Long-time behaviour of a one-dimensional BGK model: convergence to macroscopic rarefaction waves

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Abstract

For one-dimensional kinetic BGK models, regarded as relaxation models for scalar conservation laws with genuinely nonlinear fluxes, we prove that the macroscopic density converges to the rarefaction wave solution of the corresponding scalar conservation law in the long time limit, and that the phase space density approaches an equilibrium distribution with the rarefaction wave as macroscopic density. The proof requires a smallness assumption on the amplitude of the rarefaction waves and uses a micro-macro decomposition of the perturbation equation.

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1 Introduction

We study the convergence to macroscopic rarefaction waves as $t \to \infty$ of solutions of the BGK-type equation

$$\partial_t f + v\,\partial_x f = M(\rho_f, v) - f\,, \quad \text{with } t > 0\,,\ x \in \mathbb{R}\,,\ v \in \Omega \subset \mathbb{R}\,. \tag{1.1}$$

Here $f(t, x, v)$ can be interpreted (in analogy to the Boltzmann equation) as a time dependent phase space density of particles with time t, position x, and velocity v.

The function $\rho_f(t, x)$ in (1.1) is the macroscopic density corresponding to the distribution f, i.e., the zeroth order velocity moment

$$\rho_f(t, x) = \int f(t, x, v)\, dv\,. \tag{1.2}$$

We assume (Ω, dv) to be a measure space and omit here in the following to write Ω under the integral sign in integrals with respect to v. The 'Maxwellian' $M(\rho, v)$ is an equilibrium distribution satisfying the moment conditions

$$\int M(\rho, v)\, dv = \rho\,, \qquad \int v\, M(\rho, v)\, dv = a(\rho)\,, \tag{1.3}$$

for a macroscopic flux function $a(\rho)$. In addition we assume that the Maxwellian is a smooth and strictly increasing function of ρ and that it has certain decay properties with respect to v:

$$\partial_\rho M(\rho, v) > 0\,, \quad \int v^4 \partial_\rho M(\rho, v) dv < \infty\,, \quad \int \frac{v^{2k}(\partial_\rho^m M(\rho, v))^2}{\partial_\rho M(\rho^*, v)} dv < \infty\,, \tag{1.4}$$

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for all $\rho, \rho^* \in \mathbb{R}$, $k = 0, 1$, $m = 1, 2$. The last two properties here are required for technical reasons. The first and (1.3) ensure that the macroscopic limit equation (scaling with $(t, x) \to (t/\varepsilon, x/\varepsilon)$ and taking $\varepsilon \to 0$) of (1.1) is the scalar conservation law

$$\partial_t \rho + \partial_x a(\rho) = 0 \,. \tag{1.5}$$

Key to proving this limit rigorously is the first assumption in (1.4) that guarantees the construction of kinetic entropy inequalities (or H-theorems) with a macroscopic counterpart, as explained in e.g. [1]: There exists a function $\theta(f, v)$ such that $f = M(\rho, v)$ is equivalent to $\rho = \theta(f, v)$. With the primitive $\Theta(f, v)$ ($\partial_f \Theta = \theta$), solutions of (1.1) satisfy for any convex function η and letting $h(f, v) = \int_{\bar f}^{f} \eta'(\theta(g, v))\, dg$

$$\partial_t \int h(f, v)\, dv + \partial_x \int v\, h(f, v)\, dv = \int (M(\rho_f, v) - f)(\eta'(\theta(f, v)) - \eta'(\rho_f))\, dv \leq 0 \,.$$

But more to the point here, this property guarantees that the linearised collision operator is negative semidefinite in the L^2 space with weight $1/\partial_\rho M$ (evaluated at the perturbation). This can be interpreted from the kinetic entropy inequality that corresponds to the macroscopic entropy $\eta(\rho) = \rho^2/2$.

After applying the macroscopic scaling, the model (1.1) can be seen as a hyperbolic system with relaxation, (see e.g. [9] for a review, and [11] for an account of related models). Examples of smooth Maxwellians satisfying (1.3) as well as (1.4) are given in [4] for $\Omega = \mathbb{R}$. A related kinetic model for scalar conservation laws is the Perthame-Tadmor model (see [12]), although there the equilibrium distribution is not continuous.

We recall that a Chapman-Enskog expansion gives the viscous regularisation of (1.5)

$$\partial_t \rho_f + \partial_x a(\rho_f) = \partial_x (D(\rho_f) \partial_x \rho_f) \,, \tag{1.6}$$

with $D(\rho) = \int (v - a'(\rho))^2 \partial_\rho M(\rho, v)\, dv > 0$, where again the monotonicity of M is essential for parabolicity.

