WEAK SHOCKS OF A BGK KINETIC MODEL FOR ISENTROPIC GAS DYNAMICS

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Abstract. We consider a one-dimensional BGK model as a regularisation for the isentropic system of gas dynamics. Existence and dynamic stability of small amplitude travelling waves of the kinetic transport equation are proven. Their macroscopic moments approximate viscous shock profiles for the isentropic system. These results are also extended to the isothermal case.

1. Introduction

We analyse the existence and stability of small amplitude travelling wave solutions for the one-dimensional BGK equation

$$\partial_t f + v\partial_x f = M(\rho_f, m_f, v) - f, \qquad \text{with } t > 0,\ x \in \mathbb{R},\ v \in \mathbb{R}, \tag{1.1}$$

where $f(t, x, v)$ is a density of particles moving with velocity v at the time-space position (t, x). The functions $\rho_f(t, x)$ and $m_f(t, x)$ denote the macroscopic density and momentum corresponding to the distribution f, i.e. the zeroth and first order moments with respect to the velocity v

$$\rho_f(t, x) = \int f(t, x, v)dv, \qquad m_f(t, x) = \int v f(t, x, v)dv.$$

Here and in the following integrations with respect to v are over $\mathbb{R}$.
We consider a class of Maxwellians, which has been introduced by Lions, Perthame and Tadmor in [12]:

$$M(\rho, m, v) = d\left(\frac{1+2\alpha}{\alpha}\rho^{2\alpha} - \left(v - \frac{m}{\rho}\right)^2\right)_+^{\beta}, \tag{1.2}$$

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where

$$\beta = \frac{1-\alpha}{2\alpha}\,, \quad d = \frac{1}{J_\beta}\left(\frac{1+2\alpha}{\alpha}\right)^{-\frac{1}{2\alpha}}\,, \quad J_\beta = \int_{-1}^{1}(1-z^2)^\beta dz = \frac{\sqrt{\pi}\,\Gamma(\beta+1)}{\Gamma(\beta+3/2)}\,,$$

and $0 < \alpha < 1$, see e.g. also [3], [1]. The equilibrium distributions satisfy the moment conditions

$$\int M(\rho,m,v)dv = \rho\,, \qquad \int vM(\rho,m,v)dv = m\,, \tag{1.3}$$

$$\int v^2 M(\rho,m,v)dv = \frac{m^2}{\rho} + \rho^{1+2\alpha} \quad =: P(\rho,m)\,, \tag{1.4}$$

$$\int v^3 M(\rho,m,v)dv = \frac{m^3}{\rho^2} + 3\rho^{2\alpha}m\,. \tag{1.5}$$

Therefore we obtain for the collision term

$$\int \begin{pmatrix} 1 \\ v \end{pmatrix} (M(\rho_f,m_f,v) - f)\, dv \;= \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{1.6}$$

implying the conservation of mass and momentum:

$$\begin{aligned} \partial_t\rho_f + \partial_x m_f &= 0, \\ \partial_t m_f + \partial_x P_f &= 0, \qquad \text{with } P_f := \int v^2 f dv. \end{aligned}$$

With this notation $P_{M(\rho,m)} = P(\rho,m)$. A connection between the kinetic equation and continuum mechanics models can be established by the macroscopic limit, based on the rescaling $(t,x) \to (t/\epsilon, x/\epsilon)$ with $0 < \epsilon \ll 1$:

$$\epsilon(\partial_t f + v\partial_x f) = M_f - f, \tag{1.7}$$

where here and in the following we use the abbreviation $M_f(t,x,v) := M(\rho_f(t,x), m_f(t,x), v)$. In the macroscopic limit $\epsilon \to 0$, one formally obtains $f(t,x,v) \to M(\rho(t,x), m(t,x), v)$ and the system of isentropic gas dynamics

$$\partial_t \begin{pmatrix} \rho \\ m \end{pmatrix} + \partial_x \begin{pmatrix} m \\ P(\rho,m) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{1.8}$$

A correction of this system can be obtained by the Chapman-Enskog procedure, which amounts to an approximation of the error $f - M_f$ by substituting $M(\rho,m,v)$ in the left hand side of (1.7) and using (1.8) for the evaluation of the time derivatives. This leads to

$$P_f = P(\rho_f, m_f) + P_{f-M_f} \sim P(\rho,m) - \epsilon P_{\partial_t M + v\partial_x M},$$

with

$$\partial_t M + v\partial_x M \sim -\partial_\rho M \partial_x m - \partial_m M \partial_x P(\rho,m) + v\partial_\rho M \partial_x \rho + v\partial_m M \partial_x m.$$

The computation of the second order moment is facilitated by taking the derivatives of (1.4) and (1.5) with respect to ρ and m and leads to

$$P_{f-M_f} \sim -\epsilon\, D(\rho)\, \partial_x\left(\frac{m}{\rho}\right), \qquad \text{with } \; D(\rho) = 2(1-\alpha)\rho^{1+2\alpha},$$

and, thus, to the Navier-Stokes version of isentropic gas dynamics

$$\partial_t \rho + \partial_x m = 0\,, \tag{1.9}$$

$$\partial_t m + \partial_x P(\rho, m) = \epsilon\, \partial_x \left[D(\rho)\, \partial_x \left(\frac{m}{\rho} \right) \right]. \tag{1.10}$$

We recall some properties of (1.8), which for smooth solutions can be written as

$$\partial_t \begin{pmatrix} \rho \\ m \end{pmatrix} + \mathbf{A}(\rho, m) \cdot \partial_x \begin{pmatrix} \rho \\ m \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

with

$$\mathbf{A}(\rho, m) = \begin{pmatrix} 0 & 1 \\ c^2 - u^2 & 2u \end{pmatrix}. \tag{1.11}$$

Here and in the following u denotes the macroscopic velocity and c the speed of sound:

$$u(\rho, m) = \frac{m}{\rho}, \qquad c(\rho) = \sqrt{1 + 2\alpha}\, \rho^{\alpha}.$$

The eigenvalues of $\mathbf{A}$ are given by

$$\lambda_{1/2}(\rho, m) \;\; = \;\; u(\rho, m) \mp c(\rho).$$

For $\rho > 0$ (away from vacuum) $\lambda_1 < \lambda_2$ holds, and the system is strictly hyperbolic. With the corresponding right eigenvectors

$$r_1(\rho, m) = \begin{pmatrix} 1 \\ \lambda_1(\rho, m) \end{pmatrix}, \qquad r_2(\rho, m) = \begin{pmatrix} 1 \\ \lambda_2(\rho, m) \end{pmatrix}$$

one can see that the system is genuinely nonlinear, i.e.:

$$\nabla \lambda_k \cdot r_k \neq 0, \qquad k = 1, 2. \tag{1.12}$$

We consider shock wave solutions of (1.8) of the form

$$(\rho(t,x), m(t,x)) = \begin{cases} (\rho_l, m_l) & \text{for } x < st\,, \\ (\rho_r, m_r) & \text{for } x > st\,. \end{cases}$$

Here s is the shock speed and $(\rho_{l,r}, m_{l,r})$ are the constant left and right states, which (by the theory of nonlinear hyperbolic conservation laws, see e.g. [11]) are related by the Rankine-Hugoniot condition

$$s \begin{pmatrix} \rho_r - \rho_l \\ m_r - m_l \end{pmatrix} = \begin{pmatrix} m_r - m_l \\ P_r - P_l \end{pmatrix}, \tag{1.13}$$

where we denote $P_{l,r} := P(\rho_{l,r}, m_{l,r})$. For a fixed left state (ρ_l, m_l) the Hugoniot locus is defined as the set of all (ρ_r, m_r) such that (1.13) is satisfied for an appropriate s. In a neighbourhood of (ρ_l, m_l) the Hugoniot locus consists of two curves intersecting in (ρ_l, m_l). At (ρ_l, m_l) the k-th curve is tangent to $r_k(\rho_l, m_l)$ and the shock speed s takes the value $\lambda_k(\rho_l, m_l)$ (see, e.g., [10]). If (ρ_r, m_r) lies on the k-th curve of the Hugoniot locus we refer to $\{\rho_{l,r}, m_{l,r}, s\}$ as a k-shock. The Lax entropy condition

$$\lambda_k(\rho_r, m_r) < s < \lambda_k(\rho_l, m_l) \tag{1.14}$$

for k-shocks (see [10]) is a stability condition, which means that the characteristics go into the shock. One can show that the admissibility condition for a 1-shock reduces to

$$\rho_l < \rho_r\,, \tag{1.15}$$

and for a 2-shock to

$$\rho_l > \rho_r \,.$$

For simplicity we only consider 1-shocks, the procedure for 2-shocks is analogous. The entropy conditions can also be obtained from viscous regularisations, see e.g. [8]. Travelling wave solutions of the viscous system are regularisations of admissible shocks. In this paper the regularisation by the kinetic transport equation (1.1) is studied.

A (mathematical) entropy density for the system (1.8) is given by the (physical) energy density

$$\eta(\rho, m) = \frac{1}{2}\frac{m^2}{\rho} + \frac{1}{2\alpha}\rho^{1+2\alpha},$$

which, in the limit of vanishing viscosity, satisfies the entropy inequality

$$\partial_t \eta + \partial_x \Psi \le 0 \tag{1.16}$$

in the weak sense. For the entropy flux

$$\Psi(\rho, m) = \frac{1}{2}\frac{m^3}{\rho^2} + \frac{1+2\alpha}{2\alpha}\rho^{2\alpha} m$$

the relation $\nabla\Psi = \nabla\eta\,\mathbf{A}$ holds. For our choice of Maxwellians (1.2), the convex kinetic entropy corresponding to η is given by

$$H(f, v) = \frac{v^2}{2} f + \frac{1}{2d^{1/\beta}} \frac{f^{1+1/\beta}}{1+\frac{1}{\beta}}. \tag{1.17}$$

The kinetic entropy is related to the macroscopic one by

$$\begin{aligned} \eta(\rho, m) &= \int H(M(\rho, m, v), v)\,dv, \qquad (1.18)\\ \Psi(\rho, m) &= \int v H(M(\rho, m, v), v)\,dv, \end{aligned}$$

and satisfies the minimisation principle

$$\eta(\rho, m) = \min_{\substack{\int f dv = \rho \\ \int v f dv = m}} \int H(f, v)\,dv, \tag{1.19}$$

see Bouchut [1]. Multiplying (1.1) with $\partial_f H(f, v)$ gives

$$\partial_t H(f, v) + v\partial_x H(f, v) = \partial_f H(f, v)\frac{M_f - f}{\epsilon} \le \frac{H(M_f, v) - H(f, v)}{\epsilon},$$

where the last inequality comes from the convexity of H. Integration implies

$$\partial_t \int H(f, v)\,dv + \partial_x \int v H(f, v)\,dv \le 0,$$

which is the microscopic version of (1.16). For a rigorous proof see Berthelin and Bouchut [3].

