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STABILITY OF SOLITARY WAVES IN A SEMICONDUCTOR DRIFT-DIFFUSION MODEL*

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Abstract. We consider a macroscopic (drift-diffusion) model describing a simple microwave generator, consisting of a special type of semiconductor material that, when biased above a certain threshold voltage, generates charge waves. These waves correspond to travelling wave solutions of the model equation which, however, turn out to be unstable in a standard formulation of the travelling wave problem. Here a different formulation of this problem is considered, where an external voltage condition is applied in the form of an integral constraint. Global existence of this novel Cauchy problem is proven and the results of numerical experiments are presented, which suggest the stability of solitary waves. In addition, a small amplitude limit is considered, for which linearized orbital stability of solitary waves can be proven.

Key words. Gunn effect, drift-diffusion equation, solitary waves, global constraint

AMS subject classifications. 82D37, 35K55, 35B40

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1. Introduction. In this paper we consider the nondimensionalized one-dimensional semiconductor drift-diffusion model

$$\partial_t n = \partial_x(\partial_x n - v(E)\, n)\,, \tag{1.1}$$

$$\partial_x E = n - 1 \tag{1.2}$$

for $(x,t) \in \mathbb{R} \times (0,\infty)$, where $n(x,t)$ denotes the electron density and $E(x,t)$ the (negative) electric field. In the drift-diffusion equation (1.1), $v(E)$ is the field dependent drift velocity, and in the Poisson equation (1.2), the constant 1 represents the scaled constant doping concentration. The special feature of the model is the nonmonotonicity of $v(E)$, made precise below.

The system will be considered subject to the initial condition

$$n(0,x) = n_I(x) \quad \text{for all } x \in \mathbb{R}\,, \tag{1.3}$$

where initially and, thus, for all times, we assume global charge neutrality:

$$\int_{\mathbb{R}} (n_I - 1)dx = 0\,.$$

This has the consequence that the field takes the same value

$$E_\infty(t) := \lim_{|x|\to\infty} E(t,x)$$

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at $x = \pm\infty$. Instead of prescribing $E_\infty(t)$, we leave it as an unknown and pose the integral constraint

$$\int_{\mathbb{R}} (E(t,x) - E_\infty(t))\, dx = U(t)\,, \tag{1.4}$$

where the function $U(t)$ is given for $t \geq 0$.

This problem arises from a one-dimensional model of a simple microwave generator. When biased above a certain voltage threshold, the generator produces current oscillations based on dipole charge waves travelling through the semiconductor material. This is known as the Gunn effect; see [4] and [5].

The system (1.1), (1.2) subject to (1.3), (1.4) will be motivated below by scaling arguments. We start with the unscaled equations describing the flow of electrons in a piece of homogeneous n-type semiconductor material of length L (cf. [9]),

$$\partial_t n = \partial_x (D\partial_x n - v(E)\, n)\,, \quad \text{with } t > 0,\ x \in (-L, L)\,, \tag{1.5}$$

$$\varepsilon_s \partial_x E = q(n - C), \quad \text{with } x \in (-L, L)\,. \tag{1.6}$$

This is the standard unipolar drift-diffusion model where the transport of holes is neglected. The constant parameters are the diffusivity D, the permittivity ε_s of the semiconductor material, the elementary charge q, and the donor concentration $C > 0$. Since this fixed background charge density is positive, the negatively charged electrons will dominate among the mobile charges, satisfying the omission of the positively charged holes from the model. The function v stands for the drift velocity of electrons and depends on the field, thus leading to a nonlinear coupling of the system, which is supplemented by an initial condition $n(0, x) = n_I(x)$ and by Dirichlet boundary conditions for the electron concentration:

$$n(t, -L) = n(t, L) = C \quad \text{for } t > 0\,. \tag{1.7}$$

In addition, the application of an exterior (given) voltage $\bar{U}$ is described by the integral condition

$$\int_{-L}^{L} E(t,x)\, dx = \bar{U}(t)\,. \tag{1.8}$$

For standard semiconductor materials such as silicon, measurements of the drift velocity $v(E)$ yield an odd nonlinear increasing function of the field E, almost linear for small fields, and bounded from above by a velocity saturation value v_{sat}. However, there are semiconductor materials such as *gallium arsenide* (GaAs), for which the velocity v reaches a maximum at a certain threshold value of the field E_T (cf. [13]), with the profile of v decreasing for $E > E_T$ to v_{sat}; see Figure 1. This nonmonotonicity of the velocity is responsible for the existence of pulse like solutions, namely solitary (travelling) waves, which are necessary for the Gunn effect. We are interested in studying the stability of these waves.

Using L as characteristic length, L/v_{sat} as characteristic time, v_{sat} as characteristic velocity, C as characteristic electron density, and E_T as characteristic field strength, one obtains the dimensionless equations

$$\partial_t n = \partial_x (\nu \partial_x n - n v(E))\,, \tag{1.9}$$

$$\lambda^2 \partial_x E = n - 1\,, \tag{1.10}$$

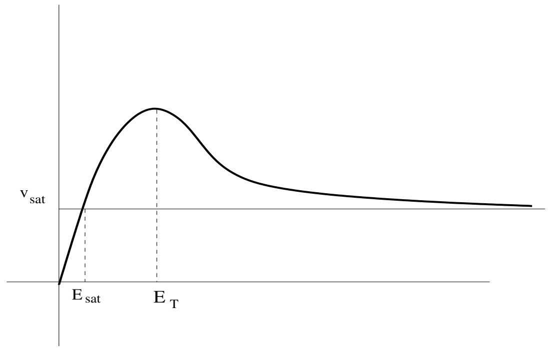


FIG. 1. *Electron drift velocity.*

subject to the conditions

$$n(t,-1) = n(t,1) = 1\,, \tag{1.11}$$

$$\int_{-1}^{1} E(t,x)\,dx = \bar{U}(t)\,, \tag{1.12}$$

where the drift velocity v is now normalized in the sense that it takes its maximum at $E = 1$ and satisfies $\lim_{E\to\infty} v(E) = 1$. The dimensionless parameters

$$\lambda^2 = \frac{\varepsilon_s E_T}{L^2 qC}\,, \quad \nu = \frac{D}{L v_{sat}}$$

are, respectively, the square of the scaled Debye length and the relative strength of diffusive and convective terms. We are interested in the case of a high doping concentration and a long device; therefore the parameters λ^2 and ν are both small. We shall make the scaling assumption that they are of the same order of magnitude and, for simplicity, actually set $\nu = \lambda^2$.