Uniqueness of weak solutions of (1.5) is guaranteed only if they are derived as limits ($D \to 0$ of such a regularized equation, see, e.g., [13]). In the case of Riemann data this defines admissibility conditions for shocks waves.

Regarding the long time behaviour of solutions of (1.6) with Riemann initial data, it is well-known that for genuinely nonlinear fluxes (i.e., $a''(\rho) \neq 0$) solutions converge either to a travelling wave solution (viscous shock profile) or to a rarefaction wave, whichever is the admissible solution of the inviscid equation (cf. [5] and [6]). In recent papers we have studied existence and long-time stability of small and large amplitude travelling wave solutions of (1.1), see [3] (small amplitude), [4] and [2] (no assumption in the amplitude). The construction of these waves relies on the assumption that the flux is genuinely nonlinear and that the far-field values of the wave satisfy the Lax shock admissibility condition, i.e. these travelling waves are regularisations of admissible shock solutions (kinetic shock profiles).

We now turn to the question, whether rarefaction waves of (1.5) describe the long time behaviour of (1.1) for appropriate initial data. As a solution of a Riemann problem, a rarefaction wave is a function of the similarity variable x/t. In other words, the rarefaction wave is a stationary solution, if the space variable x is replaced by the similarity variable. Because of the singularity of this transformation at $t = 0$, we introduce the alternative change of variables

$$\xi = \frac{x}{t+1} \,, \qquad \tau = \ln(t+1) \,, \tag{1.7}$$

where the usefulness of the rescaling of time becomes apparent in the rewritten version of (1.6):

$$\partial_\tau \rho - \xi \partial_\xi \rho + \partial_\xi (a(\rho)) = e^{-\tau} \partial_\xi (D(\rho) \partial_\xi \rho) \,. \tag{1.8}$$

When the solution satisfies appropriate far-field conditions, we may expect convergence as $\tau \to \infty$ of ρ to a rarefaction wave solution of

$$-\xi \partial_\xi \rho_\infty + \partial_\xi a(\rho_\infty) = 0 \,, \tag{1.9}$$

which is explicitly given by

$$\rho_\infty(\xi) = \begin{cases} \rho_- & \xi < a'(\rho_-) \\ (a')^{-1}(\xi) & a'(\rho_-) \le \xi \le a'(\rho_+) \\ \rho_+ & \xi > a'(\rho_+)\,. \end{cases} \tag{1.10}$$

Clearly ρ_∞ is monotone and satisfies

$$\partial_\xi a'(\rho_\infty) \ge 0\,. \tag{1.11}$$

The far-field values ρ_- and ρ_+ necessarily have to satisfy

$$(\rho_+ - \rho_-)a''(\rho) > 0 \quad \text{for } \rho \text{ between } \rho_- \text{ and } \rho_+\,.$$

In terms of the new variables, the BGK-model (1.1) becomes

$$\partial_\tau f + (v-\xi)\,\partial_\xi f = e^\tau\left(M(\rho_f, v) - f\right)\,. \tag{1.12}$$

Since for large τ, $e^{-\tau}$ plays the role of the Knudsen number (ε in the macroscopic scaling mentioned above), f formally approaches $f_{eq} = M(\rho_{eq}, v)$ for some ρ_{eq} that satisfies the macroscopic limiting equation

$$\partial_\tau \rho_{eq} - \xi\partial_\xi\rho_{eq} + \partial_\xi(a(\rho_{eq})) = 0\,. \tag{1.13}$$

So convergence to an equilibrium distribution with a rarefaction wave as macroscopic density can be hoped for.

The main difference to the analysis of travelling waves is the explicit appearance of time in the rescaled regularized equations, reflecting the fact that they are not invariant under the transformations $(t,x) \to (\lambda t, \lambda x)$ for any $\lambda \in \mathbb{R}$. We expect that the effect of the regularisations is negligible as $t \to \infty$.

For initial conditions with the far-field limits $M(\rho_\pm, v)$, we shall prove that solutions of (1.12) converge to $M(\rho_\infty, v)$ as $\tau \to \infty$ under the conditions that the initial data have some smoothness in the ξ-direction and that they are L^∞-close to an equilibrium distribution, which is constant in ξ. In other words, small perturbations of small amplitude rarefaction waves are considered.