We are interested in the BGK model as a regularisation of hyperbolic conservation laws and in particular in the construction and the dynamic stability of travelling waves. One of the first authors considering such relaxation approximations for discrete velocity spaces was Natalini [16]. Bouchut [1] gave a general framework for BGK models and concentrated together with Berthelin on the relaxation to the isentropic system in [4], [2], [3]. For small amplitude shock profile solutions of the Boltzmann equation existence was proven by Caflisch and Nicolaenko in [5] and stability by Liu and Yu [14]. Both treatments are perturbative around macroscopic limits and use micro-macro decompositions motivated by the Chapman-Enskog expansion as their main technical tool. In the spirit of these works, Cuesta and Schmeiser [7] studied small amplitude travelling wave solutions of BGK models for scalar conservation laws. In particular a Caflisch-Nicolaenko style micro-macro decomposition is used for proving the existence of kinetic shock profiles as perturbations of viscous Burgers shocks, and a Lyapunov functional in the spirit of Liu and Yu is constructed for proving dynamic stability of these shocks in a L^2 framework, which is related to entropy decay estimates for the linearized problem. In this article we aim to extend the results from the scalar case in [7] to the isentropic system and finally also to the isothermal system. The general framework outlined in [6], provides a modular approach, such that some partial results from [7] can be used here. In the stability analysis, the essential modul is the construction of a Lyapunov functional for the viscous regularization derived by Chapman-Enskog asymptotics. This confronts us with the stability problem for the isentropic Navier-Stokes system, where we were inspired by the work of Matsumura and Nishihara [15] on the system in Lagrangian coordinates. Due to the fact that the viscous Burgers equation is the generic limit problem for small amplitude waves, we are also able to use partial results from [7].

Our main results on existence and stability are presented and proven in Sections 3 and, respectively, 4.

In the remainder of this Section, we present the formal construction of small amplitude kinetic shock profiles and also give an idea of the energy method for proving stability of viscous profiles. In Section 2, linearizations of the collision operator are discussed. An essential tool is the H-theorem involving a weighted L^2-norm, where the weight is an inverse power of the reference equilibrium state. Since equilibrium distributions have compact support, it is essential to construct a reference state with a support large enough to contain the support of the whole shock profile. This technical point is one of the main differences between the present study and earlier work.

Assumptions on the data are also discussed in Section 2. Moment conditions are required for the equilibrium distributions, leading to the restriction $\alpha < 1/13$ or, in other words, the adiabatic exponent $1+2\alpha$ has to be smaller than 15/13.

1.1. **Formal construction of small amplitude kinetic shock profiles.** We look for travelling wave solutions of (1.1), depending on x and t through the travelling wave variable $\xi = x - st$, where s denotes the wave speed. The travelling wave version of (1.1) is

$$(v-s)\partial_\xi f = M_f - f\,, \qquad \text{for } \xi \in \mathbb{R}\,,\ v \in \mathbb{R}\,, \tag{1.20}$$

subject to the far-field conditions

$$f(\pm\infty, v) = M_{r,l}(v), \quad \text{for } v \in \mathbb{R}, \text{ with } M_{r,l}(v) := M(\rho_{r,l}, m_{r,l}, v). \tag{1.21}$$

We only consider small amplitude waves, so that

$$\rho_r - \rho_l \;=\; \epsilon\,, \qquad 0 < \epsilon \ll 1\,, \tag{1.22}$$

where the positivity of ϵ reflects the entropy condition for a 1-shock (1.15). Observe that by computing the zeroth and first order moments in v of (1.20) and then integrating with respect to ξ, we recover the Rankine-Hugoniot relations (1.13). After the macroscopic scaling $\xi \to \xi/\epsilon$ the travelling wave equation becomes

$$\epsilon(v-s)\partial_\xi f = M_f - f\,. \tag{1.23}$$

We repeat the Chapman-Enskog procedure and introduce the micro-macro decomposition

$$f = M_f + \epsilon^2 f^\perp, \tag{1.24}$$

where, due to (1.6),

$$\int f^\perp dv = 0, \qquad \int v f^\perp dv = 0.$$

We compute the zeroth and first order moments in v of (1.23) and integrate in ξ to obtain equations for ρ_f and m_f

$$m_f - m_l \;=\; s(\rho_f - \rho_l), \tag{1.25}$$

$$P(\rho_f, m_f) + \epsilon^2 P_{f^\perp} - P_l \;=\; s(m_f - m_l). \tag{1.26}$$

Since we are considering small amplitude waves, the macroscopic density and momentum are close to their far-field values at $\xi = -\infty$:

$$\rho_f \;=\; \rho_l + \epsilon y_1, \qquad m_f \;=\; m_l + \epsilon y_2, \tag{1.27}$$

where (1.25) implies the relation

$$y_2 = s y_1. \tag{1.28}$$

Using this and comparing $O(\epsilon)$-terms in (1.26) gives the eigenvalue equation $\nabla P_l \cdot (1, s_0)^T = s_0^2$ for the limit s_0 of s as $\epsilon \to 0$. At this point we decide for 1-shocks and set

$$s \;=\; \lambda_{1l} + \epsilon\sigma, \qquad \text{with} \;\; \lambda_{1l} := \lambda_1(\rho_l, m_l).$$

The final equation determining $y_1(\xi)$ will follow from comparing $O(\epsilon^2)$-terms in (1.26). For this purpose we need to approximate $P_{f^\perp}$. Expanding M_f in (1.24) gives

$$f = M_l + \epsilon \mathbf{F}_l \cdot r_{1l}\, y_1 + O(\epsilon^2), \;\; \text{with} \;\; \mathbf{F}_l(v) := \nabla M_l(v) \;\; \text{and} \;\; r_{1l} := r_1(\rho_l, m_l),$$

where here and in the following $\nabla M(\rho, m, v) := \nabla_{(\rho,m)} M(\rho, m, v)$. Substituting this into (1.23) we obtain

$$f^\perp \;=\; -(v - \lambda_{1l})\mathbf{F}_l \cdot r_{1l}\, \partial_\xi y_1 + O(\epsilon). \tag{1.29}$$

The computation of $P_{f^\perp}$ is facilitated by computing the gradient with respect to ρ and m of (1.5):

$$P_{f^\perp} = D_0 \partial_\xi y_1 \tag{1.30}$$

with

$$D_0 := -\int v^2 (v - \lambda_{1l}) \mathbf{F}_l dv \cdot r_{1l} = 2(1-\alpha) c_l \rho_l^{2\alpha} > 0, \quad \text{where } c_l := c(\rho_l). \tag{1.31}$$

Comparing the $O(\epsilon^2)$-terms in (1.13) and (1.26) now gives

$$D_0\partial_\xi y_1 = -2\sigma c_l y_1\,(1-y_1)\,, \quad \text{with } \sigma = -\frac{(\alpha+1)}{2\rho_l}c_l < 0. \tag{1.32}$$

Up to a scaling, this is the travelling wave form of the viscous Burgers equation. Obviously, it has a solution (unique up to shifts) connecting the far-field values $y_1(-\infty) = 0$ and $y_1(\infty) = 1$. Note that (1.32) could also have been derived from the Navier-Stokes system (1.9), (1.10) since

$$u \sim u_l - \epsilon\frac{c_l}{\rho_l}y_1 \quad \text{and} \quad D(\rho)\partial_\xi u \sim -\epsilon D(\rho_l)\frac{c_l}{\rho_l}\partial_\xi y_1 = -\epsilon D_0\partial_\xi y_1.$$

We make this approximation rigorous in Section 3, where we prove the existence of a small amplitude kinetic shock profile satisfying

$$f(\xi,v) = M_l(v) + \epsilon\mathbf{F}_l(v)\cdot r_{1l}\,y_1(\xi) + O(\epsilon^2), \tag{1.33}$$

where y_1 is the viscous Burgers profile solving (1.32).

1.2. **Stability of Navier-Stokes shock profiles.** In this section we shall deal with the stability of travelling waves of a simplified version of the Navier-Stokes system (1.9), (1.10). The simplification concerns the viscosity term, which will be approximated by its linearization around the left far-field state. Introducing the travelling wave variable $\xi = x - st$ then gives

$$\begin{aligned} \partial_t\rho - s\partial_\xi\rho + \partial_\xi m &= 0\,, \\ \partial_t m - s\partial_\xi m + \partial_\xi P(\rho,m) &= \epsilon\tilde{D}_0\partial_\xi^2(m - u_l\rho)\,, \end{aligned}$$

with $\tilde{D}_0 = D(\rho_l)/\rho_l > 0$. Travelling waves of the Navier-Stokes system are steady states $(\rho_\phi(\xi), m_\phi(\xi))$ of this system satisfying (after integration)

$$\begin{aligned} -s(\rho_\phi - \rho_l) + m_\phi - m_l &= 0\,, \\ -s(m_\phi - m_l) + P(\rho_\phi, m_\phi) - P(\rho_l, m_l) &= \epsilon\tilde{D}_0\partial_\xi(m_\phi - u_l\rho_\phi)\,. \end{aligned}$$

Elimination leads to an equation for ρ_ϕ:

$$\epsilon\tilde{D}_0(u_l - s)\partial_\xi\rho_\phi = s^2(\rho_\phi - \rho_l) - \frac{1}{\rho_\phi}[m_l + s(\rho_\phi - \rho_l)]^2 + \frac{m_l^2}{\rho_l} - \rho_\phi^{1+2\alpha} + \rho_l^{1+2\alpha}\,.$$

The entropy condition (1.14) for 1-shocks implies the positivity of the coefficient $u_l - s$. The right hand side is easily seen to be a strictly concave function of ρ_ϕ and, thus, positive between its zeroes $\rho_\phi = \rho_l$ and $\rho_\phi = \rho_r$. This shows that viscous profiles exist for all shocks. The density ρ_ϕ and the velocity $u_\phi = m_\phi/\rho_\phi$ are strictly monotone:

$$\partial_\xi\rho_\phi > 0\,, \qquad \partial_\xi u_\phi = \frac{s\rho_\phi - m_\phi}{\rho_\phi^2}\partial_\xi\rho_\phi = -\frac{\rho_l(u_l - s)}{\rho_\phi^2}\partial_\xi\rho_\phi < 0\,. \tag{1.34}$$

The stability of the shock profiles will be investigated by introducing the primitives of the deviations

$$W_\rho(t,\xi) = \int_{-\infty}^{\xi}[\rho(t,\eta) - \rho_\phi(\eta)]d\eta\,, \quad W_m(t,\xi) = \int_{-\infty}^{\xi}[m(t,\eta) - m_\phi(\eta)]d\eta\,,$$

and assuming for the initial data

$$W_\rho(0,\pm\infty) = W_m(0,\pm\infty) = 0\,.$$

This requires not only appropriate decay of $\rho(t=0)$ and $m(t=0)$ to the far-field values of the travelling wave, but also that the total mass and the total momentum of the deviation vanish initially. Whereas one of these conditions can be satisfied by choosing the shift of the travelling wave accordingly, the other one is a 'well-preparedness' assumption on the initial data, whose violation would make the problem of proving convergence to a travelling wave significantly more complicated (see, e.g., [13]).