We recall that for a given constant voltage, the homogeneous steady state solution

$$n \equiv 1, \quad E \equiv \frac{1}{2}\bar{U}$$

of (1.9), (1.10) is stable if $\bar{U} \le 2$ ($E \le 1$) and unstable if $\bar{U} > 2$ ($E > 1$); cf. [14], [1]. Stable solitary waves are expected to arise in the latter case. The appropriate space-time scaling for these waves is achieved by $(t,x) \to (t/\lambda^2, x/\lambda^2)$, which expands both the temporal and the spatial domains. It leads to (1.1)–(1.2), and the integral condition (1.12) becomes

$$\lambda^2 \int_{-\frac{1}{\lambda^2}}^{\frac{1}{\lambda^2}} E(t,x)\,dx = \bar{U}(t)\,. \tag{1.13}$$

In the "Gunn operation mode" we expect waves travelling through the device, whose typical length is of order one in terms of the new x-variable. Away from the wave, i.e., in most of the device, we expect an almost constant electric field, and we denote an approximation by $E_{1/\lambda^2}(t)$. The condition (1.13) can then be rewritten as

$$\lambda^2 \int_{-\frac{1}{\lambda^2}}^{\frac{1}{\lambda^2}} (E(t,x) - E_{1/\lambda^2}(t))\,dx = \bar{U}(t) - 2E_{1/\lambda^2}(t)\,. \tag{1.14}$$

Passing to the limit $\lambda^2 \to 0$ formally gives $E_\infty(t) = \bar{U}(t)/2$ with $E(t,x) \to E_\infty(t)$ as $|x| \to \infty$. In [15] Szmolyan considered the problem (1.1), (1.2) subject to this boundary condition and an initial condition for n. It is striking that, with standard linearization techniques, he proved that solitary waves are unstable in this case.

These results are rather unexpected if compared with the experimental evidence on Gunn diodes. The aim of this work is to study a reformulation of the problem, which seems to stabilize the solitary waves. Formally, the reformulation can be derived by introducing

$$U(t) := \lim_{\lambda\to 0} \frac{1}{\lambda^2}\left(\bar{U}(t) - 2E_{1/\lambda^2}(t)\right)$$

and passing to the limit in (1.14) after dividing by λ^2. Obviously, this leads to the integral condition (1.4).

In the language of asymptotic analysis, the assumption that the small parameters ν and λ^2 are of the same order of magnitude leads to a significant limit, since the small parameters can then be eliminated from the differential equations by the above rescaling. However, since the ratio λ^2/ν depends on both the device length and the doping concentration, situations where this ratio is either very small or very large can also be physically relevant. An asymptotic analysis of travelling waves in the former case can be found in [9]. It turns out that in this case all travelling wave solutions have a far-field value of the electric field close to E_{sat} (see Figure 1). This result can be seen as a (not very strong) physical justification of prescribing $U(t)$, since this is then close to prescribing the contact voltage $\bar{U}(t)$.

For convenience we introduce the unknown

$$e(t,x) := E(t,x) - E_\infty(t) = \int_{-\infty}^x (n(t,y)-1)\,dy \quad \text{with } t>0,\ x\in\mathbb{R}\,.$$

Substituting $n = \partial_x e + 1$ into (1.1) and integrating with respect to x gives the equation

$$\partial_t e = \partial_x^2 e - v(e+E_\infty)\,\partial_x e + v(E_\infty) - v(e+E_\infty)\,, \tag{1.15}$$

subject to the initial condition

$$e(0,x) = e_I(x) = \int_{-\infty}^x (n_I(y)-1)\,dy\,, \tag{1.16}$$

with n_I as in (1.3), and to the integral constraint (1.4), which now simply reads

$$\int_{\mathbb{R}} e(t,x)\,dx = U(t)\,. \tag{1.17}$$

Differentiation with respect to time gives

$$U'(t) = \int_{\mathbb{R}} (v(E_\infty(t)) - v(E_\infty(t) + e(t,x)))dx\,. \tag{1.18}$$

We shall solve (1.15) subject to (1.18) instead of (1.17). This will be favorable since (1.18) can be seen as an equation for E_∞ for given $U'(t)$ and e.

The formulation of the problem will be completed by specifying the precise assumptions on the drift velocity.

Assumption 1. We assume $v \in C_B^3([0,\infty))$, $v(0) = 0$, $\operatorname{sign} v'(E) = \operatorname{sign}(1-E)$, $\lim_{E\to\infty} v(E) = 1$, $\exists\, E_i > 1$ such that $\operatorname{sign} v''(E) = \operatorname{sign}(E - E_i)$. Finally, $v''' \geq 0$ on $(1, E_i)$.

The equation $v(E_{sat}) = 1$ uniquely defines $E_{sat} < 1$. We also introduce $\sigma_i = \sup_{E>0} |d^i v/dE^i(E)|$, $i = 1, 2, 3$.

The paper is organized as follows. In section 2 we review the existence of solitary waves but incorporate the condition (1.4) into the problem. It turns out that for all $U > 0$ there exists a unique (up to translation) solitary wave having $E_\infty < 1$. In section 3 we prove existence of solutions of (1.15)–(1.18) for positive $U(t)$. Actually there is also a restriction on the values of $U'(t)$, which is required to be in the range of the right-hand side of (1.18). The existence proof uses a fixed point argument involving the operator defined by solving the condition (1.18) (for given e). This operator is only locally Lipschitz in $L^1_x(\mathbb{R})$. This difficulty does not ensue in bounded domains; see [8]. There is still no general result on the stability of solitary waves. In section 4, however, we provide strong numerical evidence that we succeeded in stabilizing the travelling waves by the new formulation. Moreover, in section 5 we consider a *small* wave limit by imposing a small external voltage. We prove linear asymptotic stability of the limiting solitary waves. It turns out that the limit equation is the so-called conserved Fisher equation with a constant competition rate, a model of population dynamics with global regulation [11]. In particular, our proof shows linearized asymptotic stability of its stationary solutions.

