M(\rho_f) - f. \tag{2.22}$$

Let us denote by ϕ the travelling wave solution constructed in Section 2.1, and introduce the unknown G, given by

$$\varepsilon G = f - \phi, \quad \text{with} \quad G = \rho F + \varepsilon g, \tag{2.23}$$

here we let ρ denote ρ_G, for simplicity. We further choose ϕ such that

$$\int_\mathbb{R} \rho \, d\xi = \int_\mathbb{R} (\rho_f - \rho_\phi) \, d\xi = 0. \tag{2.24}$$

This condition fixes the shift of the travelling wave solutions, so one expects f to approach this ϕ as $t \to \infty$. We shall further assume, that the macroscopic travelling wave profiles are monotone:

$$\partial_\xi(a'(\rho_\phi)) \leq 0. \tag{2.25}$$

This condition is satisfied for the approximate wave f_{as} in Section 2.1. However, a proof for the full kinetic profile ϕ is still missing.

The deviation G satisfies the equation

$$\varepsilon \partial_t G + (v - s)\partial_\xi G = R_2 - g, \tag{2.26}$$

with

$$R_2 := \frac{1}{\varepsilon^2} \left[M(\rho_\phi + \varepsilon\rho) - M(\rho_\phi) - \varepsilon\rho F \right]. \tag{2.27}$$

Next we derive integral estimates as one would do for the purely macroscopic case. Before that, we shall split equation (2.26) into its microscopic and macroscopic part, i.e. we apply two projections, P_0 and P_1, say, where $P_0 G = F\rho$ and $P_1 G = \varepsilon g$. Application of the projection P_0 to equation (2.26) and division by F gives

$$\partial_t \rho + \frac{1}{\varepsilon}(a'(\rho_-) - s)\partial_\xi \rho + \partial_\xi \int (v - s)g \, dv = 0, \tag{2.28}$$

and application of P_1 gives

$$\varepsilon^2 \partial_t g + (v - a'(\rho_-))F\partial_\xi \rho + \varepsilon\partial_\xi P_1((v - s)g) = R_2 - g. \tag{2.29}$$

Equation (2.29) gives an expression for g. Substituting this expression into (2.28) and writing

$$r_2 := \int (v - s)R_2 \, dv = \frac{1}{\varepsilon}\{a(\rho_\phi + \varepsilon\rho) - a(\rho) - \varepsilon\rho a'(\rho_\phi)\}, \tag{2.30}$$

we arrive at the macroscopic equation

$$\partial_t \rho - \sigma\partial_\xi \rho + \partial_\xi r_2 - A\partial_\xi^2 \rho - \partial_\xi \int (v - s) \left(\varepsilon^2 \partial_t g + \varepsilon P_1((v - s)\partial_\xi g) \right) dv = 0. \tag{2.31}$$

In a direct energy approach, the term $\partial_\xi r_2$ leads to a contribution with the bad sign, which cannot be controlled. This problem is circumvented by a standard trick (see e.g. [9], [8]), introducing the new macroscopic unknown

$$W(\xi, t) = \int_{-\infty}^{\xi} \rho(x, t)\, dx. \tag{2.32}$$

The integrated macroscopic equation then reads, in terms of W,

$$\partial_t W - \sigma \partial_\xi W + r_2 - A \partial_\xi^2 W - \int (v - s)(\varepsilon^2 \partial_t g + \varepsilon\, P_1((v - s)\partial_\xi g))\, dv = 0. \tag{2.33}$$

The advantage of introducing W becomes clear, if we assume for the moment that (2.33) is purely macroscopic (the last term of the right-hand side is zero). Testing equation (2.33) with W gives the integral identity

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} \|W\|_x^2\, d\xi + A \int_{\mathbb{R}} (\partial_\xi W)^2\, d\xi + \int_{\mathbb{R}} r_2 W\, d\xi = 0.$$

We estimate the term containing r_2 by first writing

$$r_2 = \frac{1}{\varepsilon}(a'(\rho_\phi) - a'(\rho_-))\rho + \frac{1}{\varepsilon^2}(a(\rho_\phi + \varepsilon\rho) - a(\rho_\phi) - \varepsilon a'(\rho_\phi)\rho),$$

then

$$\int_{\mathbb{R}} r_2 W d\xi \leq -\frac{1}{2\varepsilon}\int_{\mathbb{R}} \partial_\xi(a'(\rho_\phi))W^2 d\xi - C_0 \|\rho\|_x^2 \|W\|_\infty, \tag{2.34}$$

after integration by parts the first term of $\int r_2 W$. The first term in (2.34) is positive by (2.25), and can be combined with similar terms which have the wrong sign.