The linearized equations for the unknowns W_ρ and W_m read

$$\begin{aligned} \partial_t W_\rho - s\partial_\xi W_\rho + \partial_\xi W_m &= 0\,, \\ \partial_t W_m - s\partial_\xi W_m + \nabla P(\rho_\phi, m_\phi)\cdot(\partial_\xi W_\rho, \partial_\xi W_m) &= \epsilon\tilde{D}_0 \partial_\xi^2 (W_m - u_l W_\rho)\,. \end{aligned}$$

The form of the viscous term suggests to introduce the variable $W_u := W_m - u_l W_\rho$:

$$\partial_t W_\rho + (u_l - s)\partial_\xi W_\rho + \partial_\xi W_u = 0\,, \tag{1.35}$$

$$\partial_t W_u + K_1(\phi)\partial_\xi W_\rho + K_2(\phi)\partial_\xi W_u = \epsilon\tilde{D}_0\partial_\xi^2 W_u\,, \tag{1.36}$$

where $K_1(\phi) := c_\phi^2 - (u_l - u_\phi)^2$ and $K_2(\phi) := u_l - s + 2(u_\phi - u_l)$. With a smallness assumption on the amplitude of the travelling wave we see that $K_1(\phi)$ and $K_2(\phi)$ are close to positive constants. Testing (1.35) with W_ρ and (1.36) with $K_1(\phi)^{-1}W_u$ implies

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int (W_\rho^2 + K_1(\phi)^{-1}W_u^2)d\xi + \int \left(-\partial_\xi \frac{K_2(\phi)}{K_1(\phi)}\right)\frac{W_u^2}{2}d\xi \qquad (1.37) \\ &+\epsilon\tilde{D}_0 \int K_1(\phi)^{-1}(\partial_\xi W_u)^2 d\xi + \epsilon\tilde{D}_0\int (\partial_\xi K_1(\phi)^{-1})W_u\partial_\xi W_u d\xi = 0. \end{aligned}$$

Since $\partial_\xi u_\phi$ and $\partial_\xi \rho_\phi$ have the same decay rate (see (1.34)) we obtain $\partial_\xi K_1(\phi) > 0$ and $\partial_\xi K_2(\phi) < 0$ uniformly in ξ, and hence the second integral in (1.37) has the right sign. We apply Young's inequality to the last term

$$\left|\int (\partial_\xi K_1(\phi)^{-1})W_u\partial_\xi W_u d\xi\right| \le \frac{1}{\sqrt{\epsilon}}\int (-\partial_\xi K_1(\phi)^{-1})\frac{W_u^2}{2}d\xi + \sqrt{\epsilon}C_2\|\partial_\xi W_u\|_{L^2}^2,$$

for a positive constant C_2. For ϵ small these terms can be controlled by the second and third integral in (1.37). Now we are able to write down the final estimate:

$$\frac{1}{2}\frac{d}{dt}\int (W_\rho^2 + K_1(\phi)^{-1}W_u^2)d\xi + \epsilon C_0\|\partial_\xi W_u\|_{L^2}^2 \le 0,$$

where C_0 is a positive constant, which implies the global existence of W_ρ and W_u in L^2. We note that due to the lack of a viscous term in W_ρ we cannot deduce asymptotic stability from this estimate.

Now we briefly explain the ideas how we are going to control the nonlinear terms in Section 4. Since the diffusion only gives a positive term in $\partial_\xi W_u$, we, leaning on [15], artificially produce also a positive term in $\partial_\xi W_\rho$. We will construct a functional J, which controls the H^2-norm of W_ρ, W_u and derive an estimate of the following form:

$$\frac{d}{dt}J + (1 - \tilde{C}(\|W_\rho\|_\infty + \|W_u\|_\infty + \gamma))(\|\partial_\xi W_\rho\|_{H^2}^2 + \|\partial_\xi W_u\|_{H^2}^2) \le 0,$$

where $\tilde{C}$ depends on $\|\partial_\xi W_\rho\|_\infty, \|\partial_\xi W_u\|_\infty$ and $\gamma > 0$ is a constant resulting from producing a linear combination of the integral estimates. By the

Sobolev-imbedding we know $H^2(\mathbb{R}) \subset C_b^1(\mathbb{R})$. If $1 > \tilde{C}(\|W_\rho\|_\infty + \|W_u\|_\infty + \gamma)$ at $t = 0$, it remains so for all times t for small enough initial data, since these coefficients in turn are controlled by J. This gives global existence and local stability of W_ρ and W_u in $H^2(\mathbb{R})$. Since also the microscopic terms will be controlled by this estimate, we can deduce the local asymptotic stability of small amplitude travelling wave solutions.

2. The linearized collision operator

The fact that equilibrium velocity distributions have compact support requires special care in our analysis, which will be based on linearization around a global (i.e. independent of ξ) equilibrium. It will be important that the support of this equilibrium includes the velocity supports of all other distributions occurring in our analysis. The velocity support of an equilibrium distribution (1.2) is determined by

$$u - \frac{c}{\sqrt{\alpha}} \le v \le u + \frac{c}{\sqrt{\alpha}}\,. \tag{2.1}$$

The formal approximation of a shock profile computed in Section 1.1 has the monotonicity properties $\partial_\xi \rho_f > 0$, $\partial_\xi u_f \sim -\frac{c_l}{\rho_l}\partial_\xi \rho_f < 0$, implying that the left hand side of (2.1) is strictly decreasing. The same is true for the right hand side of (2.1) by

$$\partial_\xi \left(u_f + \frac{c_f}{\sqrt{\alpha}} \right) \sim \frac{c_l}{\rho_l}(\sqrt{\alpha} - 1)\partial_\xi \rho_f < 0\,.$$

So neither the support of M_l is contained in the support of M_r nor vice versa, excluding both of them as candidates for the required global equilbrium. We shall construct a constant state $(\hat\rho, \hat m)$ such that the support of $\hat M := M(\hat\rho, \hat m)$ includes the supports of M_f for the formal approximation of the shock profile for all ξ. Actually, the support of the travelling wave stays a $O(\epsilon)$ distance away from the boundaries of *supp* $\hat M$. We choose

$$\hat u = u_l, \qquad \hat c = c_r(1 + \epsilon/\rho_r), \tag{2.2}$$

which defines $\hat\rho$ and $\hat m$ uniquely. Then for ϵ small $\hat M$ has the desired properties, i.e. there exist positive δ_1, δ_2 such that

$$\hat u - \frac{\hat c}{\sqrt{\alpha}} = u_r - \frac{c_r}{\sqrt{\alpha}} - \epsilon\delta_1, \qquad \hat u + \frac{\hat c}{\sqrt{\alpha}} = u_l + \frac{c_l}{\sqrt{\alpha}} + \epsilon\delta_2,$$

where asymptotically $\delta_1 \sim \frac{c_l}{\rho_l}\left(\frac{1}{\sqrt{\alpha}} - 1\right)$, $\delta_2 \sim \frac{c_l}{\rho_l}\left(\sqrt{\alpha} + \frac{1}{\sqrt{\alpha}}\right)$.

From now on we linearize around the Maxwellian $\hat M$ with the support

$$\Omega := \left[\hat u - \frac{\hat c}{\sqrt{\alpha}}, \hat u + \frac{\hat c}{\sqrt{\alpha}}\right].$$

For the macroscopic wave profiles we use the following alternative expansions to (1.27), (1.28)

$$\begin{aligned} \rho_f &= \hat\rho + \epsilon\hat y_1 = \hat\rho + \epsilon\left(y_1 + \frac{\rho_l - \hat\rho}{\epsilon}\right), \\ m_f &= \hat m + \epsilon\hat y_2 = \hat m + \epsilon\left(s y_1 + \frac{m_l - \hat m}{\epsilon}\right). \end{aligned} \tag{2.3}$$

Also the wave speed s is expanded around $\hat{\lambda}_1 := \lambda_1(\hat{\rho}, \hat{m})$:

$$s = \lambda_{1l} + \epsilon\sigma = \hat{\lambda}_1 + \epsilon\hat{\sigma}, \qquad \text{with } \hat{\sigma} \sim \frac{1+\alpha}{2\hat{\rho}}\hat{c}.$$

We introduce the notation $\mathbf{F} := \nabla\hat{M}$. Clearly $\int \mathbf{F} dv = (1,0)^T$, $\int v\mathbf{F} dv = (0,1)^T$. Then the linearized collision operator reads

$$\mathcal{L}f = \mathbf{F} \cdot \begin{pmatrix} \rho_f \\ m_f \end{pmatrix} - f. \tag{2.4}$$

We define the inner product

$$\langle f, g \rangle_v := \frac{1}{2\beta d^{\frac{1}{\beta}}} \int f\, g\, \hat{M}^{\frac{1}{\beta}-1} dv, \qquad \text{for } \mathit{supp}\, f, \mathit{supp}\, g \subset \Omega, \tag{2.5}$$

where the weight is the second derivative of the kinetic entropy $H''(\hat{M})$. The induced norm and space we denote by $(L^2_v, \|.\|_v)$.

The following relations appear several times:

$$\langle \mathbf{F}, f \rangle_v = \int f\, \nabla H'(\hat{M})\, dv = \frac{1}{\hat{\rho}} \begin{pmatrix} \hat{c}^2 + \hat{u}^2 & -\hat{u} \\ -\hat{u} & 1 \end{pmatrix} \cdot \begin{pmatrix} \rho_f \\ m_f \end{pmatrix}, \tag{2.6}$$

where $\hat{c} = c(\hat{\rho})$ and $\hat{u} = u(\hat{\rho}, \hat{m})$. We note that the matrix on the right hand side is the Hessian of $\hat{\eta} = \eta(\hat{M})$.

This relation also holds in general for any choice of Maxwellians with twice differentiable kinetic entropies. The inner product in v is defined as above with the weight being the second derivative of H. The minimization principle (1.19) has the further consequence that $H'(M(\rho, m))$ is linear in the collision invariants, i.e. there exists a vector $b_{(\rho,m)} \in \mathbb{R}^n$ such that $H'(M(\rho, m)) = b_{(\rho,m)} \cdot (1, v)^T$. If we take the gradient of the first relation in (1.18) it turns out that $b_{(\rho,m)} = \nabla\eta(\rho, m)$, such that $H'(M(\rho, m)) = \nabla\eta(\rho, m) \cdot (1, v)^T$. Taking again the gradient of this equality we discover $\langle \mathbf{F}, f \rangle_v = \mathcal{H}(\hat{\eta})(\rho_f, m_f)^T$.

With respect to the above weighted inner product in v the linearized collision operator $\mathcal{L}$ is symmetric and negative semidefinite

$$\langle \mathcal{L}f, f \rangle_v \leq 0.$$

The symmetry follows directly from (2.6). To see the semidefiniteness we write

$$\begin{aligned} \langle \mathcal{L}f, f \rangle_v &= \left\langle \mathbf{F} \cdot \begin{pmatrix} \rho_f \\ m_f \end{pmatrix} - f, f - \mathbf{F} \cdot \begin{pmatrix} \rho_f \\ m_f \end{pmatrix} \right\rangle_v \\ &\quad + \left\langle \mathbf{F} \cdot \begin{pmatrix} \rho_f \\ m_f \end{pmatrix}, \mathbf{F} \cdot \begin{pmatrix} \rho_f \\ m_f \end{pmatrix} - f \right\rangle_v . \end{aligned}$$

The first term is nonpositive and the second one vanishes because of (2.6).