2. Solitary waves. In this section we prove existence of solitary waves subject to the constraint (1.4). Let $\xi =: x - ct$ be the travelling wave variable, where $c > 0$ is the wave speed. Then a solitary wave solution $(E(\xi), n(\xi))$ of (1.1)–(1.2) is a solution of

$$n' = n(v(E) - c) - v(E_\infty) + c\,,$$
$$E' = n - 1$$

that satisfies

$$n \to 1 \quad \text{and} \quad E \to E_\infty \qquad \text{as } |\xi| \to \infty\,. \tag{2.1}$$

A straightforward computation using both differential equations leads to

$$\frac{n-1}{n} n' - (v(E) - v(E_\infty))E' = \frac{(n-1)^2}{n}(v(E_\infty) - c)\,.$$

Since the right-hand side does not change sign, integration with respect to ξ and the far-field conditions imply that a solution exists only if $c = v(E_\infty)$ holds, which we assume in the following:

$$n' = n(v(E) - v(E_\infty)), \tag{2.2}$$
$$E' = n - 1. \tag{2.3}$$

We incorporate the condition (1.4), which in the travelling wave variable reads

$$\int_{\mathbb{R}} (E(\xi) - E_\infty)\, d\xi = U, \tag{2.4}$$

where U is a given constant, and E_∞ will be determined as part of the solution of (2.1)–(2.4). The main result of this section is the following theorem.

THEOREM 2.1. *For each $U > 0$ there exists a solution (n, E, E_∞) of (2.1)–(2.4) which is unique up to translation in ξ and satisfies $E_{sat} < E_\infty < 1$. The far-field value E_∞ of the field is a strictly decreasing function of U, satisfying*

$$\lim_{U\to 0} E_\infty(U) = 1 \quad \text{and} \quad \lim_{U\to\infty} E_\infty(U) = E_{sat}\,. \tag{2.5}$$

Before we prove the theorem we recall the existence result of (2.1)–(2.3) for a given value of E_∞.

LEMMA 2.2. *For every $E_\infty \in (E_{sat}, 1)$, there exists a unique (up to translation in ξ) solution (n, E) of (2.1)–(2.3) that satisfies $E > E_\infty$. The total charge density $n-1$ has one simple zero, to the left of which it is positive (and negative to the right).*

This lemma is just a reformulation of the existence result that appears in [15]. The proof uses the fact that (2.2), (2.3) is a conservative system and uses the first integral relation

$$n - \log n - 1 = \int_{E_\infty}^{E} (v(y) - v(E_\infty))\, dy\,. \tag{2.6}$$

Proof of Theorem 2.1. By Lemma 2.2 it is sufficient to prove that the relation between E_∞ and U is one-to-one. With the solution (n, E) of (2.1)–(2.3) for given $E_\infty \in (E_{sat}, 1)$, we define

$$\mathcal{U}(E_\infty) := \int_{\mathbb{R}} (E(\xi) - E_\infty) d\xi\,.$$

The derivative can be written as $\mathcal{U}' := \int_{\mathbb{R}} (\hat{E}(\xi) - 1) d\xi$, where we define $\hat{E} = dE/dE_\infty$ and $\hat{n} = dn/dE_\infty$. The latter satisfy the equations

$$\hat{E}' = \hat{n}\,, \qquad \frac{n-1}{n}\hat{n} = (v(E) - v(E_\infty))\hat{E} - v'(E_\infty)(E - E_\infty)\,,$$

by differentiating (2.3) and (2.6) with respect to E_∞. Let us, without loss of generality, fix the point where $n-1$ changes sign at $\xi = 0$, i.e., $n(0) = 1$. The second equation above implies that

$$\hat{E}(0) = v'(E_\infty)\frac{E(0) - E_\infty}{v(E(0)) - v(E_\infty)}\,.$$

The properties of v, $E_\infty < 1$, and $E > E_\infty$ imply that $\hat{E}(0) < 1$. Away from $\xi = 0$, $\hat{E}$ solves

$$\begin{aligned} \hat{E}' = &\frac{n}{n-1}[v(E) - v(E_\infty)](\hat{E} - 1) \\ &+ \frac{n}{n-1}[v(E) - v(E_\infty) - v'(E_\infty)(E - E_\infty)]\,. \end{aligned}$$

The term in the second line is negative for large negative ξ and positive for large positive ξ. This implies $\hat{E} < 1$ for large $|\xi|$. Extrema of $\hat{E}$ away from $\xi = 0$ satisfy $\hat{E} = v'(E_\infty)\frac{E - E_\infty}{v(E) - v(E_\infty)} < 1$ analogously to the above. This shows that $\hat{E}(\xi) < 1$ for all ξ and, thus, $\mathcal{U}'(E_\infty) < 0$.

The assertion (2.5) then also follows since the amplitude of the wave tends to zero for $E_\infty \to 1$ and to infinity for $E_\infty \to E_{sat}$. ∎

3. Existence. In this section existence of solutions of (1.15), (1.16), (1.18) will be proven for given bounded $U(t) \in C_B^1(\mathbb{R}_+)$ and for initial data e_I satisfying

$$e_I \in L_x^1(\mathbb{R}) \cap L_x^\infty(\mathbb{R}), \quad e_I(x) > 0 \text{ a.e. in } x\,. \tag{3.1}$$

Clearly $U(t)$ is fixed by $U(0) = \int_{\mathbb{R}} e_I(x)\,dx > 0$ and by $U'(t)$ appearing in (1.18).

Assumption 2. There are positive constants δ and K, such that

$$0 < \delta \le U(t) \le K \quad \text{and} \quad \|e_I\|_\infty \le K\,,$$

where $\|\cdot\|_p$ denotes the norm in $L_x^p(\mathbb{R})$.

The derivative $U'(t)$ will have to be small enough as specified below. We start by the derivation of an a priori estimate.

PROPOSITION 3.1. *For solutions of* (1.15), (1.16), (1.18), $\|e(t,\cdot)\|_\infty \le C(\sigma_1)K$ *with* $C(\sigma_1) = \sqrt{2}\max\{2, c\sqrt{\sigma_1}\}$ *holds for all* $t \ge 0$.