Similarly, we get integral identities from equations (2.33), (2.31) and its derivatives with respect to ξ. In terms of W, we have the following estimates

$$\frac{1}{2}\frac{d}{dt}\|W\|_x^2 - \frac{1}{2\varepsilon}\int_{\mathbb{R}} \partial_\xi(a'(\rho_\phi))W^2 d\xi + (A - C_0\|W\|_\infty)\|\partial_\xi W\|_x^2 + R^0 \leq 0, \tag{2.35}$$

$$\frac{1}{2}\frac{d}{dt}\|\partial_\xi^k W\|_x^2 + (A - \frac{\delta}{2})\|\partial_\xi^{k+1} W\|_x^2 - \frac{C_k}{2\delta}\|\partial_\xi^k W\|_x^2 + R^k \leq 0 \text{ for } k = 1, 2, \tag{2.36}$$

where

$$R^k := -\int \partial_\xi^k W\, \partial_\xi^k \left\{ \int (v - s)(\varepsilon P_1((v - s)\partial_\xi g + \varepsilon^2 \partial_t g)dv \right\}\, d\xi = O(\varepsilon), \tag{2.37}$$

for $k = 0, 1, 2$, $\delta \in (0, 2A)$ and the constants C_k for $k = 1, 2$ depend on $\|\rho\|_\infty$.

Up to now, we have obtained estimates on the macroscopic part only. The idea consists of combining the estimates for $k = 0, 1, 2$ to get an estimate on the H^2 norm of W. If we ignore the microscopic parts of (2.35) and (2.36), i.e. the R^k terms, and set $\delta = A$, we get the following estimate on the H^2-norm of W (H^1-norm of ρ)

$$\frac{1}{2}\frac{d}{dt}\left(\|W\|_x^2 + \gamma_1\|\partial_\xi W\|_x^2 + \gamma_2\|\partial_\xi^2 W\|_x^2\right)$$

$$+ \left(A - C\|W\|_\infty - \frac{\gamma_1 C_1}{2A}\right)\|\partial_\xi W\|_x^2 + \left(\frac{A\gamma_1}{2} - \frac{\gamma_2 C_2}{2A}\right)\|\partial_\xi^2 W\|_x^2 \leq 0, \tag{2.38}$$

The L^∞-norm of W is controlled by the H^1-norm. Then the magnitude of the constants C_k are controlled locally in t by the estimate (2.38) itself. Hence, starting with initial data such that the initial W_0 is small enough, and choosing γ_1 and γ_2 small enough to ensure that $(A - C\|W\|_\infty - \gamma_1 C_1/2A) \geq 0$ and $(A\gamma_1/2 - \gamma_2 C_2/2A) \geq 0$, implies that the functional $\|W\|_x^2 + \gamma_1 \|\partial_\xi W\|_x^2 + \gamma_2 \|\partial_\xi \xi W\|_x^2$ decreases with t for all $t > 0$, i.e the estimate implies global existence in t, as well as asymptotic stability of macroscopic travelling waves in H^1 ($\rho = \partial_\xi W$).

To achieve an analogous result for the full equation, the argument is similar, but one has to take care of the contribution of the microscopic part. The basic estimate for the microscopic part being

$$\frac{d}{dt}\{\|\partial_\xi^k \rho\|_x^2 + \varepsilon^2 \|\partial_\xi^k g\|_{x,v}^2\} + \{\|\partial_\xi^k g\|_{x,v}^2 - \|\partial_\xi^k R_2\|_{x,v}^2\} \leq 0, \tag{2.39}$$

for $k = 0, 1, 2$ (obtained from equation (2.29)). The quadratic terms (involving R_2) are estimated by assuming $\|\rho\|_\infty < \infty$. The next step is to combine the macroscopic integral estimates (2.35) and (2.36) with the microscopic estimate (2.39), and play a similar game combining the estimates for different order derivatives. The final result is the following