The standard norms and spaces of functions of ξ we denote with $(L^2_\xi, \|.\|_\xi)$, $(H^k_\xi, \|.\|_{H^k_\xi})$, $(L^\infty_\xi, \|.\|_\infty)$. The Hilbert space $L^2_{\xi,v}$ is then naturally defined by the scalar product

$$\langle f, g \rangle_{\xi,v} = \int_{\mathbb{R}} \langle f, g \rangle_v \, d\xi, \qquad \text{where } \mathit{supp}\ f, \mathit{supp}\ g \subset \Omega,$$

with the induced norm $\|.\|_{\xi,v}$. Similarly the spaces $H^k_\xi(L^2_v)$ of functions, whose derivatives in ξ up to order k are in L^2_v, are defined by

$$\|f\|_{H^k_\xi(L^2_v)} = \left(\|f\|^2_{\xi,v} + \cdots + \|\partial^k_\xi f\|^2_{\xi,v}\right)^{1/2}.$$

For the existence and stability proofs we need the following property for Maxwellians $M(\rho,m)$ with $supp\, M(\rho,m) \subset \Omega$:

$$\sup_\xi \left| \int \left(\partial^j_\rho \partial^k_m M(\rho,m,v)\right)^2 \hat{M}^{\frac{1}{\beta}-1} dv \right| < \infty, \tag{2.7}$$

for $j+k = 0,\dots,4$. In order to guarantee that this holds, we have to make a technical assumption and restrict in the following α to values

$$0 < \alpha < \frac{1}{13}.$$

To see that this implies (2.7) we first observe that it is sufficient to show the uniform boundedness of

$$\int_{supp\, M(\rho,m)} |p(v)| \left(\frac{c^2}{\alpha} - (v-u)^2\right)^{2(\beta-n)} \left(\frac{\hat{c}^2}{\alpha} - (v-\hat{u})^2\right)^{1-\beta} dv, \tag{2.8}$$

for $n = 0,\dots,4$. Here $p(v)$ is a polynomial in v, which can also be neglected since the integration is over a bounded domain. The assumption $supp\, M(\rho,m) \subset \Omega$ implies

$$\left(\frac{c}{\sqrt{\alpha}} + u - v\right)\left(\frac{c}{\sqrt{\alpha}} - u + v\right) \le \left(\frac{\hat{c}}{\sqrt{\alpha}} + \hat{u} - v\right)\left(\frac{\hat{c}}{\sqrt{\alpha}} - \hat{u} + v\right),$$

for all $v \in supp\, M(\rho,m)$ and $\xi \in \mathbb{R}$. Hence, assuming for the moment $\beta > 1$, the integral in (2.8) is bounded by

$$\int \left(\frac{c^2}{\alpha} - (v-u)^2\right)_+^{\beta+1-2n} dv.$$

A transformation of variable leads to the Beta function and hence (2.7) is valid only if $\beta + 1 - 2n > -1$, i.e. $\beta > 6$ or equivalently $0 < \alpha < 1/13$.

Moreover, if $supp\ f \subset \Omega$, then

$$\left| \int v^k f dv \right| \le C\|f\|_v, \qquad \text{for } k \le 3, \tag{2.9}$$

and clearly also

$$\|\rho_f\|_{H^k_\xi} \le C\|f\|_{H^k_\xi(L^2_v)}, \qquad \|m_f\|_{H^k_\xi} \le C\|f\|_{H^k_\xi(L^2_v)}. \tag{2.10}$$

3. Existence of small amplitude travelling waves

As in the formal asymptotics we want to expand f in powers of ϵ. Similarly to [7], we construct a formal asymptotic approximation

$$f_{as} := M(\rho,m) + \epsilon^2 f^\perp[\rho,m], \tag{3.1}$$

where ρ, m are yet undetermined and the leading term of $f^\perp$ is chosen as in (1.29) with a correction term of $O(\epsilon)$

$$f^\perp[\rho,m] := -\frac{1}{\epsilon}(v-s)\partial_\xi M(\rho,m) + \epsilon\left(\frac{1}{\epsilon^2}\partial_\xi(P(\rho,m) - sm)\partial_m \hat{M}\right). \tag{3.2}$$

This choice gives $\rho^\perp = -\frac{1}{\epsilon}\partial_\xi(m - s\rho)$ and $m^\perp = 0$. The residual is

$$\epsilon^3 h := \epsilon(v-s)\partial_\xi f_{as} - M_{f_{as}} + f_{as}\,. \tag{3.3}$$

We again pose the far-field conditions

$$f_{as}(\pm\infty, v) = M_{r,l}(v)\,, \tag{3.4}$$

and require for ρ, m the relation

$$m - m_l \quad = \quad s(\rho - \rho_l), \tag{3.5}$$

which implies $\rho^\perp = 0$ and also $\int h dv = 0$. Therefore

$$\rho_{as} = \rho, \quad m_{as} = m.$$

Moreover we determine ρ_{as}, m_{as} such that $\int vh dv = 0$. This condition is equivalent to the differential equation

$$\frac{1}{\epsilon}\partial_\xi\left(\int v(v-s)^2 M_{f_{as}} dv - (2\hat{u} - s)(P(\rho_{as}, m_{as}) - s m_{as})\right) \tag{3.6}$$
$$= \frac{1}{\epsilon^2}\left(P(\rho_{as}, m_{as}) - P_l - s(m_{as} - m_l)\right).$$

With the ansatz

$$\rho_{as} = \rho_l + \epsilon y_1, \qquad m_{as} = m_l + \epsilon y_2, \tag{3.7}$$

equation (3.5) implies $y_2 = s y_1$ and therefore (3.6) becomes an ODE for y_1 (which is at leading order the viscous Burgers equation (1.32)):

$$(D_0 + \epsilon\mathcal{N}_1(y_1))\partial_\xi y_1 = -2\sigma c_l y_1(1 - y_1) + \epsilon\mathcal{N}_2(y_1),$$

where $\mathcal{N}_1(y_1)$ and $\mathcal{N}_2(y_1)$ are bounded if y_1 is bounded and $\mathcal{N}_2(0) = \mathcal{N}_2(1) = 0$. Hence for small ϵ a bounded smooth monotone solution y_1 connecting the values $y_1(-\infty) = 0$ to $y_1(\infty) = 1$ exists, which is made unique by the initial condition

$$\rho_{as}(0) = \frac{\rho_l + \rho_r}{2}. \tag{3.8}$$

Since y_1 is uniformly bounded in $\xi \in \mathbb{R}$, the same holds for $\partial_\xi y_1$. Differentiation shows that $\partial_\xi^k y_1$ for $k = 2, \dots, 4$ are bounded and therefore clearly also $\rho_{as}, m_{as}, \partial_\xi^k \rho_{as}/\epsilon, \partial_\xi^k m_{as}/\epsilon$, where $k = 1, \dots 4$. Moreover the decay of all these terms is exponential as $\xi \to \pm\infty$.

As in Section 2 the monotonicity of ρ_{as} and m_{as} imply *supp* $M(\rho_{as}, m_{as}) \subset \Omega$, from which we deduce *supp* $f_{as} \subset \Omega$.

Lemma 3.1. *The asymptotic profile f_{as} satisfies the far-field conditions (3.4), and the travelling wave equation (1.23) up to the residual $\epsilon^3 h$, where h is bounded uniformly in $H_\xi^2(L_v^2)$ and fulfills the moment conditions*

$$\int h dv = 0, \qquad \int vh dv = 0. \tag{3.9}$$

Proof. The far-field conditions and (3.9) result directly from the construction of f_{as}. The boundedness of h in $H_\xi^2(L_v^2)$ we obtain from (2.7) and from the exponential decay of all terms, which allows us to integrate in ξ. □

3.1. **The micro-macro decomposition of the correction term.** We introduce the correction term

$$\epsilon^2 g = f - f_{as}$$

with

$$\epsilon^2 \rho_g = \rho_f - \rho_{as}, \qquad \epsilon^2 m_g = m_f - m_{as}.$$

Observe that due to (1.25) and (3.5) the relation

$$m_g = s\rho_g \tag{3.10}$$

holds. The travelling wave problem for the correction term g can be written as follows

$$\epsilon(v-s)\partial_\xi g - \mathcal{L}g = (\mathbf{F}_{f_{as}} - \mathbf{F}) \cdot \begin{pmatrix} 1 \\ s \end{pmatrix} \rho_g \tag{3.11}$$

$$+\frac{1}{\epsilon^2}\left(M_{f_{as}+\epsilon^2 g} - M_{f_{as}} - \epsilon^2 \mathbf{F}_{f_{as}} \cdot \begin{pmatrix} 1 \\ s \end{pmatrix} \rho_g \right) + \epsilon h,$$

subject to

$$g(\pm\infty, v) = 0 \qquad \text{for all } v \in \Omega. \tag{3.12}$$

Here we denoted $\mathbf{F}_{f_{as}} := \nabla M_{f_{as}}$. The left hand side is the linearization of the travelling wave equation (1.23). On the right hand side we have a linear term of order $O(\epsilon)$, a nonlinear term of order $O(\epsilon^2)$ and the residual term. Lemma 3.1, (3.12) and integration with respect to ξ give

$$\int (v-s) g dv = 0, \qquad \int (v-s)^2 g dv = 0,$$

where the first relation is just (3.10). We introduce a decomposition of g into a macroscopic and into a microscopic part, which was done by Caflisch and Nicolaenko in [5] for the Boltzmann equation

$$g(\xi, v) = z(\xi)\Phi(v) + \epsilon w(\xi, v), \tag{3.13}$$

where Φ is chosen such that to leading order $\mathcal{L}\Phi = \epsilon(v-s)\tau\Phi$ for a constant τ and moreover

$$\int (v-s)\Phi dv = 0, \qquad \int (v-s)^2 \Phi dv = 0. \tag{3.14}$$

The choice

$$\Phi(v) = \mathbf{F} \cdot \begin{pmatrix} 1 \\ s \end{pmatrix} + \epsilon \frac{1}{\hat{D}_0} \hat{\sigma}(2\hat{c} - \epsilon\hat{\sigma})(v - \hat{\lambda}_1) \mathbf{F} \cdot \begin{pmatrix} 1 \\ \hat{\lambda}_1 \end{pmatrix} \tag{3.15}$$

with $\hat{D}_0 := 2(1-\alpha)\hat{\rho}^{2\alpha}\hat{c}$ is sufficient for all required properties. We denote

$$\hat{D} := -\int (v-s)^3 \Phi dv = \hat{D}_0 + O(\epsilon).$$

Since

$$\langle (v-s)\Phi, \Phi \rangle_v = \epsilon \frac{\hat{D}}{\hat{D}_0} \frac{\hat{\sigma}\hat{c}}{\hat{\rho}} (2\hat{c} - \epsilon\hat{\sigma}) \neq 0,$$

the decomposition of g is made unique by the orthogonality condition

$$\langle (v-s)\Phi, w \rangle_v = 0.$$

Moreover by computing the zeroth and first order moments of (3.15) we see that the macroscopic density and momentum of Φ are the constant values

$$\rho_\Phi = 1, \quad m_\Phi = s,$$

which together with (3.10) give

$$s\rho_w = m_w. \tag{3.16}$$

The definition of the micro-macro decomposition implies the following properties of z and w:

Lemma 3.2. *If g satisfies (3.11) and (3.12), then*

$$w(\pm\infty, v) \equiv 0, \qquad z(\pm\infty) = 0 \tag{3.17}$$

and

$$\int (v-s)^k w(\xi, v)\, dv = 0, \quad \text{for all } \xi \in \mathbb{R} \text{ and } k = 1, 2, 3. \tag{3.18}$$

Substitution of (3.13) into (3.11) and division by ϵ gives

$$(v-s)\Phi\partial_\xi z - \Lambda z + \epsilon(v-s)\partial_\xi w - \mathcal{L}w = \epsilon\Gamma\rho_w + \epsilon R(\rho_g) + h, \tag{3.19}$$

where

$$\Lambda = \frac{1}{\epsilon}\left(\mathbf{F}_{f_{as}} \cdot \begin{pmatrix} 1 \\ s \end{pmatrix} - \Phi\right), \qquad \Gamma = \frac{1}{\epsilon}(\mathbf{F}_{f_{as}} - \mathbf{F}) \cdot \begin{pmatrix} 1 \\ s \end{pmatrix},$$

$$R(\rho_g) = \frac{1}{\epsilon^4}\left(M_{f_{as}+\epsilon^2 g} - M_{f_{as}} - \epsilon^2\mathbf{F}_{f_{as}} \cdot \begin{pmatrix} 1 \\ s \end{pmatrix}\rho_g\right).$$