Proof. The proof follows the idea of a similar result in [7]. Multiplying (1.15) by e^{p-1} for $p \ge 2$ and integration gives the estimate

$$\frac{d}{dt}\int_{\mathbb{R}} e^p\,dx \le -4\frac{(p-1)}{p}\int_{\mathbb{R}} (\partial_x e^{p/2})^2\,dx + p\sigma_1 \int_{\mathbb{R}} e^p\,dx\,. \tag{3.2}$$

We observe that, by interpolation,

$$\|e_I\|_p \le \|e_I\|_\infty^{(p-1)/p}\,\|e_I\|_1^{1/p} \le K\,.$$

Our aim is to derive a uniform-in-p and uniform-in-time estimate on $\|e(t,\cdot)\|_p$ for a sequence of p such that $p \to \infty$. We use the Nash inequality [10]

$$\|u\|_2^3 \le c\|u\|_1^2 \|\partial_x u\|_2$$

in one space dimension with $u = e^{p/2}$; thus, with the notation $z_p(t) = \|e(t,\cdot)\|_p^p$,

$$\frac{dz_p}{dt} \le p\sigma_1 z_p \left(1 - \frac{\tilde{c}(p-1)}{p^2}\frac{z_p^2}{z_{p/2}^4}\right), \tag{3.3}$$

where $\tilde{c} = 4/(c^2\sigma_1)$. Starting with $z_1(t) = U(t) \le K$, the above inequality can be used recursively for obtaining bounds M_k for $z_{2^k}(t)$. Suppose $z_{2^{k-1}}(t) \le M_{k-1}$; then

$$z_{2^k}(t) \le M_k = \max\left\{K^{2^k}, \frac{2^k}{\sqrt{\tilde{c}(2^k-1)}}M_{k-1}^2\right\}\,.$$

Let us now examine the sequence M_k, defined by the recursion and by $M_0 = K$. Since, obviously, $M_{k-1} \ge K^{2^{k-1}}$ and $2^k/\sqrt{2^k-1} \ge 1$,

$$K^{2^k} \le \frac{2^k}{\sqrt{2^k-1}}M_{k-1}^2$$

holds. Thus, we make the upper bound M_k larger by the new definition

$$M_k = B2^{(k+1)/2}M_{k-1}^2\,, \quad M_0 = K\,, \quad B := \max\{1, \tilde{c}^{-1/2}\}\,,$$

where we have used $2^k/\sqrt{2^k-1} \le 2^{(k+1)/2}$. This recursion can be solved explicitly:

$$M_k = (\sqrt{2}\,B)^{a_k} 2^{b_k/2} K^{2^k}\,,$$

where $a_k = \sum_{n=0}^{k-1} 2^n = 2^k - 1 < 2^k$ and $b_k = \sum_{n=0}^{k-1}(k-n)2^n = 2^{k+1} - 2 - k < 2^{k+1}$. Thus, since $B \geq 1$,

$$M_k \leq (2\sqrt{2}\,BK)^{2^k}\,,$$

and hence

$$\|e(t,\cdot)\|_{2^k} \leq \sqrt{2}\,K\,\max\{2, c\sqrt{\sigma_1}\} \quad \text{for all } k\,.$$

The proof is completed by passing to the limit $k \to \infty$. ∎

Now we prepare a decoupled solution approach and examine (1.18) as an equation for $E_\infty(t)$.

PROPOSITION 3.2. *Let the function $e \in L_x^1(\mathbb{R}) \cap L_x^\infty(\mathbb{R})$ satisfy $\|e\|_1 \geq \gamma > 0$ and $\|e\|_\infty \leq M$. Then the function $F(E;e) := \int_{\mathbb{R}}(v(E) - v(E+e(x)))dx$ is strictly increasing on $(0,\bar{E})$ with*

$$\bar{E}(\gamma, M) = 1 - \frac{v'(1+M)\gamma}{2M^2\sigma_3} > 1\,.$$

Furthermore,

$$F(0;e) \leq -v(M)\frac{\gamma}{M}\,, \quad F(\bar{E};e) \geq \frac{3v'(1+M)^2\gamma^2}{8M^3\sigma_3}\,,$$

$$F'(E;e) \geq -\frac{v'(1+M)\gamma}{2M} \quad \text{for } 0 \leq E \leq \bar{E}\,.$$

Proof. By the convexity of v' on $(0,E_i)$ and by the fact that v' is increasing and negative on (E_i,∞), the secant between E and $E+M$ lies above the graph of v' for $E \leq 1$. Therefore

$$\begin{aligned} F'(E) &\geq \int_{\mathbb{R}} \left(v'(E) - v'(E)\left(1 - \frac{e}{M}\right) - v'(E+M)\frac{e}{M}\right)dx \\ &= \int_{\mathbb{R}} (v'(E) - v'(E+M))\frac{e}{M}dx \geq (v'(E) - v'(E+M))\frac{\gamma}{M} \end{aligned}$$

for $0 \leq E \leq 1$. Again by the same properties of v', the right-hand side takes its minimum value for $E = 1$, so $F'(E) \geq -v'(1+M)\gamma/M$ for $0 \leq E \leq 1$.

Since

$$|F''(E)| \leq \int_{\mathbb{R}} |v''(E) - v''(E+e)|dx \leq \sigma_3 M\,,$$

the derivative of F for $E > 1$ can be estimated by

$$F'(E) \geq -v'(1+M)\frac{\gamma}{M} - (E-1)\sigma_3 M\,,$$

proving that F is increasing on $(0,\bar{E})$ and the lower bound on F' in the statement of the proposition. The lower bound for $F(\bar{E})$ is obtained by integrating the above inequality from $E = 1$ to $E = \bar{E}$ and using that $F(1) > 0$, which holds, obviously, since v has its maximum at $E = 1$.

For estimating $F(0) = -\int_{\mathbb{R}} v(e)dx$, we use the L^∞-bound on e and the fact that secants between the origin and other points on the graph of v lie below the graph by the properties of v:

$$F(0) \leq -\int_{\mathbb{R}} v(M)\frac{e}{M}dx \leq -v(M)\frac{\gamma}{M}\,,$$

where the second inequality is due to the lower bound on the L^1-norm of e. ∎

On the other hand, we consider the problem for e with given E_∞. In this case, the integral of e will not necessarily be equal to $U(t)$, which was the basis of the proof of Proposition 3.1. As a consequence, the estimates below are not uniform in time.

PROPOSITION 3.3. *Let $E_\infty(t)$ be given. Then the problem* (1.15), (1.16) *for e has a unique positive solution satisfying*

$$\int_{\mathbb{R}} e(t,x)dx \ge U(0)e^{-t\sigma_1} \quad \text{and} \quad e(t,x) \le Ke^{t\sigma_1}\,, \quad x \in \mathbb{R},\ t > 0\,.$$

Proof. Existence and uniqueness are standard results for semilinear parabolic equations. Positivity is a consequence of the maximum principle. The first estimate follows easily from integration of (1.15). The upper bound in the second estimate is a supersolution. $\square$

We are now ready to formulate the main existence result.

THEOREM 3.4. *Let $M = C(\sigma_1)K$ denote the bound from Proposition* 3.1 *and let*

$$-v(M)\frac{\delta}{M} < U'(t) < \frac{3v'(1+M)^2\delta^2}{8M^3\sigma_3}\,, \quad t \ge 0\,.$$

Then the problem (1.15)–(1.18) *has a unique global solution satisfying* $0 < E_\infty(t) < \bar{E}(\delta, M)$ *and* $0 < e(t,x) \le M$.