Theorem 1.1 *Let the equilibrium distribution $M(\rho, v)$ satisfy (1.4). Let f be a global solution of (1.12) satisfying $f(t=0) = f_0$. Let the macroscopic flux $a(\rho)$ be smooth and genuinely nonlinear, and let the far-field densities $\rho_\pm \in \mathbb{R}$ satisfy $a'(\rho_-) < a'(\rho_+)$. Let*

$$\int_{\Omega\times\mathbb{R}} \frac{(f_0 - M(\rho^*, v))^2 + (\partial_\xi f_0)^2}{\partial_\rho M(\rho_+, v)} dv\, d\xi < \infty \tag{1.14}$$

for a smooth $\rho^*(\xi)$, constant outside a bounded interval and satisfying $\rho^*(\pm\infty) = \rho_\pm$. Then there exists a $\delta > 0$ such that under the further condition

$$M(\rho_+ - \delta, v) \le f_0(\xi, v) \le M(\rho_+ + \delta, v)\,, \tag{1.15}$$

(implying $|\rho_+ - \rho_-| \le \delta$)

$$\int_{\Omega\times\mathbb{R}} \frac{(f(\tau,\cdot,\cdot) - M(\rho_\infty, v))^2}{\partial_\rho M(\rho_+, v)} dv\, d\xi \le ce^{-\kappa\tau} \qquad \forall\, \tau > 0\,, \tag{1.16}$$

holds with $c, \kappa > 0$, where $\kappa \le 1$ and depends on δ.

We shall follow an approach similar to that in [3]. A micro-macro decomposition of the perturbation will be introduced. Then we derive L^2-estimates for both the microscopic and the macroscopic contributions, which can be combined into a Lyapunov functional. These computations would involve second order derivatives of ρ_∞, which is only Lipschitz continuous. Therefore ρ_∞ will be replaced by a smooth approximation converging to ρ_∞ as $\tau \to \infty$. This idea has been introduced by Matsumura and Nishihara [8]. The precise definition of the smooth approximation and its properties will be given in Section 3. This section also contains some assumptions, results and notation that are used in our convergence proof. In the following section we prove convergence to rarefaction waves for the viscous regularization (1.6). We shall use the macroscopic estimate later in Section 4, where the desired convergence result is proven.

We remark that convergence to rarefaction waves has been proved for some hyperbolic systems with relaxation by energy methods without assumption on the amplitude of the wave, see for instance [7] and [10]. This is possible by employing clever combinations of estimates which are not readily available for the current problem. The main reason is that the velocity space is rather general and not a finite set of velocities as it can be interpreted for hyperbolic systems with relaxation.

2 The macroscopic estimate

In this section we present a proof by means of L^2-estimates of convergence to rarefaction waves for solutions of (1.8). We recall that convergence to rarefaction waves has been proved by Oleinik in [6] by direct methods. The macroscopic estimate derived here will be used in Section 4, and will be combined with estimates on the microscopic terms. We use the notation $(L^\infty_\xi(\mathbb{R}), \|\cdot\|_\infty)$, $(L^2_\xi(\mathbb{R}), \|\cdot\|_\xi)$.

Let ρ be the solution of (1.8), subject to the initial condition

$$\rho(\tau = 0) = \rho_0\,, \qquad \text{with} \quad \rho_+ - \delta \le \rho_0 \le \rho_+ + \delta\,, \quad \rho_0(\pm\infty) = \rho_\pm\,.$$

We recall that (1.8) satisfies a maximum principle, implying

$$\rho_+ - \delta \le \rho(\tau,\xi) \le \rho_+ + \delta \quad \text{for all } \tau \ge 0\,,\ \xi \in \mathbb{R}\,. \tag{2.1}$$

The error $\tilde\rho := \rho - \rho_\infty$ satisfies

$$\partial_\tau \tilde\rho - \xi \partial_\xi \tilde\rho + \partial_\xi \left(a'(\rho_\infty)\tilde\rho\right) + \partial_\xi r(\rho_\infty, \tilde\rho) = e^{-\tau}\partial_\xi\left[D(\rho_\infty + \tilde\rho)(\partial_\xi\tilde\rho + \partial_\xi\rho_\infty)\right] \tag{2.2}$$

with the nonlinear correction of the linearized flux

$$r(\rho_\infty, \tilde\rho) := a(\rho_\infty + \tilde\rho) - a(\rho_\infty) - a'(\rho_\infty)\tilde\rho\,. \tag{2.3}$$

The fact that ρ_∞ lies between ρ_+ and ρ_- and (2.1) imply

$$\|\tilde\rho\|_\infty \le 2\delta\,. \tag{2.4}$$

As a consequence, $0 < \underline{D} \le D(\rho_\infty + \tilde\rho) \le \overline{D} < \infty$ holds.