These terms are formally $O(1)$, such that the ϵ-powers in (3.19) show the expected orders of magnitude. Moreover

$$\int \varphi dv = 0, \qquad \int v\varphi dv = 0 \qquad \text{for} \quad \varphi \in \{R, \Gamma, \Lambda, h\}.$$

First we derive an equation for z by multiplying the equation with $-(v-s)^2$ and integrating with respect to v:

$$\hat{D}\partial_\xi z + \psi(\xi) z = -\int (v-s)^2(\epsilon\hat{\Gamma}\rho_w + \epsilon R(\rho_g) + h)dv, \tag{3.20}$$

where

$$\psi(\xi) = \int (v-s)^2 \Lambda dv = \frac{1}{\epsilon}\left(\nabla P(\rho_{as}, m_{as}) \cdot \begin{pmatrix} 1 \\ s \end{pmatrix} - s^2\right),$$

$$\hat{\Gamma} = \frac{1}{\epsilon}\mathbf{F}_{f_{as}} \cdot \begin{pmatrix} 1 \\ s \end{pmatrix}.$$

Here we have used (3.14), Lemma 3.1 and Lemma 3.2. Equation (3.20) does not contain derivatives of w and becomes independent of w as $\epsilon \to 0$. Expanding $\psi(\xi)$ shows that $\psi \sim 2c_l\sigma(1-2y_1)$, where y_1 is the profile of ρ_{as} (3.7). Thus to leading order equation (3.20) is the inhomogenous linearized viscous Burgers equation. In particular there exist constants $\gamma, \bar{\xi} > 0$ such that

$$\psi(\xi) \le -\gamma \;\text{ for } \xi \le -\bar{\xi} \qquad \text{and} \qquad \psi(\xi) \ge \gamma \;\text{ for } \xi \ge \bar{\xi}. \tag{3.21}$$

To derive an equation for the microscopic term w we substitute (3.20) into (3.19), which is the same as applying the projection

$$\Pi f := f + \frac{(v-s)\Phi}{\hat{D}} \int (v-s)^2 f dv$$

to (3.19), giving

$$\epsilon(v-s)\partial_\xi w - \mathcal{L}w = \Pi\Lambda z + \epsilon\tilde{\Gamma}\rho_w + \epsilon\Pi R + \Pi h, \tag{3.22}$$

where

$$\tilde{\Gamma} \;=\; \Pi\Gamma + \frac{(v-s)\Phi}{\hat{D}}\frac{1}{\epsilon}\int (v-s)^2 \mathbf{F} dv \cdot \begin{pmatrix} 1 \\ s \end{pmatrix}.$$

The idea of the following manipulation of the equation for w is again from Caflisch and Nicolaenko [5] and was also used in [7] for the scalar conservation law. Since $\mathcal{L}$ is only negative semidefinite, we introduce a new operator $\mathcal{M}$, which is strictly negative and coincides with $\mathcal{L}$ on the set of functions satisfying the moment conditions in (3.18):

$$\mathcal{M}w \;:=\; \mathcal{L}w - v(v-s)\mathbf{F} \cdot \langle v(v-s)\mathbf{F}, w\rangle_v . \tag{3.23}$$

Using (2.6) we see that

$$\langle v(v-s)\mathbf{F}, w\rangle_v = \frac{1}{\hat{\rho}} \begin{pmatrix} \hat{c}^2 + \hat{u}^2 & -\hat{u} \\ -\hat{u} & 1 \end{pmatrix} \int \begin{pmatrix} 1 \\ v \end{pmatrix} v(v-s) w \, dv.$$

Lemma 3.3. *On the set of functions w with $m_w = s\rho_w$, the operator $\mathcal{M}$ satisfies the following properties:*

(i) *$\mathcal{M}$ is symmetric with respect to $\langle , \rangle_v$.*
(ii) *$\mathcal{M}$ coincides with $\mathcal{L}$ on the set of functions w with*

$$\int (v-s)^k w dv \equiv 0, \qquad \text{for } k = 1, 2, 3.$$

(iii) *$\mathcal{M}$ is negative definite in L^2_v. There exists a constant $\kappa > 0$, such that*

$$-\langle \mathcal{M}w, w\rangle_v \geq \kappa \|w\|_v^2. \tag{3.24}$$

Proof. We proceed as in [5] and [7]. Since $\mathcal{L}$ is symmetric, the same holds for $\mathcal{M}$. The second property is obvious.
Now it remains to show the estimate (3.24). We decompose

$$w = \mathbf{F} \cdot \begin{pmatrix} 1 \\ s \end{pmatrix} \rho_w + w^\perp.$$

This implies $\mathcal{L}w = -w^\perp$, $\langle \mathcal{L}w, w\rangle_v = -\|w^\perp\|_v^2$ and $\|w\|_v^2 = \left\|\mathbf{F} \cdot (1,s)^T \rho_w\right\|_v^2 + \|w^\perp\|_v^2$. With the relations in (2.6) and (1.5) we see that $\langle v(v-s)\partial_m \hat{M}, F \cdot$

$(1,s)^T\rangle_v = -\hat{D}_0/\hat{\rho} + O(\epsilon) =: -\hat{D}_1/\hat{\rho} < 0$ and obtain

$$\begin{aligned}
&-\langle \mathcal{M}w, w\rangle_v = \|w^\perp\|_v^2 + |\langle v(v-s)\mathbf{F}, w\rangle_v|^2 \\
&\geq \|w^\perp\|_v^2 + \frac{1}{\hat{\rho}^2}\left(\hat{D}_1\rho_w + \hat{u}\int v(v-s)w^\perp dv - \int v^2(v-s)w^\perp dv\right)^2 \\
&= \|w^\perp\|_v^2 + \gamma\frac{\hat{D}_1^2}{\hat{\rho}^2}\rho_w^2 - \frac{\gamma}{1-\gamma}\frac{1}{\hat{\rho}^2}\left(\hat{u}\int v(v-s)w^\perp dv - \int v^2(v-s)w^\perp dv\right)^2 \\
&\quad + \frac{(1-\gamma)}{\hat{\rho}^2}\left[\hat{D}_1\rho_w + \frac{1}{1-\gamma}\left(\hat{u}\int v(v-s)w^\perp dv - \int v^2(v-s)w^\perp dv\right)\right]^2
\end{aligned}$$

According to (2.9) there exists a $C > 0$ such that

$$-\langle \mathcal{M}w, w\rangle_v \geq \gamma\frac{\hat{D}_1^2}{\hat{\rho}^2}\rho_w^2 + \|w^\perp\|_v^2\left(1 - \frac{\gamma}{1-\gamma}C\right) \geq \kappa\|w\|_v^2$$

for a $\kappa > 0$ and $\gamma \in (0,1)$ sufficiently small. □

Instead of solving (3.20), (3.22) subject to (3.17), we now replace the operator $\mathcal{L}$ by $\mathcal{M}$

$$\epsilon(v-s)\partial_\xi w - \mathcal{M}w = \Pi\Lambda z + \epsilon\tilde{\Gamma}\rho_w + \epsilon\Pi R + \Pi h \tag{3.25}$$

and show the existence of a solution of (3.20), (3.25) together with (3.17). The equivalence of these problems is not obvious.

Lemma 3.4. *The function $g = z\Phi + \epsilon w$ is a solution of (3.11) and (3.12), iff z and w solve (3.20), (3.25) subject to (3.17).*

Proof. Let g be a solution of (3.11), (3.12). According to Lemma 3.2 and Lemma 3.3 (ii) we obtain $\mathcal{L}w = \mathcal{M}w$, and therefore (z,w) solves (3.20), (3.25), (3.17).
Conversely let now (z,w) be a solution of (3.20), (3.25), (3.17). We have to show $\int (v-s)^k w dv \equiv 0$ for $k = 1,2,3$, such that $\mathcal{L}$ and $\mathcal{M}$ coincide. Note that

$$\int v^i \varphi dv = 0,\ i = 0,1 \qquad \Rightarrow \qquad \int (v-s)^j \Pi\varphi dv = 0,\ j = 0,1,2,$$

and thus these relations hold for $\varphi \in \{\Lambda, \Gamma, R, h\}$. Calculating the corresponding moments of (3.25) we obtain the following linear system of ordinary differential equations

$$\epsilon\begin{pmatrix} \partial_\xi \int (v-s)wdv \\ \partial_\xi \int v(v-s)wdv \\ \partial_\xi \int v^2(v-s)wdv \end{pmatrix} = \mathbf{B}\cdot\begin{pmatrix} \int (v-s)wdv \\ \int v(v-s)wdv \\ \int v^2(v-s)wdv \end{pmatrix},$$

where $\mathbf{B}$ is a matrix with constant coefficients. We know that $\int v^k(v-s)w(\pm\infty, v)dv = 0$ for $k = 0,1,2$. Hence the only possible solution is

$$\int v^k(v-s)w(\xi,v)dv \equiv 0 \qquad \text{for } k = 0,1,2.$$

□

3.2. **Existence.** We now show the existence of a solution of the problem (3.20), (3.25), (3.17) by first solving the linear and finally the full nonlinear system of differential equations for the decomposition of g. The solvability of a similar problem was already shown in [7]. Here the results are just repeated. We start with the linear problem and regard the right hand sides of equations (3.20), (3.25) as given inhomogenities

$$\partial_\xi z + \psi(\xi) z = h_z, \qquad \text{with } h_z \in H^1_\xi, \tag{3.26}$$

$$\epsilon(v-s)\partial_\xi w - \mathcal{M} w = h_w, \qquad \text{with } h_w \in H^2_\xi(L^2_v). \tag{3.27}$$

To prove the stability result in Section 4, we need L^∞_ξ-bounds for the macrosocopic profiles of the travelling wave and for their first derivatives. Hence we look for solutions in the spaces H^2_ξ and $H^2_\xi(L^2_v)$. This requires homogenous far-field conditions and already provides uniqueness for the solution of (3.27). Equation (3.26) has a one parameter set of solutions, which is due to the arbitrary shift of travelling wave solutions. We pose the initial condition

$$z(0) = z_0. \tag{3.28}$$

The following result was shown in [7]:

Lemma 3.5. *The unique solution z of (3.26), (3.28) is bounded by*

$$\|z\|_{H^2_\xi} \le C(|z_0| + \|h_z\|_{H^1_\xi}).$$

The variation of constant formula gives the mild formulation of the unique solution of (3.26), (3.28). From the properties of ψ given in (3.21) one can deduce the L^2-continuous dependence of z on z_0 and h_z. From (3.26) we get the same estimate for $\partial_\xi z$. Using the uniform boundedness of $\partial_\xi \psi$, differentiation of (3.26) finally gives the bound for $\partial^2_\xi z$.

Lemma 3.6. *There exists a unique solution $w \in H^2_\xi(L^2_v)$ of (3.27). For this solution the bound*

$$\|\partial^k_\xi w\|_{\xi,v} \le \frac{1}{\kappa}\|\partial^k_\xi h_w\|_{\xi,v}, \qquad \text{for } k = 0,1,2,$$

holds, where κ is the same as in (3.24).