Remark 3.5. It seems unsatisfactory that the bounds on $U(t)$ (in Assumption 2) and on its derivative (in the formulation of the theorem) are required. However, examples of nonexistence of a solution for data violating such bounds are easily constructed. The range of the function $F(E_\infty, e(t,\cdot))$ (the right-hand side of (1.18)) as a function of E_∞ is a subset of $(-\sigma_1 U(t), \sigma_1 U(t))$. Therefore it is a necessary condition for the existence of a solution that $U'(t)$ lies in this interval for all t. The more restrictive bounds of the theorem guarantee stable (unique) solvability. For an example of nonexistence see the following section.

Proof. The first step is the construction of a local solution by a fixed point iteration on E_∞ acting on the set $\mathcal{E} := \{E(t) \in L_t^\infty((0,T)) :\ 0 \le E(t) \le \bar{E}\}$ with $T > 0$. For a given $E \in \mathcal{E}$, we first solve the problem (1.15), (1.16) with E_∞ replaced by E. By Proposition 3.3, this problem has a unique solution $e[E]$ satisfying

$$Ke^{t\sigma_1} \ge \int_{\mathbb{R}} e[E](t,x)dx \ge U(0)e^{-t\sigma_1} \ge \delta e^{-T\sigma_1} =: \gamma_T$$

and

$$e[E](t,x) \le Ke^{t\sigma_1} \le Me^{T\sigma_1} =: M_T$$

for $0 \le t \le T$. With Proposition 3.2, the range of $F(\cdot\,; e[E])$ includes the interval $(-v(M_T)\frac{\gamma_T}{M_T}, \frac{3v'(1+M_T)^2\gamma_T^2}{8M_T^3\sigma_3})$. For T small enough, this in turn includes the range of $U'(t)$ as given in the formulation of the theorem. Therefore the equation $F(\hat{E}; e[E]) = U'$ has a unique solution $\hat{E} = \mathcal{F}(E) \in [0, \bar{E}]$ which completes the definition of the fixed point operator $\mathcal{F} :\ \mathcal{E} \to \mathcal{E}$.

We shall prove that, for T small enough, $\mathcal{F}$ is a contraction and start with the mild formulation of (1.15), (1.16):

$$\begin{aligned} e(t,\cdot) = G(t,\cdot) * e_I &+ \int_0^t \partial_x G(t-s,\cdot) * [V(E(s)+e(s,\cdot)) - V(E(s))]ds \\ &+ \int_0^t G(t-s,\cdot) * [v(E(s)) - v(E(s)+e(s,\cdot))]ds\,, \end{aligned}$$

where $G(t,x) = (4\pi t)^{-1/2} e^{-x^2/(4t)}$ is the fundamental solution of the one-dimensional heat equation, $*$ denotes convolution with respect to x, and V is a primitive of v. For estimating the difference between $e_1 = e[E_1]$ and $e_2 = e[E_2]$, we start with

$$\begin{aligned}
&|v(E_1) - v(E_1 + e_1) - v(E_2) + v(E_2 + e_2)| \\
&\le \left| \int_{E_2}^{E_1} (v'(E) - v'(E + e_1)) dE \right| + |v(E_2 + e_2) - v(E_2 + e_1)| \\
&\le e_1\, \sigma_2 |E_1 - E_2| + \sigma_1 |e_1 - e_2| \,,
\end{aligned}$$

and, analogously,

$$\begin{aligned}
&|V(E_1) - V(E_1 + e_1) - V(E_2) - V(E_2 + e_2)| \\
&\le e_1 \sigma_1 |E_1 - E_2| + \sigma_0 |e_1 - e_2| \,.
\end{aligned}$$

We shall also use the properties

$$\int_{\mathbb{R}} G(t,x)\, dx = 1 \,, \qquad \int |\partial_x G(t,x)|\, dx = \frac{1}{\sqrt{t\pi}} \quad \text{for all } t > 0$$

of the fundamental solution as well as the convolution inequality $\|f * g\|_1 \le \|f\|_1 \|g\|_1$. A combination of these ingredients leads to an estimate of the form

$$\begin{aligned}
&\sup_{0<t<T} \|e_1(t,\cdot) - e_2(t,\cdot)\|_1 \\
&\le c\sqrt{T} \left(\sup_{0<t<T} \|e_1(t,\cdot) - e_2(t,\cdot)\|_1 + \sup_{0<t<T} |E_1(t) - E_2(t)| \right)
\end{aligned}$$

for $T \le 1$. It is an obvious consequence that the map $E \mapsto e[E]$ from $\mathcal{E}$ to $L_t^\infty((0,T), L_x^1(\mathbb{R}))$ is Lipschitz continuous with an arbitrarily small Lipschitz constant for small enough T.

Denoting $\hat{E}_1 = \mathcal{F}(E_1)$ and $\hat{E}_2 = \mathcal{F}(E_2)$, then $F(\hat{E}_i; e_i) = U'(t)$ holds for $i = 1, 2$. The difference of the two equations can be written as

$$F'(\tilde{E}; e_1)(\hat{E}_1 - \hat{E}_2) + \int_{\mathbb{R}} [v(\hat{E}_2 + e_2) - v(\hat{E}_2 + e_1)] dx = 0 \,,$$

with $\tilde{E}$ between $\hat{E}_1$ and $\hat{E}_2$. This implies the estimate

$$\sup_{0<t<T} |\hat{E}_1(t) - \hat{E}_2(t)| \le -\frac{2M\sigma_1}{v'(1+M)\gamma} \sup_{0<t<T} \|e_1(t,\cdot) - e_2(t,\cdot)\|_1 \,,$$

proving Lipschitz continuity also for the second step of the fixed point map. This concludes the proof of existence and uniqueness of a local solution.

Since solutions satisfy the uniform-in-time bounds $0 < e \le M$ and $\int_{\mathbb{R}} e\, dx \ge \delta$ and the above construction of local solutions works for initial conditions satisfying these bounds, the solution actually exists for all times, concluding the proof. ◻

4. Numerical results. In this section we present numerical experiments approximating (1.1)–(1.4) by solving the initial value problem for (1.15) subject to (1.18). In the time iteration we solve alternatively (1.18) and (1.15); for a given bounded positive initial condition e_I with finite mass we find the corresponding initial value of E_∞

by solving (1.18), this value is then used in (1.15) to get e in the next time step, and so on.