Lemma 2.1 *There exists a constant $C > 0$ such that*

$$\frac{d}{d\tau}\|\tilde\rho\|_\xi^2 + (1 - C\delta)\,\|\tilde\rho\|_\xi^2 + e^{-\tau}\underline{D}\|\partial_\xi\tilde\rho\|_\xi^2 \le e^{-\tau}\overline{D}\,\|\partial_\xi\rho_\infty\|_\xi^2 \tag{2.5}$$

Proof. We test (2.2) with $\tilde\rho$

$$\begin{aligned}&\frac{1}{2}\frac{d}{d\tau}\int_{\mathbb{R}}\tilde\rho^2\,d\xi + \frac{1}{2}\int_{\mathbb{R}}(1 + \partial_\xi a'(\rho_\infty))\tilde\rho^2\,d\xi + \int_{\mathbb{R}}\tilde\rho\,\partial_\xi r(\rho_\infty,\tilde\rho)d\xi \\ &= -e^{-\tau}\int_{\mathbb{R}} D(\rho_\infty + \tilde\rho)\,(\partial_\xi\tilde\rho)^2\,d\xi - e^{-\tau}\int_{\mathbb{R}} D(\rho_\infty + \tilde\rho)\,\partial_\xi\rho_\infty\partial_\xi\tilde\rho\,d\xi\,.\end{aligned} \tag{2.6}$$

The second term can be estimated from below by (1.11). We write the term involving r as

$$\begin{aligned}&\int_{\mathbb{R}}\tilde\rho\,\partial_\xi r(\rho_\infty,\tilde\rho)d\xi = -\int_{\mathbb{R}} r(\rho_\infty,\tilde\rho)\partial_\xi\tilde\rho\,d\xi = \int_{\mathbb{R}}\int_0^{\tilde\rho}\partial_{\rho_\infty} r(\rho_\infty,\rho')d\rho'\partial_\xi\rho_\infty d\xi \\ &= \int_{\mathbb{R}} a'''(\hat\rho)\frac{\tilde\rho^3}{6}\partial_\xi\rho_\infty d\xi\,,\end{aligned}$$

with $\hat{\rho}$ between ρ_∞ and $\rho_\infty + \tilde{\rho}$. The smoothness of a, the boundedness of the densities, and the Lipschitz continuity of ρ_∞ imply the existence of a constant $C > 0$ such that

$$\left| \int_{\mathbb{R}} \tilde{\rho}\, \partial_\xi r(\rho_\infty, \tilde{\rho}) d\xi \right| \le \frac{C\delta}{2} \|\tilde{\rho}\|_\xi^2 \,.$$

In the last term of (2.6) $ab \le a^2/2 + b^2/2$ is used. □

In order to get convergence it is enough to choose δ small enough.

Theorem 2.2 *Let ρ be a smooth solution of (1.8) satisfying*

$$\rho(t=0) - \rho_\infty \in L^2_\xi \quad \textit{and} \quad \|\rho(t=0) - \rho_+\|_\infty =: \delta < 1/C$$

with the constant C from the previous lemma. Then there exists a constant $C_1 > 0$, independent from δ, such that

$$\|\rho(\tau, \cdot) - \rho_\infty\|_\xi^2 \le \frac{C_1}{\delta} \left(e^{(C\delta - 1)\tau} - e^{-\tau} \right) . \tag{2.7}$$

Proof. By Lemma 2.1 the left hand side $y(\tau)$ of (2.7), writing $y(\tau) = \|\tilde{\rho}\|_\xi^2$, satisfies the differential inequality

$$\dot{y} + (1 - C\delta) y \le e^{-\tau} C_2 \,,$$

for $C_2 > 0$ independent of δ. The statement of the theorem is a consequence of the Gronwall lemma. □

3 Preliminaries and notation

The analysis of the following section is based on linearizaton of (1.1) around the far-field equilibrium state $M(\rho_+, v)$. The linearized collision operator is given by

$$\mathcal{L} f := \rho_f F - f \,, \quad \text{with } F(v) = \partial_\rho M(\rho_+, v) \,.$$

It is easily seen that $\mathcal{L}$ is symmetric and negative semidefinite with respect to the scalar product

$$\langle f, g \rangle_v = \int \frac{fg}{F} \, dv \,.$$

The corresponding weighted L^2-space is denoted by $(L^2_v, \|\cdot\|_v)$. Just as in [3], we also consider the space $L^2_{\xi,v}$ defined by the scalar product

$$\langle f, g \rangle_{\xi,v} = \int_{\mathbb{R}} \langle f, g \rangle_v d\xi \,,$$

with the induced norm $\|\cdot\|_{\xi,v}$. Finally, H^k_ξ, $k \ge 0$ denotes the standard L^2-based Sobolev spaces for functions of ξ (with $\|\cdot\|_\xi := \|\cdot\|_{L^2_\xi}$).