Sketch of the proof. We can apply the proof given in [7], which is based on a discretisation of the velocity variable. The latter yields a system of ordinary differential equations. Since the discretised version of the operator $\mathcal{M}$ is again symmetric and negative definite we can deduce the existence and uniqueness of a bounded solution which converges to zero as $\xi \to \pm\infty$. Choosing the quadrature formula appropriate the solution of the discretised problem is bounded by $\frac{1}{\kappa}\|h_w\|_\xi$, implying weak convergence. Then we can pass to the limit. Also the estimate carries over to the solution w of (3.27). The estimates for the derivatives are then derived from the differentiated equation in the same way. □

We now apply a fix-point argument to solve the nonlinear equations (3.20), (3.25) subject to the initial condition $z(0) = z_0$, which is related to the initial condition for the original unknown by

$$\langle (v-s)\Phi, f - f_{as}\rangle_v (\xi = 0) = \epsilon^2 \langle (v-s)\Phi, \Phi\rangle_v z_0. \tag{3.29}$$

For the contraction argument we need bounds for the right hand sides of (3.20), (3.25).

Lemma 3.7. (i) *The operator* $\Pi : L^2_v \to L^2_v$ *is bounded.*

(ii) *There exists a constant* $C > 0$*, such that*

$$\|\Pi\Lambda z\|_{H^2_\xi(L^2_v)} \le C\|z\|_{H^2_\xi}, \qquad \|\tilde\Gamma\rho_w\|_{H^2_\xi(L^2_v)} \le C\|w\|_{H^2_\xi(L^2_v)},$$
$$\|\int (v-s)^2\hat\Gamma dv\rho_w\|_{H^2_\xi} \le C\|\rho_w\|_{H^2_\xi}.$$

(iii) *There exists a constant* $K > 0$*, such that for all* ρ_1, ρ_2 *with* $\|\rho_1\|_{H^2_\xi}$, $\|\rho_2\|_{H^2_\xi} \le \frac{K}{\epsilon}$ *we obtain for the nonlinearity*

$$\|R(\rho_1) - R(\rho_2)\|_{H^2_\xi(L^2_v)} \le C(\|\rho_1\|_{H^2_\xi} + \|\rho_2\|_{H^2_\xi})\|\rho_1 - \rho_2\|_{H^2_\xi}. \tag{3.30}$$

Proof. The proof of (i) is straightforward using (2.9).
For the estimates on the linear terms in (ii) we use (i), inequality (2.10), the boundedness of $\|\hat y_1\|_\infty, \|\hat y_2\|_\infty$, (see (2.3)), and of $\|\partial^k_\xi \rho_{as}/\epsilon\|_\infty, \|\partial^k_\xi m_{as}/\epsilon\|_\infty$ for $k = 1, 2$.
We expand the nonlinearities

$$R(\rho_1) - R(\rho_2) = (1, s)\cdot \mathcal{H}(M_1)(1, s)^T(\rho_2 + \vartheta_1(\rho_1 - \rho_2))(\rho_1 - \rho_2),$$

where $M_1 = M\left((\rho_{as}, m_{as}) + \epsilon^2\vartheta_2(\rho_2 + \vartheta_1(\rho_1 - \rho_2))(1, s)\right)$ and $0 \le \vartheta_1, \vartheta_2 \le 1$. Due to the construction of Ω there exists a constant $K > 0$ such that *supp* $M_1 \subset \Omega$ for $\|\rho_1\|_\infty, \|\rho_2\|_\infty \le K/\epsilon$. If now ρ_1, ρ_2 are bounded in H^2_ξ by a constant of order $O(\epsilon^{-2})$, which is guaranteed by our assumption, we obtain the given estimate by differentiation, using (2.7) and the one-dimensional Sobolev imbedding. □

We define the following norm in $H^2_\xi(L^2_v)$:

$$\|g\| := \|z\|_{H^2_\xi} + \epsilon\|w\|_{H^2_\xi(L^2_v)},$$

and observe that $\|g\|_{H^2_\xi(L^2_v)} \le C\|g\|$ for a positive constant C.

Theorem 3.1. *For every* $z_0 \in \mathbb{R}$ *and for* ϵ *small enough, there exists a solution* f *of (1.23) satisfying (3.29), unique in a ball in* $(H^2_\xi(L^2_v), \|.\|)$ *with center* f_{as} *and an* $O(\epsilon)$ *radius. It satisfies*

$$\|f - M(\rho_{as}, m_{as})\|_{H^2_\xi(L^2_v)} = O(\epsilon^2),$$

where (ρ_{as}, m_{as}) *is the solution of (3.5), (3.6) and (3.8). More precisely*

$$f = M(\rho_{as}, m_{as}) + \epsilon^2 f^\perp[\rho_{as}, m_{as}] + \epsilon^2 z\Phi + \epsilon^3 w,$$

where $\|z\|_{H^2_\xi}$ *and* $\|w\|_{H^2_\xi(L^2_v)}$ *are uniformly bounded as* $\epsilon \to 0$.

Proof. We proceed as in [7]. As a consequence of Lemma 3.7 (ii) we can extend the results from Lemma 3.5 and Lemma 3.6 to the full linear problem

$$\begin{aligned} \hat D\partial_\xi z + \psi(\xi)z &= -\epsilon\int (v-s)^2\hat\Gamma\rho_w dv + h_z, \\ \epsilon(v-s)\partial_\xi w - \mathcal{M}w &= \Pi\Lambda z + \epsilon\tilde\Gamma\rho_w + h_w, \end{aligned}$$

with inhomogenities h_z, h_w and $z(0) = z_0$. Applying the solution operator to the nonlinearities and residual terms in (3.20) and (3.25) gives a fixed point problem $(z, w) = \mathcal{G}(z, w)$. Due to Lemma 3.7 (iii) the fixed point operator is bounded by $\|\mathcal{G}(z, w)\| \le c(1 + \epsilon\|(z, w)\|^2)$ if $\|(z, w)\| \le C/\epsilon$ for some $C > 0$. This implies that for ϵ small enough $\mathcal{G}$ maps both the ball with radius $2c$ and the ball with radius $\epsilon^{-1}\min\{1/(2c), C\}$ into themselves. Also $\mathcal{G}$ is a contraction on the ball with an $O(\epsilon^{-1})$ radius. We conclude that for ϵ small the fixed point problem has a solution with $\|(z, w)\| \le 2c$, which is unique in a ball with an $O(\epsilon^{-1})$ radius. Knowing this and returning to the fixed point equation for w, also the boundedness of $\|w\|_{H^2_\xi(L^2_v)}$ follows. □

The monotonicity of ρ_f can be deduced in the same way as it was done in [7]:

Lemma 3.8. *Let the assumptions of (the existence) Theorem 3.1 hold and let f be the solution of (1.23) with initial condition (3.29). Then $\rho_f(\xi)$ is strictly increasing.*

The proof relies on the fact that the map $z_0 \mapsto \rho_f(0)$ is invertible for ϵ small, meaning that the travelling wave can also be made locally unique by prescribing the value of $\rho_f(0)$ instead of z_0. This argument can of course be repeated for every $\xi_0 \in \mathbb{R}$ instead of the origin. Now assuming ρ_f is not strictly monotone would lead to the periodicity of f as a consequence of the uniqueness result, which contradicts the far-field conditions.

4. Local stability of small amplitude travelling waves

In this section we prove the asymptotic, dynamic stability of small amplitude travelling waves constructed before. The methods we apply are commonly used for conservation laws with diffusion terms. This motivates to introduce the parabolic scaling $\xi \to \frac{\xi}{\epsilon}, t \to \frac{t}{\epsilon^2}$ in equation (1.20). Let f be the solution of

$$\epsilon^2\partial_t f + \epsilon(v - s)\partial_\xi f = M_f - f \tag{4.1}$$

with the far-field conditions

$$f(t, \xi = \pm\infty, v) = M_{r,l}(v).$$

Let ϕ be the travelling wave solution as in Theorem 3.1. For given initial data $f_0(\xi, v) = f(0, \xi, v)$ we fix the shift in the travelling wave ϕ such that

$$\int_{\mathbb{R}} [\rho_{f_0}(\xi) - \rho_\phi(\xi)]\, d\xi = 0. \tag{4.2}$$

In addition we restrict ourselves to initial data satisfying

$$\int_{\mathbb{R}} [m_{f_0}(\xi) - m_\phi(\xi)]\, d\xi = 0. \tag{4.3}$$

This way we guarantee

$$\int_{\mathbb{R}} [\rho_f(t, \xi) - \rho_\phi(\xi)]\, d\xi = \int_{\mathbb{R}} [m_f(t, \xi) - m_\phi(\xi)]\, d\xi = 0,\ \forall\ t \ge 0. \tag{4.4}$$

Introducing the perturbation G by

$$\epsilon G = f - \phi, \quad \rho := \rho_G, \quad m := m_G,$$

we obtain

$$\epsilon\partial_t G + (v - s)\partial_\xi G = \frac{1}{\epsilon^2}\,[M_{\phi+\epsilon G} - M_\phi] - \frac{1}{\epsilon}G. \tag{4.5}$$

Motivated by the work of Liu and Yu [14] we, as in [7], apply a micro-macro decomposition to the deviation G

$$G = \mathbf{F} \cdot \begin{pmatrix} \rho \\ m \end{pmatrix} + \epsilon g. \tag{4.6}$$

Then the norm of G satisfies

$$\|\partial_\xi^k G\|_{\xi,v}^2 = \frac{1}{\hat\rho}\left[\hat c^2 \|\partial_\xi^k \rho\|_\xi^2 + \left\|\partial_\xi^k (m - \hat u \rho)\right\|_\xi^2\right] + \epsilon^2 \|\partial_\xi^k g\|_{\xi,v}^2. \tag{4.7}$$

Macroscopic equations for ρ and m are obtained by computing the zeroth and first order moments of equation (4.5)

$$\epsilon\partial_t \rho + \partial_\xi(m - s\rho) = 0, \tag{4.8}$$

$$\epsilon\partial_t m + \partial_\xi\left(\nabla\hat P \cdot \begin{pmatrix} \rho \\ m \end{pmatrix} - sm\right) + \epsilon\partial_\xi P_g = 0, \tag{4.9}$$

where $\hat P := P_{\hat M} = P(\hat\rho, \hat m)$ and as before $P_g = \int v^2 g\, dv$. Next we apply the microscopic projection $-\mathcal{L}G = \epsilon g$ to (4.5) to get an equation for g

$$\epsilon^2\partial_t g - \partial_\xi\left(\mathbf{F} \cdot \left[\left(\hat{\mathbf{A}} - v\mathbf{I}\right)\begin{pmatrix} \rho \\ m \end{pmatrix}\right]\right) - \epsilon\partial_\xi \mathcal{L}((v-s)g) = R_2(\rho, m) - g \tag{4.10}$$

with the Jacobian $\hat{\mathbf{A}} := \mathbf{A}(\hat\rho, \hat m)$ given in (1.11) and the nonlinearity

$$\begin{aligned} R_2(\rho, m) &= \frac{\mathbf{F}_\phi - \mathbf{F}}{\epsilon} \cdot \begin{pmatrix} \rho \\ m \end{pmatrix} + \tilde R_2(\rho, m), \\ \tilde R_2(\rho, m) &= \frac{1}{\epsilon^2}\left[M_{\phi+\epsilon G} - M_\phi - \epsilon \mathbf{F}_\phi \cdot \begin{pmatrix} \rho \\ m \end{pmatrix}\right], \end{aligned}$$

split into its linear and purely quadratic part.