Theorem 2.2 *Let f be a solution of (2.22) subject to initial data f_0 satisfying (1.12), and let ϕ be a travelling wave solution such that (2.24) holds. Then, under the assumtptions (2.4), (2.5) and (2.25), and if the initial data is chosen such that $W_0 \in \mathcal{H}_v^3$ is small enough, the functional H given by*

$$H = H_0 + \gamma_1 H_1 + \gamma_2 H_2,$$

with, for $k = 0, 1, 2$,

$$H_k = \frac{1}{2}\|\partial_\xi^k W\|_x^2 - \varepsilon^2 \int \partial_x^k W \int (v - s)\partial_\xi^k g \, dv \, d\xi + 2\varepsilon^3 C\|\partial_\xi^k g\|_{x,v}^2 + 2\varepsilon C\|\partial_\xi^k \rho\|_x^2,$$

is decreasing with respect to t for small enough values of ε, γ_1 and γ_2. In particular, small amplitude travelling waves satisfying (2.24) are locally asymptotically stable in $\mathcal{H}_v^2$, in the sense that

$$\varepsilon\|G\|_{\mathcal{H}_v^1} = \|f - \phi\|_{\mathcal{H}_v^1} \to 0 \quad \text{as } t \to \infty.$$

3   Existence of big travelling waves

In this section we prove existence of big travelling waves, i.e. solutions of (1.11) subject to (1.12). Throughout this section $v \in \mathbb{R}$. The main result is the following,

Theorem 3.1 *Let $M(\rho, v)$ be continuous with respect to $v \in \mathbb{R}$ and continuously differentiable with respect to $\rho \in \mathbb{R}$ with $\partial_\rho M > 0$. Let*

$$\int |v^2 M(\rho, v)| dv < \infty$$

for every $\rho \in \mathbb{R}$. Then there exists a solution $f(\xi, v)$ of the travelling wave equation (1.11) with continuous macroscopic density $\rho_f(\xi)$ satisfying the far-field conditions

(1.12) in the following sense: There exist sequences $\xi_n \to \infty$ and $\eta_n \to -\infty$ such that

$$f(\xi_n, v) \to M(\rho_+, v), \qquad f(\eta_n, v) \to M(\rho_-, v), \quad v\text{-}a.e.$$

Proof. We introduce the notation $\mu = v - s$, $g(\xi, \mu) = f(\xi, \mu + s) - M(\rho_-, \mu + s)$, and

$$\hat{M}(\rho, \mu) = M(\rho + \rho_-, \mu + s) - M(\rho_-, \mu + s).$$

Notice that $\hat{M}(\rho, \mu)$ inherits the monotonicity with respect to ρ from M and also satisfies

$$\int \hat{M}(\rho, \mu) d\mu = \rho, \qquad \int \mu \hat{M}(\rho, \mu) d\mu = a(\rho + \rho_-) - a(\rho_-) - s\rho.$$

In terms of the new variables, the problem (1.11), (1.12) reads

$$\mu \partial_\xi g = \hat{M}(\rho_g, \mu) - g \qquad \text{for } \xi, \mu \in \mathbb{R}, \tag{3.1}$$

subject to

$$g(-\infty, \mu) = 0, \quad g(+\infty, \mu) = \hat{M}(\rho_+ - \rho_-, \mu) \qquad \text{for } \mu \in \mathbb{R}. \tag{3.2}$$

The first step is to solve an approximate problem (the *slab problem*) on a finite ξ-interval:

$$\mu \partial_\xi g^L = \hat{M}(\rho^L, \mu) - g^L \qquad \text{for } -L < \xi < L, \ \mu \in \mathbb{R}, \tag{3.3}$$

subject to inflow boundary conditions

$$g^L(-L, \mu) = 0 \qquad \text{for } \mu > 0, \tag{3.4}$$

$$g^L(L, \mu) = \hat{M}(\rho_+ - \rho_-, \mu) \qquad \text{for } \mu < 0, \tag{3.5}$$

where $\rho^L = \rho_{g^L}$. Once this problem is solved, we shall pass to the limit $L \to \infty$.