To avoid differentiating the rarefaction wave, we shall consider a smooth approximation of ρ_∞ that satisfies (1.5). We use the regularisation introduced in [8]. They consider w to be the solution of the following initial value problem

$$\begin{aligned} &\partial_t w + w \partial_x w = 0 \,, \\ &w(0, x) = \tfrac{1}{2} \left[(w^+ - w^-) + (w^+ - w^-) \tanh(x) \right] \,. \end{aligned} \tag{3.1}$$

They also prove the following properties of w

Lemma 3.1 ([8]) *If $w^+ > w^-$, then (3.1) has a unique global solution satisfying*

(i) $w^- < w(t, x) < w^+$, $\partial_x w(t, x) > 0$ *for* $t \le 0$, $x \in \mathbb{R}$.

(ii) *For any* $p \in [0,\infty]$, *there exist a positive constant* C *(depending only on* p*) such that for* $t \geq 0$,

$$\|\partial_x w(t,\cdot)\|_{L^p_x} \leq C \min\{|w^+ - w^-|, |w^+ - w^-|(1+t)^{-1+\frac{1}{p}}\},$$
$$\|\partial_x^k w(t,\cdot)\|_{L^p_x} \leq C \min\{|w^+ - w^-|, (1+t)^{-1}\} \quad \text{for } k = 2, 3.$$

(iii) *If* w_∞ *denotes the rarefaction wave of (3.1) then*

$$\lim_{t\to\infty} \sup_{x\in\mathbb{R}} |w(x,t) - w_\infty(x/t)| = 0.$$

The approximation we use is defined by setting $\rho_{as} := (a')^{-1}(w)$. We need the following properties

Lemma 3.2 *Let* w *satisfy (3.1) with* $w^+ = (a')^{-1}(\rho_+)$ *and* $w^- = (a')^{-1}(\rho_-)$, *and let*

$$\rho_{as}(\tau,\xi) := (a')^{-1}(w(e^\tau - 1, \xi e^\tau)).$$

Then

(i) ρ_{as} *is a smooth solution of (1.13) with values between* ρ_+ *and* ρ_- *satisfying*

$$\partial_\xi(a'(\rho_{as}(\tau,\xi))) > 0 \qquad \text{for } \tau \geq 0.$$

(ii) *There exists* $C > 0$ *such that for every* $\tau \geq 0$

$$\|\partial_\xi \rho_{as}(\tau,\cdot)\|_\infty \leq C|\rho_+ - \rho_-|, \qquad \|\partial_\xi^2 \rho_{as}(\tau,\cdot)\|_\infty \leq Ce^\tau,$$
$$\|\partial_\xi \rho_{as}(\tau,\cdot)\|_\xi \leq C|\rho_+ - \rho_-|, \qquad \|\partial_\xi^2 \rho_{as}(\tau,\cdot)\|_\xi \leq Ce^{\tau/2}.$$

(iii) *There exists* $C > 0$ *such that for every* $0 < \delta < 1$, $\tau \geq 0$,

$$\|\rho_{as}(\tau,\cdot) - \rho_\infty\|_\xi^2 \leq \frac{C}{\delta} e^{(\delta-1)\tau}, \qquad \lim_{\tau\to\infty} \|\rho_{as}(\tau,\cdot) - \rho_\infty\|_\infty = 0.$$

Proof. The proof follows easily from Lemma 3.1 except the L^2 estimate in (iii). Note however, that the proof of Theorem 2.2 also works for smooth solutions of the hyperbolic equation ($D = 0$). By the L^∞ convergence result, $\rho(\tau,\xi) := \rho_{as}(\tau_0 + \tau, \xi)$ satisfies the assumptions of Theorem 2.2 with arbitrary δ for τ_0 large enough. □

4 Convergence to macroscopic rarefaction waves

In this section Theorem 1.1 will be proven. With ρ_{as} as in Lemma 3.2 we define

$$f_{as}(\tau,\xi,v) := M(\rho_{as}(\tau,\xi), v).$$

The perturbation

$$G := f - f_{as}$$

satisfies

$$\partial_\tau G + (v - \xi)\partial_\xi G = e^\tau [M(\rho_{as} + \rho_G) - M(\rho_{as}) - G] - (v - a'(\rho_{as}))\partial_\xi M(\rho_{as}), \tag{4.1}$$

where

$$\partial_\tau f_{as} + (v - \xi)\partial_\xi f_{as} = (v - a'(\rho_{as}))\partial_\xi M(\rho_{as}) \tag{4.2}$$

has been used. We introduce the micro-macro decomposition of G according to the linearized collision operator around the far field equilibrium $M(\rho_+, v)$:

$$G = \rho F + g \qquad \text{with} \quad \rho := \rho_G\,, \quad g = -\mathcal{L}G\,,$$

where the notation has been introduced in the previous section. Since the maximum principle can be applied to the BGK model (1.1), it also holds for the equation (1.12) in terms of the variables (τ, ξ), with the consequence

$$M(\rho_+ - \delta, v) \le f(\tau, \xi, v) \le M(\rho_+ + \delta, v)\,, \qquad \forall\, \tau,\, \xi,\, v\,.$$

Since the same estimate holds for f_{as},

$$\|\rho(\tau, \cdot)\|_\infty \le 2\delta\,, \qquad \text{for } \tau \ge 0 \tag{4.3}$$

holds (using the monotonicity of M with respect to ρ).