We discretize the equations on a domain $(0, L)$ and impose Neumann boundary conditions for (1.15). The scheme treats the second order term implicitly (backward Euler) and the first order term explicitly (forward Euler) in time. Also, the first order term is discretized in space by first order upwinding. For a given $U'(t)$ we approximate the integral (1.18) in the interval $[0, L]$ as a Riemann integral by using the trapezoidal rule. At each time step k a unique solution of the discretized equation

$$\int_0^L \{v(E_\infty^{k+1}) - v(E_\infty^{k+1} + e^k)\}dx - U'(t_k) = 0$$

is achieved by using the MATLAB implemented routine fzero, where the starting guess is E_∞^k.

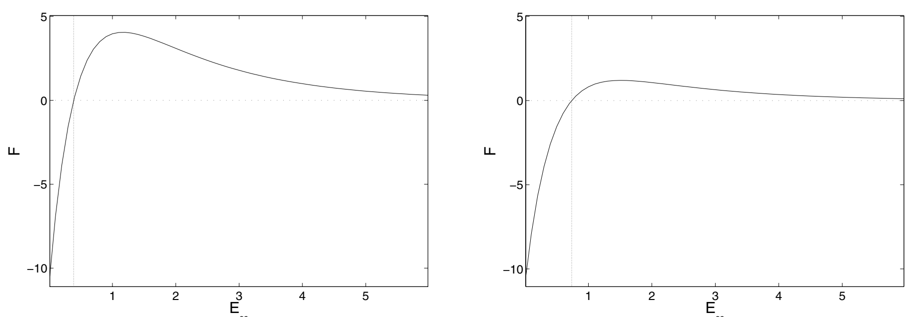
In all examples below we have taken $L = 200$, the spatial step $h = 0.1$, and the time step $\tau = 0.01$. As electron velocity function we use

$$v(E) = c\,e^{-a\,E} - d\,e^{-b\,E} + 1\,, \tag{4.1}$$

with

$$a = \ln(6)/3\,,\ b = 4\ln(6)/3\,,\ c = 2,\ \text{and}\ d = 3.$$

This v is normalized according to Assumption 1.



(a) $F(E_\infty, e_I)$ for e_I as in (4.2) with $l = 5$. (b) $F(E_\infty, e_I)$ for e_I as in (4.2) with $l = 1$.

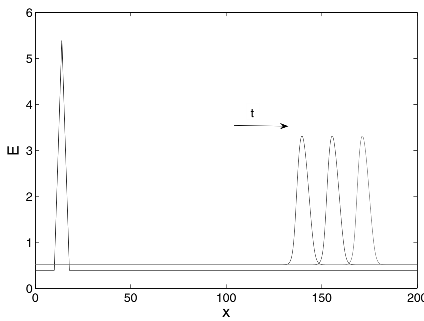
FIG. 2. *The function F computed for the initial data* (4.2).

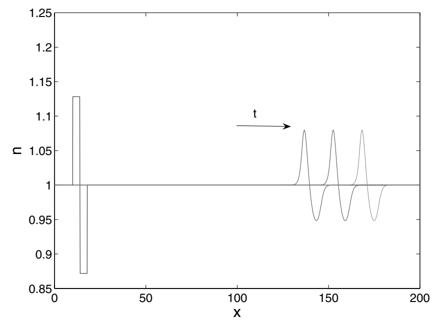
As initial condition we take the piecewise linear function

$$e_I(x) = \begin{cases} 0 & \text{if } 0 \le x \le 10 \text{ or } x > 18, \\ \frac{l}{4}x - \frac{5}{2}l & \text{if } 10 < x \le 14, \\ -\frac{l}{4}x + \frac{9}{2}l & \text{if } 14 < x \le 18; \end{cases} \tag{4.2}$$

here l is the maximum of e_I giving the initial voltage $U(0) = 4l$. The function $E_\infty \to F(E, e_I)$ for e_I with $l = 1$ and $l = 5$, respectively, is plotted in Figure 2(a). Observe that the values at which F vanishes are, respectively, $E_\infty(0) \approx 0.77$ and $E_\infty(0) \approx 0.37$; i.e., the smaller the integral of e, the closer is E_∞ to 1, as expected for solitary waves (see Theorem 2.1). Since the speed of the solitary waves is given by $c = v(E_\infty)$, we expect the profiles to move to the right faster for smaller values of l. From now on we take $l = 5$ in (4.2); in this case $U(0) = 20$.

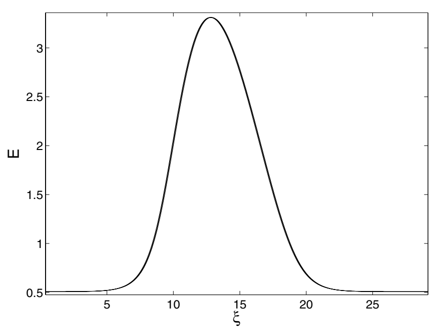
We start with examples for constant U. Figures 3(a) and 3(b) show, respectively, electric field and electron concentration profiles at $t = 0$ and $t = 90, 100, 110$. Figures 3(c) and 3(d) show the same solutions against the moving variable $\xi = x - ct$, where the speed $c = v(E_\infty(t))$ is evaluated at $t = 110$. The profiles at times $t = 90, 100, 110$ overlap in this frame, suggesting the stability of solitary waves.

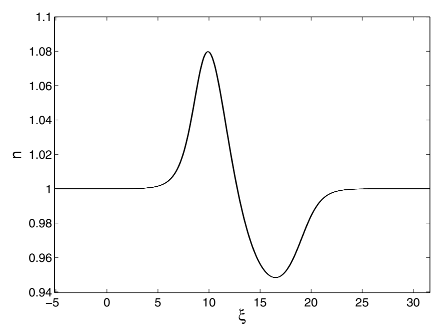




(a) The electric field profile $E = e + E_\infty$ at $t = 0, 90, 100, 110$.

(b) The electron concentration profile n at $t = 0, 90, 100, 110$.





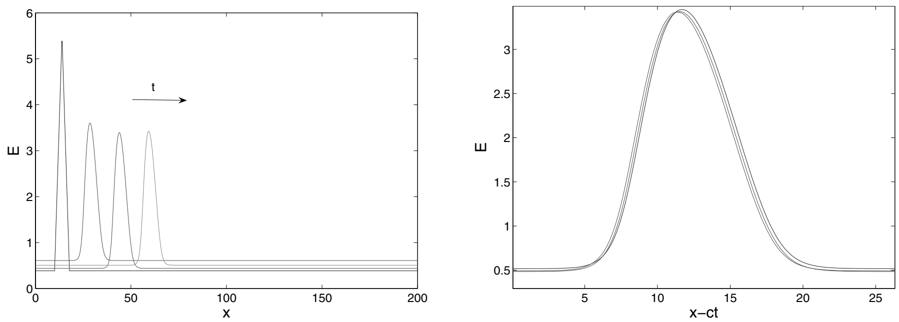
(c) The electric field profile $E = e + E_\infty$ at $t = 90, 100, 110$ against the travelling wave coordinate ξ.