As in the Chapman-Enskog approximation we compute the last term in (4.9) using (4.10)

$$P_g = q(\rho, m) - \epsilon^2 \partial_t P_g + \epsilon\partial_\xi P_{\mathcal{L}((v-s)g)} - D\partial_\xi(m - \hat u\rho), \tag{4.11}$$

with the constant $D := 2(1-\alpha)\hat\rho^{2\alpha} > 0$ and the nonlinearity $q(\rho, m) := P_{R_2(\rho,m)}$. According to (4.7) and the diffusion term in (4.11) it is convenient (as already mentioned in the introduction) to define

$$W_\rho(t,\xi) = \int_{-\infty}^{\xi} \rho(t,\xi)\, d\xi, \qquad W_u(t,\xi) = \int_{-\infty}^{\xi} [m(t,\xi) - \hat u\, \rho(t,\xi)]\, d\xi.$$

The assumptions (4.2) and (4.3) assure that $W_\rho(t, \pm\infty) = W_u(t, \pm\infty) = 0$ for $t \geq 0$. Integrating (4.8), (4.9) with respect to ξ gives the macroscopic equations

$$\partial_t W_\rho + \frac{1}{\epsilon}\left[(\hat c - \epsilon\hat\sigma)\partial_\xi W_\rho + \partial_\xi W_u\right] = 0, \tag{4.12}$$

$$\partial_t W_u + \frac{1}{\epsilon}\left[\hat c^2 \partial_\xi W_\rho + (\hat c - \epsilon\hat\sigma)\partial_\xi W_u\right] + P_g = 0. \tag{4.13}$$

Observe that the second equation is obtained by a linear combination of (4.8),(4.9). Substituting (4.11) we get the equivalent equation for (4.13)

$$\begin{aligned} \partial_t W_u + \frac{1}{\epsilon}\left[\hat c^2 \partial_\xi W_\rho + (\hat c - \epsilon\hat\sigma)\partial_\xi W_u\right] + q - D\partial_\xi^2 W_u \\ = \epsilon^2 \partial_t P_g - \epsilon\partial_\xi P_{\mathcal{L}((v-s)g)}. \end{aligned} \tag{4.14}$$

Splitting q as R_2 into its linear and purely quadratic term, the equation can again be reformulated to

$$\partial_t W_u + \frac{1}{\epsilon}\left[K_1(\phi)\partial_\xi W_\rho + K_2(\phi)\partial_\xi W_u\right] + \tilde{q} - D\partial_\xi^2 W_u \\ = \epsilon^2 \partial_t P_g - \epsilon \partial_\xi P_{\mathcal{L}((v-s)g)}, \qquad (4.15)$$

where $\tilde{q} = P_{\tilde{R}_2}$ and

$$K_1(\phi) := c_\phi^2 - (u_\phi - \hat{u})^2, \quad K_2(\phi) := \hat{c} - \epsilon\hat{\sigma} + 2(u_\phi - \hat{u}). \qquad (4.16)$$

In the following we will switch between the three different representations of the second macroscopic equation. We will need the signs of K_1, K_2 and of their derivatives. From Lemma 3.8 we know that the density of the travelling wave is strictly increasing, which also implies $\partial_\xi u_\phi < 0$, see (1.34). Since all components have the same decay rate we get (for ϵ small)

$$\frac{\hat{c}^2}{2} < K_1(\phi) < 2\hat{c}^2, \qquad \partial_\xi K_1(\phi) > 0, \qquad \partial_\xi\left(K_1(\phi)^{-1}\right) < 0, \quad (4.17)$$

$$\frac{\hat{c}}{2} < K_2(\phi) < 2\hat{c}, \qquad \partial_\xi K_2(\phi) < 0. \qquad (4.18)$$

We recall the uniform boundedness of $\partial_\xi K_1(\phi), \partial_\xi K_2(\phi)$ by a constant of $O(\epsilon)$ resulting from Theorem 3.1.

We start with deriving estimates for the macroscopic part of the system. For controlling the nonlinear terms, L_ξ^∞-bounds of ρ, m are needed, which we shall control in H_ξ^1. This means we need to control the H_ξ^2-norm of W_1, W_2 and therefore we give integral estimates for their derivatives up to second order in the following.

Taylor expansion of $\tilde{R}_2$ gives

$$\tilde{R}_2(\rho, m) = (\rho, m)\cdot\mathcal{H}(M_{\phi+\epsilon\vartheta G})\begin{pmatrix}\rho\\ m\end{pmatrix}, \quad \text{for a} \quad \vartheta \in (0,1). \qquad (4.19)$$

For $\|R_2\|_v$ to be well defined we have to guarantee *supp* $\tilde{R}_2 \subset \Omega$. Due to the construction of Ω this is only true for sufficiently small $\|\rho\|_\infty, \|m\|_\infty$. We make this smallness assumption for the moment and prove it in the stability result at the end of this section. The nonlinear terms satisfy the estimate

$$\|R_2\|^2_{H_\xi^k(L_v^2)} + \|\tilde{R}_2\|^2_{H_\xi^k(L_v^2)} + \|q\|^2_{H_\xi^k} + \|\tilde{q}\|^2_{H_\xi^k} \qquad (4.20)$$
$$\le \tilde{C}\left[\|\partial_\xi W_\rho\|^2_{H_\xi^k} + \|\partial_\xi W_u\|^2_{H_\xi^k}\right] \text{ for } k = 0, 1.$$

Here and in the following $\tilde{C}$ depends on $\|\rho\|_\infty, \|m\|_\infty$. By differentiating (4.19), using (2.7), the Sobolev imbedding and the smoothness of the travelling wave, the estimate for $\tilde{R}_2$ is obtained. Then the bound for R_2 is deduced easily, and finally the estimates for q, $\tilde{q}$ are an immediate consequence of (2.9).

We now test (4.12) with W_ρ and (4.15) with $K_1^{-1}W_u$, which amounts to a symmetrization of the system and yields the cancellation of the mixed terms containing $W_\rho \partial_\xi W_u$ and $W_u \partial_\xi W_\rho$:

$$\frac{1}{2}\frac{d}{dt}[\|W_\rho\|_\xi^2 + \|K_1^{-1/2}W_u\|_\xi^2] + \frac{1}{2\epsilon}\int \partial_\xi\left(-K_2\,K_1^{-1}\right)W_u^2\,d\xi + \int K_1^{-1}W_u\,\tilde{q}\,d\xi$$
$$+D\|K_1^{-1/2}\partial_\xi W_u\|_\xi^2 + D\int \partial_\xi(K_1^{-1})W_u\partial_\xi W_u\,d\xi$$
$$= \epsilon^2\partial_t\int K_1^{-1}W_u\,P_g\,d\xi + \epsilon\int K_1^{-1}[\hat{c}^2\partial_\xi W_\rho + (\hat{c}-\epsilon\hat{\sigma})\partial_\xi W_u + \epsilon P_g]P_g\,d\xi$$
$$+\epsilon\int(\partial_\xi K_1^{-1}W_u + K_1^{-1}\partial_\xi W_u)P_{\mathcal{L}((v-s)g)}d\xi.$$

Here we have just used integration by parts and equation (4.13) for the substitution of $\partial_t W_u$ on the right hand side. We note that $\|K_1^{-1/2}\partial_\xi^k W\|_\xi^2 \geq 1/(2\hat{c}^2)\|\partial_\xi^k W\|_\xi^2$ and the second term has the favourable sign since $\partial_\xi(K_2K_1^{-1}) \leq 0$ for all ξ. On the left hand side we estimate the purely quadratic nonlinearity by

$$\left|\int K_1^{-1}W_u\,\tilde{q}\,d\xi\right| \leq C\|W_u\|_\infty\int|\tilde{q}|\,d\xi \leq \tilde{C}\|W_u\|_\infty\left[\|\partial_\xi W_\rho\|_\xi^2 + \|\partial_\xi W_u\|_\xi^2\right]. \tag{4.21}$$

The triangle inequality is used for

$$\left|\int \partial_\xi\left(K_1^{-1}\right)W_u\partial_\xi W_u\,d\xi\right| \leq \int|\partial_\xi\left(K_1^{-1}\right)|\,W_u^2\,d\xi + \epsilon C\|\partial_\xi W_u\|_\xi^2,$$

where we recall that $\|\partial_\xi\rho_\phi\|_\infty, \|\partial_\xi m_\phi\|_\infty \leq \epsilon C$. We now turn to the right hand side. Recalling (2.9), with the Cauchy-Schwarz inequality we can bound the last two integrals by

$$\epsilon C(\|g\|_{\xi,v}^2 + \|\partial_\xi W_\rho\|_\xi^2 + \|\partial_\xi W_m\|_\xi^2) + \frac{\epsilon}{2}\int|\partial_\xi\left(K_1^{-1}\right)|\,W_u^2\,d\xi.$$

Summarizing, we obtain for ϵ small enough

$$\frac{1}{2}\frac{d}{dt}\left[\|W_\rho\|_\xi^2 + \|K_1^{-1/2}W_u\|_\xi^2 - \epsilon^2\int K_1^{-1}W_uP_gd\xi\right] + \int\kappa W_u^2d\xi \tag{4.22}$$
$$+\left(\frac{D}{4\hat{c}^2} - \tilde{C}\|W_u\|_\infty\right)\|\partial_\xi W_u\|_\xi^2 \leq (\tilde{C}\|W_u\|_\infty + \epsilon C)\|\partial_\xi W_\rho\|_\xi^2 + \epsilon C\|g\|_{\xi,v}^2$$

where $\kappa(\phi) = \frac{1}{2\epsilon}(\partial_\xi(-K_2K_1^{-1}) + \epsilon(D+\epsilon)\partial_\xi K_1^{-1}) \geq 0$ for all ξ. The lack of a diffusion term in W_ρ will be overcome with another combination of the macroscopic equations. We observe that

$$\frac{d}{dt}\int W_u\partial_\xi W_\rho d\xi = \int[\partial_t(W_u\partial_\xi W_\rho) - \partial_\xi(W_u\partial_t W_\rho)]d\xi$$
$$= \int[\partial_t W_u\partial_\xi W_\rho - \partial_t W_\rho\partial_\xi W_u]d\xi.$$

Corresponding to the right hand side we now combine the equations (4.12), (4.13) yielding

$$\epsilon\frac{d}{dt}\int W_u\partial_\xi W_\rho d\xi + \hat{c}^2\|\partial_\xi W_\rho\|_\xi^2 - \|\partial_\xi W_u\|_\xi^2 \tag{4.23}$$
$$= -\epsilon\int\partial_\xi W_\rho P_g d\xi \leq \epsilon(\|\partial_\xi W_\rho\|_\xi^2 + C\|g\|_{\xi,v}^2).$$

A linear combination of (4.22) and (4.23) implies the following lemma:

Lemma 4.1. *Let W_ρ, W_u be the solution of the system (4.12), (4.15) and ϵ be small enough. Then there exist constants C and $\tilde{C}$ such that, for any $\alpha_0 > 0$,*

$$\frac{d}{dt}J_0 + \left(\frac{\alpha_0 \hat{c}^2}{2} - \tilde{C}\|W_u\|_\infty\right)\|\partial_\xi W_\rho\|_\xi^2 + \left(\frac{D}{4\hat{c}^2} - \alpha_0 - \tilde{C}\|W_u\|_\infty\right)\|\partial_\xi W_u\|_\xi^2 \le \epsilon C\|g\|_{\xi,v}^2 \tag{4.24}$$

with $J_0 = \frac{1}{2}\left[\|W_\rho\|_\xi^2 + \|K_1^{-1/2}W_u\|_\xi^2 - \epsilon^2\int K_1^{-1}W_u\, P_g d\xi + 2\epsilon\alpha_0\int W_u\partial_\xi W_\rho d\xi\right]$.