Equations for ρ and g are obtained by projection of (4.1), where the macroscopic projection amounts to integration with respect to v, and $-\mathcal{L}$ is the microscopic projection:

$$\partial_\tau \rho + (a'(\rho_+) - \xi)\partial_\xi \rho + \partial_\xi m_g = 0\,, \tag{4.4}$$

$$\begin{aligned}&\partial_\tau g + (v - \xi)\partial_\xi g - \partial_\rho M(\rho_+)\partial_\xi m_g + (v - a'(\rho_+))\partial_\rho M(\rho_+)\partial_\xi \rho \\ &= e^\tau\{R[\rho_{as}, \rho] + \Lambda[\rho_{as}]\rho - g\} - (v - a'(\rho_{as}))\partial_\xi M(\rho_{as})\,,\end{aligned} \tag{4.5}$$

with the microscopic flux $m_g = \int vg\, dv$, the nonlinear correction

$$R[\rho_{as}, \rho] := [M(\rho_{as} + \rho) - M(\rho_{as}) - \partial_\rho M(\rho_{as})\rho]\,, \tag{4.6}$$

for a linearization around $M(\rho_{as})$, and

$$\Lambda[\rho_{as}] = \partial_\rho M(\rho_{as}) - F\,,$$

the correction due to actually linearizing around the constant (in ξ) state $M(\rho_+)$.

Now the macroscopic equation is rewritten by using the basic idea of the Chapman-Enskog expansion: For large values of τ, the dominating term on the right hand side of (4.5) is used to compute g and, after multiplication with v and integration,

$$\begin{aligned} m_g \;\; = \;\; & r(\rho_{as}, \rho) + (a'(\rho_{as}) - a'(\rho_+))\rho \\ & - e^{-\tau}[\partial_\tau m_g + \partial_\xi P_g - (\xi + a'(\rho_+))\partial_\xi m_g + D_+\partial_\xi \rho + D_{as}\partial_\xi \rho_{as}]\,.\end{aligned} \tag{4.7}$$

with $D_+ := D(\rho_+)$, $D_{as}(\tau, \xi) := D(\rho_{as}(\tau, \xi))$, the second order moment $P_g = \int v^2 g\, dv$, and $r = \int vR\, dv$ has been defined in (2.3). Substituting (4.7) into (4.4) we obtain the macroscopic equation

$$\begin{aligned}&\partial_\tau \rho - \xi\partial_\xi \rho + \partial_\xi(a'(\rho_{as})\rho) + \partial_\xi r(\rho_{as}, \rho) - e^{-\tau} D_+ \partial_\xi^2 \rho \\ &= e^{-\tau}\, \partial_\xi[D_{as}\partial_\xi \rho_{as} + \partial_\tau m_g + \partial_\xi P_g - (\xi + a'(\rho_+))\partial_\xi m_g]\,,\end{aligned} \tag{4.8}$$

Formally, this is a promising result, since we already know how to deal with the left hand side, which is analogous to the macroscopic model analyzed in Section 2. The perturbation on the right hand side however, although formally of order $e^{-\tau}$, contains second order derivatives of the microscopic part g, and it is not obvious at the moment how to control these. We start by deriving integral estimates as we did for the purely macroscopic case.

Lemma 4.1 *There exists a positive constant C, such that*

$$\begin{aligned}&\frac{d}{d\tau}\left(\|\rho\|_\xi^2 - 2e^{-\tau}\int \rho\partial_\xi m_g\, d\xi\right) \\ &+ (1 - C\delta - e^{-\tau})\|\rho\|_\xi^2 + e^{-\tau} D_+ \|\partial_\xi \rho\|_\xi^2 \le e^{-\tau} C\left(\|\partial_\xi g\|_{\xi,v}^2 + 1\right)\,.\end{aligned} \tag{4.9}$$