(d) The electron concentration profile n at $t = 90, 100, 110$ against ξ.

FIG. 3. *Numerical solutions for constant U. Figures* 3(a) *and* 3(c) *show electric field values, and for completeness those corresponding to the electron concentration are shown to the right in Figures* 3(b) *and* 3(d). *The wave speed used above has been computed by using the value of E_∞ at $t = 110$; here $E_\infty(110) \approx 0.5$ and $c \sim v(E_\infty) \approx 1.58$.*

For nonconstant U we first choose $U'(t) = 4\sin(4t)/(1 + t/10)$. Thus initially $U'(0) = 0$, and $U'(t)$ oscillates about this value, while the amplitude of the oscillations decays to 0 as $t \to \infty$. Figure 4(a) shows electric field profiles initially and at times $t = 10, 20, 30$. In Figure 4(b) electric field profiles are shown at times $t = 90, 80, 110$ against the variable $x - ct$. Since $U(t) \to const.$ as $t \to \infty$, we have taken $c = v(E_\infty(t))$ for $t = 110$, as before. The profiles now do not overlap precisely, but are fairly close to each other, again suggesting convergence to a solitary wave as $t \to \infty$ with wave speed $c = \lim_{t\to\infty} v(E_\infty(t))$.

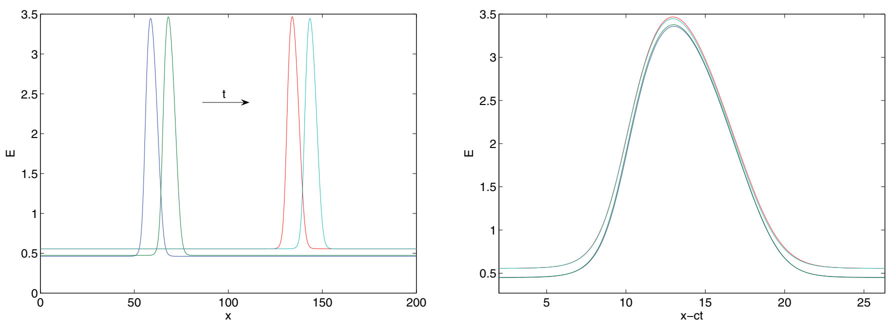
We now consider a t-periodic U, simply choosing $U'(t) = \sin(t)$. Although U does not approach a constant value as $t \to \infty$ and convergence to solitary waves is not expected, the solution profiles move to the right with an apparently constant speed. Figure 5(a) shows the solution profiles at $t = 31, 37$ (left) and at $t = 79, 85$ (right), i.e., profiles at, roughly, the beginning and the end of two time periods. The two



(a) Electric field profile at $t = 0, 10, 20, 30$.

(b) Electric field profile at $t = 90, 100, 110$ against $x - ct$ with $c = v(E_\infty(110)) \approx 1.55$.

FIG. 4. *Numerical solutions with* $U'(t) = 4\sin(4t)/(1 + t/10)$. *Only electric field profiles are shown. Figure* 4(a) *shows profiles at early time steps, where the amplitude of the oscillations of* $U'(t)$ *is appreciated. In Figure* 4(b) *late time steps are shown in the moving frame* ξ.



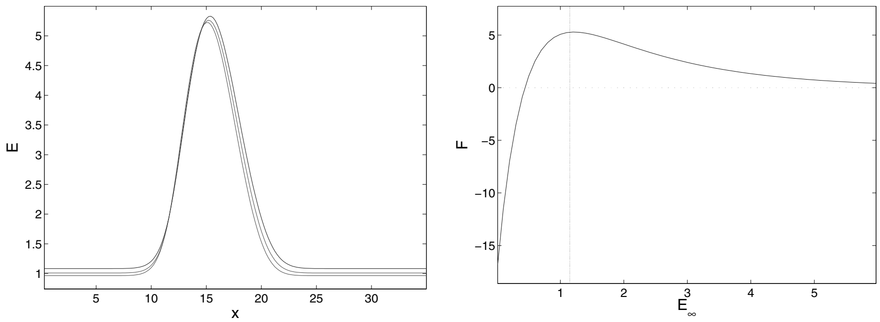
(a) Electric field profile at $t = 31, 37$ (left) and at $t = 79, 85$ (right).

(b) Electric field profile at against $x - ct$ with average speed $c \approx 1.57$.

FIG. 5. *Numerical solutions with* $U'(t) = \sin(t)$. *Only electric field profiles are shown. Figure* 5(a) *shows profiles at times* $t = 31, 37$ *(left) and at* $t = 79, 85$ *(right). Figure* 5(b) *shows the same profiles as Figure* 5(a) *against the coordinate* $x - ct$ *with the average speed* $c = \sum_{t_k=50}^{100} v(E_\infty(t_k))/5000 \approx 1.57$.

profiles to the left are almost a translation of each other, so are the two profiles on the right. This indicates that, as $t \to \infty$, a t-periodic "translating speed" is reached, presumably given by $c = v(E_\infty(t))$. To support this idea, we have computed the "averaged" speed of the solution at late time steps, including at least one period, namely $c = \sum_{t_k=50}^{100} v(E_\infty(t_k))/5000 \approx 1.57$. Figure 5(b) shows well-centered profiles against the moving coordinate with the average speed; these are at times $t = 51, 57$ and at $t = 79, 85$ (on top).

Finally, as an illustration of nonexistence we take $U'(t) = t^2 + 3.8$, so that initially U' is close to the maximum of F; see Figure 2(b). In this case the (numerical) solution ceases to exist at $t = 1.23$; i.e., $U'(1.22)$ exceeds the maximum of F. Electric field profiles for $t < 1.23$ are shown in Figure 6(a). The function F for e at $t = 1.2$ is shown in Figure 6(b). Observe that the maximum of F is approximately attained at $E_\infty = 1.1$ and that the solution (e, E_∞) has $E_\infty(1.2) \approx 1.084$.



(a) Electric field profile at $t = 1, 1.1, 1.2$. (b) The function F for e at $t = 1.2$.