Proposition 3.2 *With the assumptions of theorem 3.1 the slab problem (3.3)–(3.5) has a solution with continuous macroscopic density $\rho^L(\xi)$, satisfying*

$$0 \leq g^L(\xi, \mu) \leq \hat{M}(\rho_+ - \rho_-, \mu) \qquad \forall \xi, \mu \in \mathbb{R}, \tag{3.6}$$

and, thus, $0 \leq \rho^L(\xi) \leq \rho_+ - \rho_-$.

Proof. The proof is based on the fixed point map

$$\mathcal{T} : \theta \mapsto \int g_\theta \, d\mu,$$

where g_θ solves

$$\mu \partial_\xi g_\theta = \hat{M}(\theta, \mu) - g_\theta,$$

subject to the boundary conditions (3.4), (3.5), i.e.,

$$g_\theta(\xi, \mu) = \begin{cases} \frac{1}{\mu} \int_{-L}^{\xi} \hat{M}(\theta(s), \mu) e^{(s-\xi)/\mu} ds & \text{for } \mu > 0, \\ \hat{M}(\rho_+ - \rho_-, \mu) e^{(L-\xi)/\mu} + \frac{1}{-\mu} \int_{\xi}^{L} \hat{M}(\theta(s), \mu) e^{(s-\xi)/\mu} ds & \text{for } \mu < 0. \end{cases}$$

A straightforward estimate, using the monotonicity of $\hat{M}$, shows that $\mathcal{T}$ maps the set $\{\rho \in C([-L, L]) : 0 \le \rho \le \rho_+ - \rho_-\}$ into itself.

Compactness of $\mathcal{T}$ is the consequence of a velocity averaging lemma (cf. [5]). $\square$

Due to the translation invariance of the problem, before going to the limit $L \to \infty$, we have to fix the shift of the profile to make sure that in the limit the conditions (3.4) and (3.5) are satisfied. By using (3.6) and the boundary conditions it can be seen that

$$\rho^L(-L) \le \int_{-\infty}^0 \hat{M}(\rho_+ - \rho_-, \mu)d\mu \le \rho^L(L).$$

and continuity of ρ^L implies that there exists a $\xi^L \in (-L, L)$ such that $\rho^L(\xi^L) = \int_{-\infty}^0 \hat{M}(\rho_+ - \rho_-, \mu)d\mu$. The strict montonicity of $\hat{M}$ with respect to the first argument implies

$$0 < \rho^L(\xi^L) = \int_{-\infty}^0 \hat{M}(\rho_+ - \rho_-, \mu)d\mu < \rho_+ - \rho_-.$$

The next step is to extend the solution of the slab problem to the whole ξ-domain $\mathbb{R}$. We let g_1^L denote the extension of g^L, defined as follows:

$$g_1^L(\xi, \mu) = \begin{cases} g^L(-L, \mu) & \text{if } \xi < -L, \\ g^L(\xi, \mu) & \text{if } -L < \xi < L, \\ g^L(L, \mu) & \text{if } \xi > L. \end{cases}$$

In order to take the limit $L \to \infty$, let $L_n \to \infty$ as $n \to \infty$, let also $\xi_n := \xi^{L_n}$ and $g_n(\xi, \mu) = g_1^{L_n}(\xi + \xi_n, \mu)$, so that $\rho_{g_n}(0) = \int_{-\infty}^0 \hat{M}(\rho_+ - \rho_-, \mu)d\mu$. From the bound (3.6) we have that $g_n \to g$ in $L^\infty(\mathbb{R} \times \mathbb{R})$ weak* (restricting to a subsequence). Also, by velocity averaging, $\rho_{g_n} \to \rho_g$ uniformly on compact intervals with

$$\rho_g(0) = \int_{-\infty}^0 \hat{M}(\rho_+ - \rho_-, \mu)d\mu. \tag{3.7}$$

To prove that g satisfies equation (3.1), it is necessary to prove that the shifted interval $[-L_n + \xi_n, L_n + \xi_n]$ tends to $\mathbb{R}$. If we assume $L_n + \xi_n \to l < \infty$ as $n \to \infty$, then by passing to the limit in the differential equation in the distributional sense, g satisfies (3.1) for $\xi > -l$. Since, by velocity averaging, we can pass to the limit in terms of the form $\int g_n(-l, \mu)\Phi(\mu)d\mu$ for arbitrary test functions Φ, $g(-l, \mu) = 0$ holds for $\mu > 0$.