Proof. We test (4.8) with ρ. Observing that all the terms in the left-hand side can be estimated in the same way as in Lemma 2.1, we obtain

$$\frac{1}{2}\frac{d}{d\tau}\|\rho\|_\xi^2 + \frac{1}{2}(1-C\delta)\|\rho\|_\xi^2 + e^{-\tau}D_+\|\partial_\xi\rho\|_\xi^2 \le e^{-\tau}\|\partial_\xi\rho\|_\xi\|D_{as}\partial_\xi\rho_{as}\|_\xi$$
$$+\frac{d}{d\tau}\left(e^{-\tau}\int\rho\partial_\xi m_g\,d\xi\right) + e^{-\tau}C\|\partial_\xi\rho\|_\xi(\|\partial_\xi m_g\|_\xi + \|\partial_\xi P_g\|_\xi)$$
$$+e^{-\tau}\|\rho\|_\xi\|\partial_\xi m_g\|_\xi + e^{-\tau}\|\partial_\xi m_g\|_\xi^2\,. \tag{4.10}$$

Since by our assumptions on the equilibrium distribution, F has velocity moments of order up to four, an application of the Cauchy-Schwarz inequality shows

$$|\partial_\xi m_g|,\ |\partial_\xi P_g| \le C\|\partial_\xi g\|_v\,.$$

This fact as well as the boundedness of D_{as} in L^∞_ξ and of $\partial_\xi\rho_{as}$ in L^2_ξ (see Lemma 3.2) complete the proof. □

Two difficulties have to be overcome when using this result. First, the term under the time derivative is indefinite and, second, the ξ-derivative of the microscopic part occurs on the right hand side. As a remedy an estimate for the new unknown

$$W := e^{-\tau}\partial_\xi G\,,$$

will be derived. Note that, by orthogonality,

$$\|W\|_{\xi,v}^2 = \|e^{-\tau}\partial_\xi\rho\|_\xi^2 + \|e^{-\tau}\partial_\xi g\|_{\xi,v}^2\,. \tag{4.11}$$

Differentiating (4.1) with respect to ξ and multiplying by $e^{-\tau}$ we get the following equation for W

$$\partial_\tau W + (v-\xi)\partial_\xi W = \partial_\xi\left(R[\rho_{as},\rho] + \Lambda[\rho_{as}]\rho - g\right) - e^{-\tau}\partial_\xi[(v-a'(\rho_{as}))\partial_\xi M(\rho_{as})]\,. \tag{4.12}$$

Lemma 4.2 *There exists a positive constant C such that*

$$\frac{d}{d\tau}\|W\|_{\xi,v}^2 + \|W\|_{\xi,v}^2 + e^{-\tau}\|\partial_\xi g\|_{\xi,v}^2 \le C\left(e^{-\tau}\delta^2\|\rho\|_{H^1_\xi}^2 + e^{-2\tau}\right)\,. \tag{4.13}$$

Proof. We compute the scalar product of (4.12) with W to obtain

$$\frac{1}{2}\frac{d}{d\tau}\|W\|_{\xi,v}^2 + \frac{1}{2}\|W\|_{\xi,v}^2 = e^{-\tau}\left\langle\partial_\xi(R[\rho_{as},\rho] + \Lambda[\rho_{as}]\rho - g), \partial_\xi g\right\rangle_{\xi,v}$$
$$-e^{-2\tau}\left\langle\partial_\xi\left((v-a'(\rho_{as}))\partial_\rho M(\rho_{as})\,\partial_\xi\rho_{as}\right), \partial_\xi g\right\rangle_{\xi,v}\,. \tag{4.14}$$

We have used that the integral with respect to v of the right hand side of (4.12) vanishes. For estimating the first term on the right hand side, we compute

$$\begin{aligned}|\partial_\xi(R[\rho_{as},\rho] + \Lambda[\rho_{as}]\rho)| &= |[\partial_\rho M(\rho_{as}+\rho) - \partial_\rho M(\rho_{as})]\partial_\xi\rho_{as} + [\partial_\rho M(\rho_{as}+\rho) - F]\partial_\xi\rho| \\ &\le C\delta\left[|\partial_\rho^2 M(\rho')\rho| + |\partial_\rho^2 M(\rho'')\partial_\xi\rho|\right]\,,\end{aligned}$$

where Lemma 3.2 has been used as well as the fact that both ρ_- and $\rho_{as}+\rho$ are $O(\delta)$ away from ρ_+. Now the assumptions on the equilibrium distribution can be employed to show that the first term on the right hand side of (4.14) can be estimated from above by

$$e^{-\tau}\left(C\delta^2\|\rho\|_{H^1_\xi}^2 - \frac{3}{4}\|\partial_\xi g\|_{\xi,v}^2\right)\,.$$

For the last term in (4.14) we have

$$\partial_\xi\left[(v-a'(\rho_{as}))\partial_\rho M(\rho_{as})\,\partial_\xi\rho_{as}\right] = (v-a'(\rho_{as}))\partial_\rho^2 M(\rho_{as})(\partial_\xi\rho_{as})^2$$
$$-a''(\rho_{as})\partial_\rho M(\rho_{as}))(\partial_\xi\rho_{as})^2 + (v-a'(\rho_{as}))\partial_\rho M(\rho_{as})\,\partial_\xi^2\rho_{as} \tag{4.15}$$

Again, Lemma 3.2 and the assumptions on M are used to show that the norm $\|\cdot\|_{\xi,v}$ of this term is bounded by $Ce^{\tau/2}$. So the last term in (4.14) can be bounded from above by

$$\frac{1}{4}e^{-\tau}\|\partial_\xi g\|_{\xi,v}^2 + 2C^2e^{-2\tau}\,.$$

The proof is completed by combining our results. □

After these preparations we are ready to prove decay of the macroscopic part of the perturbation.