FIG. 6. *Numerical solutions for* $U'(t) = t^2 + 3.8$ *and the function* F *for* e *evaluated at* $e(1.2, x)$. *In this case the numerical solution ceases to exist at* $t = 1.23$ *when the value of* $U'(t)$ *exceeds the maximum of* F.

5. Small wave limit: Linearized stability. In this section we prove linearized stability of *small* solitary waves. We consider a small given constant voltage:

$$U = \varepsilon \ll 1 .$$

We derive the limit $\varepsilon \to 0$ formally. From Theorem 2.1, solitary waves have $E_\infty \sim 1$ as $\varepsilon \to 0$, hence also $c \sim v(1)$ as $\varepsilon \to 0$. The amplitude of the waves is also small by (2.6). With this in mind we introduce the moving coordinate $\xi = x - v(1)t$ and the scaling

$$e = \varepsilon^2 e_1 \, , \ E_\infty = 1 - \varepsilon^2 E_1 \, , \ \tau = \varepsilon^2 t \, , \ \eta = \varepsilon \xi \, .$$

Then, in (1.15), after dividing by ε^4 and formally passing to the limit $\varepsilon \to 0$, we obtain

$$\partial_\tau e_1 = \partial_\eta^2 e_1 + \frac{v''(1)}{2}(2E_1 e_1 - e_1^2) \tag{5.1}$$

and, from (1.17) and (1.18),

$$\int_{\mathbb{R}} e_1 \, d\eta = 1 \, , \quad 2E_1 = \int_{\mathbb{R}} e_1^2 \, d\eta \, . \tag{5.2}$$

As mentioned in the introduction, problem (5.1)–(5.2) is the conserved Fisher equation; see [11]. We now look at stability of stationary solutions to (5.1), since these are the limiting profiles of solitary waves as $\varepsilon \to 0$.

With the abbreviation

$$\kappa := -v''(1) > 0 \, ,$$

the family of stationary solutions is given explicitly by

$$\bar{e}(\eta) = \frac{\kappa}{48} \operatorname{sech}^2\left(\frac{\kappa}{24}(\eta + C)\right) \, , \quad \bar{E} = \frac{\kappa}{144} \, , \tag{5.3}$$

with the shift $C \in \mathbb{R}$.

We observe that rescaling with

$$\eta \to \kappa^{-1}\eta\,,\ e_1 \to \kappa e_1\,,\ E_1 \to \kappa E_1\,,\ \tau \to \kappa^{-2}\tau\,,$$

we can set $\kappa = 1$ in (5.1), with no changes in (5.2).

Denoting perturbations of e_1 and E_1 by u and A, respectively, the linearized problem (with $\kappa = 1$) reads

$$\partial_\tau u = \partial_\eta^2 u + (\bar e - \bar E)u - \bar e A[u]\,, \tag{5.4}$$

$$\int_{\mathbb{R}} u\, d\eta = 0\,, \qquad A[u] = \int_{\mathbb{R}} \bar e\, u\, d\eta\,, \tag{5.5}$$

with

$$\bar e(\eta) = \frac{1}{48}\operatorname{sech}^2\left(\frac{\eta}{24}\right)\,, \quad \bar E = \frac{1}{144}\,, \tag{5.6}$$

where, without loss of generality, the shift has been set to zero. Note that there is a one-dimensional family of stationary solutions spanned by $u = \bar e'$, $A = 0$. This fact corresponds to the translation invariance of the nonlinear problem.

THEOREM 5.1. *The family of stationary solutions of* (5.4), (5.5) *is asymptotically stable: for an initial condition u_0 satisfying*

$$\int_{\mathbb{R}} u_0\, \bar e'\, d\eta = 0\,,$$

the solution of (5.4), (5.5) *subject to $u(\tau = 0) = u_0$ satisfies*

$$\|u(\tau,\cdot)\|_2 \le e^{\mu\tau}\|u_0\|_2 \quad \textit{with} \quad \mu \le -\frac{1}{192} < 0\,.$$

Proof. The linearized operator can be written as the sum of two self-adjoint operators on the space $L_0^2(\mathbb{R}) = \{u \in L^2(\mathbb{R}) :\ \int_{\mathbb{R}} u\, d\eta = 0\}$ equipped with the L^2-inner product $\langle\cdot,\cdot\rangle$:

$$\mathcal{L}u = \mathcal{L}_1 u + \mathcal{L}_2 u\,, \qquad \mathcal{L}_1 u = \partial_\eta^2 u + (\bar e - \bar E)u\,,\ \mathcal{L}_2 u = -\bar e A[u]\,.$$

Obviously, $\mathcal{L}_2$ is nonpositive: $\langle \mathcal{L}_2 u, u\rangle = -A[u]^2 \le 0$.

The spectrum of $\mathcal{L}_1$ considered on all of $L^2(\mathbb{R})$ can be computed explicitly; see [6]: we obtain the essential spectrum $(-\infty, -\bar E]$ and the isolated eigenvalues

$$\lambda_1 = -\frac{3}{4}\bar E = -\frac{1}{192}\,, \quad \lambda_2 = 0\,, \quad \lambda_3 = \frac{5}{4}\bar E\,.$$

This can be obtained by using (5.6) and transforming the linear eigenvalue problem for $\mathcal{L}_1$ into a hypergeometric equation; see [3] for details. In the computation of λ_1 we also used (5.6).

The eigenfunction corresponding to λ_3 has $\int_{\mathbb{R}} u\, d\eta \ne 0$, since, according to the Sturm–Liouville theory (see, e.g., [2]), the eigenfunction corresponding to the largest eigenvalue does not change sign. This implies that in the restricted space $L_0^2(\mathbb{R})$ we actually have $\langle \mathcal{L}_1 u, u\rangle \le 0$.

Finally, $\bar e'$ is the eigenfunction corresponding to λ_2 and $\ker(\mathcal{L}_1) = \operatorname{span}\{\bar e'\}$. If P is the spectral projection onto $\ker(\mathcal{L}_1)$ then for $u \in L^2(\mathbb{R})$ satisfying (5.5), we have

$$\langle \mathcal{L}_1(I-P)u, (I-P)u\rangle \le \lambda_1 \|(I-P)u\|_2^2\,. \tag{5.7}$$

Since $\mathcal{L}_1$ is self-adjoint, P can be expressed as $Pu = \langle u, \bar e'\rangle\, \bar e'$.

By choosing the initial condition u_0 of (5.4), (5.5) such that $Pu_0 = 0$, it is easily checked that also $Pu = 0$ for all $t > 0$, which finishes the proof. □

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