Integration of (3.1) with respect to μ shows that $A = \int \mu g \, d\mu$ is a constant. Evaluation at $\xi = -l$ gives

$$A = \int_{-\infty}^0 \mu g(-l, \mu)d\mu \le 0.$$

On the other hand, multiplication of (3.1) by μ and integration leads to

$$\partial_\xi \int \mu^2 g \, d\mu = a(\rho_- + \rho_g) - a(\rho_-) - s\rho_g - A \ge -A, \tag{3.8}$$

where the inequality is a consequence of (1.9). By integration with respect to ξ from $-l$ to X,

$$A(X + l) \geq \int \mu^2 g(-l, \mu) \, d\mu - \int \mu^2 g(X, \mu) \, d\mu \, ,$$

where the right hand side is bounded uniformly in X. Passage to the limit $X \to \infty$ gives $A \geq 0$, and, thus, $A = 0$. As a consequence, $g(-l, \mu) = 0$ for all μ.

Since g is a bounded solution of (3.1), it satisfies

$$g(\xi, \mu) = \frac{1}{\mu} \int_\infty^\xi e^{(s-\xi)/\mu} \hat{M}(\rho_g(s), \mu) ds \qquad \text{for } \mu < 0 \, .$$

Evaluation at $\xi = -l$ gives

$$0 = \int_{-l}^\infty e^{(s+l)/\mu} \hat{M}(\rho_g(s), \mu) ds \qquad \text{for } \mu < 0 \, ,$$

with the consequence $\rho_g(\xi) = 0$ for $\xi \geq -l$. This, however, contradicts (3.7), hence $L_n + x_n \to \infty$ follows. Similarly one proves that $L_n - x_n \to \infty$, and therefore g satisfies equation (3.1) for all $x \in \mathbb{R}$.

The last step of the proof consists of checking that $\rho_g(-\infty) = 0$ and $\rho_g(+\infty) = \rho_+ - \rho_-$ in a suitable sense. For the argument we need an entropy inequality. Since $\hat{M}$ is a strictly increasing function of ρ, there exists a function $\varphi(g, \mu)$, such that $g = \hat{M}(\rho, \mu)$ is equivalent to $\rho = \varphi(g, \mu)$. Let $\Phi(g, \mu)$ satisfy $\partial_g \Phi(g, \mu) = \varphi(g, \mu)$, then

$$\partial_\xi \int \mu \Phi(g) \, d\mu = \int (\hat{M}(\rho_g) - g)(\varphi(g) - \rho_g) \, d\mu \leq 0 \, . \tag{3.9}$$

Hence, by the boundedness of g and upon integrating with respect to ξ over $\mathbb{R}$,

$$\int_{-\infty}^\infty \int (\hat{M}(\rho_g) - g)(\rho_g - \varphi(g)) \, d\mu \, d\xi < \infty \, .$$

Then there exist sequences $\xi_n \to \infty$ and $\eta_n \to -\infty$ such that

$$\int (\hat{M}(\rho_g) - g)(\rho_g - \varphi(g)) \, d\mu \to 0 \, .$$

As a consequence there exist values ρ_∞ and $\rho_{-\infty}$ such that

$$g(\xi_n, \mu) \to \hat{M}(\rho_\infty, \mu), \qquad g(\eta_n, \mu) \to \hat{M}(\rho_{-\infty}, \mu), \quad \mu\text{-a.e.}$$

modulo subsequences. Finally, using (3.8),

$$\partial_x \int \mu^2 g \, d\mu = a(\rho_- + \rho_g) - a(\rho_-) - s\rho_g \geq 0, \tag{3.10}$$

and, thus,

$$\int \mu^2 \hat{M}(\rho_{-\infty}\mu) \, d\mu \leq \int \mu^2 \hat{M}(\rho_g(0)\mu) \, d\mu \leq \int \mu^2 \hat{M}(\rho_\infty, \mu) \, d\mu \, .$$

Also by (3.10) and an argument as above we get that $\rho_{\pm\infty}$ have to satisfy $a(\rho_- + \rho_{\pm\infty}) - a(\rho_-) - s\rho_{\pm\infty} = 0$ and therefore $\rho_{-\infty} = 0$ and $\rho_\infty = \rho_+ - \rho_-$. $\qquad \square$

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