Lemma 4.3 *Under the assumptions of Theorem 1.1,*

$$\|\rho(\tau,\cdot)\|_\xi^2 \le ce^{-\kappa\tau}\,, \qquad \tau \ge 0\,.$$

Proof. We introduce

$$\begin{aligned} H &= \|\rho\|_\xi^2 - 2e^{-\tau}\int \rho\partial_\xi m_g\,d\xi + \alpha\|W\|_{\xi,v}^2 \\ &\ge \|\rho\|_\xi^2 - C\|\rho\|_\xi\|e^{-\tau}\partial_\xi g\|_{\xi,v} + \alpha\|e^{-\tau}\partial_\xi\rho\|_\xi^2 + \alpha\|e^{-\tau}\partial_\xi g\|_{\xi,v}^2 \\ &\ge \frac{1}{2}\left(\|\rho\|_\xi^2 + \|e^{-\tau}\partial_\xi g\|_{\xi,v}^2\right) + \alpha\|e^{-\tau}\partial_\xi\rho\|_\xi^2\,, \end{aligned}$$

for $\alpha \ge (C^2+1)/2$. Combining the results of Lemmas 4.1 and 4.2 gives

$$\begin{aligned} &\frac{dH}{d\tau} + \alpha\|W\|_{\xi,v}^2 + \alpha e^{-\tau}\|\partial_\xi g\|_{\xi,v}^2 + \left(1 - C\delta - e^{-\tau}\right)\|\rho\|_\xi^2 + e^{-\tau}D_+\|\partial_\xi\rho\|_\xi^2 \\ &\le \alpha C\delta^2e^{-\tau}\left(\|\rho\|_\xi^2 + \|\partial_\xi\rho\|_\xi^2\right) + Ce^{-\tau}\|\partial_\xi g\|_{\xi,v}^2 + Ce^{-\tau} \end{aligned}$$

By choosing α and τ large enough and δ small enough, we easily obtain an inequality of the form

$$\frac{dH}{d\tau} + \kappa H \le Ce^{-\tau}\,,$$

with $\kappa > 0$, implying the result, with κ replaced by $\min\{1,\kappa\}$, by an application of the Gronwall lemma. □

Since the functional H in the above proof only controls the macroscopic part of the perturbation, an additional step is required. To get control of the microscopic part, we derive energy estimates from the full kinetic perturbation equation (4.1). Testing (4.1) by G gives

$$\begin{aligned} &\frac{1}{2}\frac{d}{d\tau}\|G\|_{\xi,v}^2 + \frac{1}{2}\|G\|_{\xi,v}^2 \\ &= e^\tau\left(\langle R[\rho_{as},\rho] + \Lambda[\rho_{as}]\rho, g\rangle_{\xi,v} - \|g\|_{\xi,v}^2\right) - \langle (v - a'(\rho_{as}))\partial_\xi M(\rho_{as}), G\rangle_{\xi,v}\,. \end{aligned}$$

By the properties of ρ_{as}, R, and Λ, this immediately implies

$$\begin{aligned} \frac{d}{d\tau}\|G\|_{\xi,v}^2 + \|G\|_{\xi,v}^2 &\le 2e^\tau\left(C\|\rho\|_\xi\|g\|_{\xi,v} - \|g\|_{\xi,v}^2\right) + C\|G\|_{\xi,v} \\ &\le e^\tau\left(C\|\rho\|_\xi^2 - \|G\|_{\xi,v}^2\right) + \|G\|_{\xi,v}^2 + C\,, \end{aligned}$$

being equivalent to

$$\frac{d}{d\tau}\|G\|_{\xi,v}^2 + e^\tau\|G\|_{\xi,v}^2 \le e^\tau C\left(\|\rho\|_\xi^2 + e^{-\tau}\right)\,.$$

The Gronwall lemma implies that $\|G\|_{\xi,v}^2$ decays like $\|\rho\|_\xi^2 + e^{-\tau}$. This completes the proof of Theorem 1.1 since, by Lemma 3.2 (iii), it is sufficient to estimate the distance of the solution to ρ_{as} instead of ρ_∞.

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