We now turn to the estimate for the first order derivatives. It is derived by testing the derivatives of (4.12) and (4.14) with $\hat{c}^2\partial_\xi W_\rho$ and $\partial_\xi W_u$ respectively. Since there is no contribution of the travelling wave terms in the integration by parts, the derivation is even simpler. The term with the nonlinearity we treat in a different way:

$$\left|\int \partial_\xi q\, \partial_\xi W_u d\xi\right| = \left|\int q\, \partial_\xi^2 W_u d\xi\right| \le \frac{D}{4}\|\partial_\xi^2 W_u\|_\xi^2 + \tilde{C}(\|\partial_\xi W_\rho\|_\xi^2 + \|\partial_\xi W_u\|_\xi^2).$$

Following the procedure above, where now the differentiated versions of the equations (4.12), (4.13) are used, we arrive at

Lemma 4.2. *Let W_ρ, W_u be the solution of the system (4.12), (4.14) and ϵ be small enough. Then there exist constants C and $\tilde{C}$ such that, for any $\alpha_1 > 0$,*

$$\frac{d}{dt}J_1 + \frac{\alpha_1\hat{c}^2}{2}\|\partial_\xi^2 W_\rho\|_\xi^2 + \left(\frac{D}{2} - \alpha_1\right)\|\partial_\xi^2 W_u\|_\xi^2 \le \tilde{C}(\|\partial_\xi W_\rho\|_\xi^2 + \|\partial_\xi W_u\|_\xi^2) + \epsilon C\|\partial_\xi g\|_{\xi,v}^2 \tag{4.25}$$

with $J_1 = \frac{1}{2}\left[\|\hat{c}^2\partial_\xi W_\rho\|_\xi^2 + \|\partial_\xi W_u\|_\xi^2 - \epsilon^2\int \partial_\xi W_u\, P_{\partial_\xi g} d\xi + 2\epsilon\alpha_1\int \partial_\xi W_u \partial_\xi^2 W_\rho d\xi\right]$.

For controlling the microscopic terms we use the full kinetic perturbation equation.

Lemma 4.3. *Let G, decomposed as in (4.6), be the solution of (4.5). Then there exists a $\tilde{C}$ such that, for $k = 0, 1$,*

$$\frac{d}{dt}\|\partial_\xi^k G\|_{\xi,v}^2 + \|\partial_\xi^k g\|_{\xi,v}^2 \le \tilde{C}\left[\|\partial_\xi W_\rho\|_{H_\xi^k}^2 + \|\partial_\xi W_u\|_{H_\xi^k}^2\right].$$

Proof. We take the inner product in ξ and v of the kth derivative of (4.5) with $\partial_\xi^k G$ and divide by ϵ

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\partial_\xi^k G\|_{\xi,v}^2 &= \frac{1}{\epsilon}\langle \partial_\xi^k(R_2 - g), \partial_\xi^k G\rangle_{\xi,v} = \langle \partial_\xi^k R_2, \partial_\xi^k g\rangle_{\xi,v} - \|\partial_\xi^k g\|_{\xi,v} \\ &\le \tilde{C}(\|\partial_\xi W_\rho\|_{H_\xi^k}^2 + \|\partial_\xi W_m\|_{H_\xi^k}^2) - \frac{1}{2}\|\partial_\xi^k g\|_\xi^2.\end{aligned}$$

□

Now we are able to prove the main result of this section.

Theorem 4.1. *Let the assumptions of Theorem 3.1 hold and let ϕ be the travelling wave solution. Let $f_0(\xi,v)$ be the initial datum for (4.1) and let*

$$\begin{aligned} W_{\rho,0}(\xi) &= \frac{1}{\epsilon}\int_{-\infty}^{\xi} [\rho_{f_0}(\eta)-\rho_\phi(\eta)]\,d\eta, \\ W_{u,0}(\xi) &= \frac{1}{\epsilon}\int_{-\infty}^{\xi} [(m_{f_0}(\eta)-m_\phi(\eta))-\hat{u}(\rho_{f_0}(\eta)-\rho_\phi(\eta))]\,d\eta. \end{aligned}$$

Moreover let $f_0-\phi\in H^2_\xi(L^2_v)$, (implying $f_0(\pm\infty,v)=\phi(\pm\infty,v)$,) and $W_{\rho,0},W_{u,0}\in L^2_\xi$, which ensures assumption (4.2) and (4.3). Let

$$\|W_{\rho,0}\|_{L^2_\xi}+\|W_{u,0}\|_{L^2_\xi}+\frac{1}{\epsilon}\|f_0-\phi\|_{H^1_\xi(L^2_v)}\le\delta \tag{4.26}$$

for a δ small enough, which is independent from ϵ. Then for ϵ small enough equation (4.1) with initial data f_0 has a unique global solution. In particular, small amplitude travelling waves are locally stable in the sense that

$$\lim_{t\to\infty}\int_t^\infty \|f(\tau,.)-\phi(.)\|^2_{H^1_\xi(L^2_v)}d\tau=0.$$

Proof. The main idea is to construct a Lyapunov functional, which is decaying in time. Recall (4.1), (4.2) and define

$$J:=J_0+\gamma_1J_1+\gamma_2\|G\|^2_{H^1_\xi(L^2_v)},$$

where the ϵ-independent constants $\gamma_1,\gamma_2>0$ and $\alpha_0,\alpha_1>0$ will be chosen below. For any positive choice of the constants the functional J is bounded from above and below by

$$\|W_\rho\|^2_{H^2_\xi}+\|W_u\|^2_{H^2_\xi}+\epsilon^2\|g\|^2_{H^1_\xi(L^2_v)}. \tag{4.27}$$

Hence by Sobolev imbedding there exists a constant C_0 such that $\|W_\rho\|_\infty+\|W_u\|_\infty+\|\rho\|_\infty+\|m\|_\infty\le C_0\sqrt{J}$. We now combine the estimates from Lemma 4.1, Lemma 4.2, and Lemma 4.3 to get

$$\begin{aligned} \frac{d}{dt}J+\frac{\gamma_2}{2}\|g\|^2_{H^1_\xi(L^2_v)}+\left(\frac{\alpha_0\,\hat{c}^2}{2}-\tilde{C}(\|W_u\|_\infty+\gamma_1+\gamma_2)\right)\|\partial_\xi W_\rho\|^2_\xi \\ +\left(\frac{D}{4\hat{c}^2}-\alpha_0-\tilde{C}(\|W_u\|_\infty+\gamma_1+\gamma_2)\right)\|\partial_\xi W_u\|^2_\xi \\ +\left(\gamma_1\frac{\alpha_1\hat{c}^2}{2}-\gamma_2\tilde{C}\right)\|\partial^2_\xi W_\rho\|^2_\xi+\left(\gamma_1\frac{D}{2}-\gamma_1\alpha_1-\gamma_2\tilde{C}\right)\|\partial^2_\xi W_u\|^2_\xi\le\ 0 \end{aligned}$$

We choose $L=C_0\sqrt{J(0)}$ small enough, such that there exist constants $\alpha_0,\alpha_1,\gamma_1,\gamma_2>0$ satisfying

$$\frac{\alpha_0\,\hat{c}^2}{2}>\tilde{C}(L)(L+\gamma_1+\gamma_2),\qquad \gamma_1\frac{\alpha_1\hat{c}^2}{2}>\gamma_2\tilde{C}(L), \tag{4.28}$$

$$\frac{D}{4\hat{c}^2}>\alpha_0+\tilde{C}(L)(L+\gamma_1+\gamma_2),\qquad \gamma_1\frac{D}{2}>\gamma_1\alpha_1+\gamma_2\tilde{C}(L). \tag{4.29}$$

Then the coefficients of $\|\partial^{k+1}_\xi W_\rho\|^2_\xi$, $\|\partial^{k+1}_\xi W_u\|^2_\xi$, $k=0,1$, are positive initially. Since J controls the L^∞_ξ-norms of W_ρ,W_u and their derivatives, these coefficients stay positive. Hence there is a constant $C_1 > 0$ such that

$$\frac{d}{dt} J \leq -C_1 \|G\|^2_{H^1_\xi(L^2_v)}, \qquad \text{for all } t \geq 0,$$

showing that J is a Lyapunov functional. The proof is now completed by integration with respect to t. □

Appendix. Isothermal Case

In this appendix we show, how the existence and stability results can be extended to the isothermal system

$$\partial_t \begin{pmatrix} \rho \\ m \end{pmatrix} + \partial_x \begin{pmatrix} m \\ \frac{m^2}{\rho} + \rho \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{A.1}$$

A Maxwellian satisfying (1.3)-(1.4) for $\alpha = 0$ and therefore leading to the isothermal system is given by

$$M(\rho, m, v) = \frac{\rho}{\sqrt{2\pi}} e^{-(v-u)^2/2}, \qquad v \in \mathbb{R},$$

see Bouchut [1] and references therein. Observe that in this case the speed of sound is constant, $c(\rho) \equiv 1$. Kinetic and macroscopic entropies are given by

$$H(f, v) = \frac{v^2}{2} f + f \ln f\,, \qquad \eta(\rho, m) = \frac{1}{2}\frac{m^2}{\rho} + (\rho \ln \rho - \rho \ln \sqrt{2\pi}),$$

see again [1]. Now the equilibrium distributions do not have compact support, which simplifies the procedure and allows to linearize around the asymptotic state at $\xi = -\infty$. Again the quadratic approximation of the kinetic entropy close to equilibrium gives the weight for the inner product

$$\langle f, g \rangle_v = \int_{\mathbb{R}} f\, g\, \frac{1}{M_l} dv\,.$$

Then one can show that all conditions from Section 2 are satisfied and therefore the proofs for existence and stability carry over to this case. Only one difference in Section 4 is important to mention. Since the sound speed is constant, the derivative of $K_1(\phi)$ corresponding to (4.16) is now of $O(\epsilon^2)$ and has a different sign

$$\partial_\xi K_1(\phi) = -2(u_\phi - u_l)\partial_\xi u_\phi < 0.$$

Therefore the first macroscopic estimate corresponding to (4.24) has to be derived differently. We test equation (4.13) with $K_1(\phi)W_\rho$ and (4.15) with W_u to obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}[\|K_1^{1/2} W_\rho\|^2_\xi + \|W_u\|^2_\xi] + \int \tilde{q}\, W_u d\xi + D\|\partial_\xi W_u\|^2_\xi \\
&+\frac{1}{2\epsilon}\int \left[2(-\partial_\xi K_1)W_\rho W_u + (1-\epsilon\sigma)(-\partial_\xi K_1)W_\rho^2 + (-\partial_\xi K_2)W_u^2\right] d\xi \\
&= \epsilon^2 \frac{d}{dt}\int W_u P_g d\xi + \epsilon \int [\partial_\xi W_\rho + (1-\epsilon\sigma)\partial_\xi W_u + \epsilon P_g] P_g\, d\xi \\
&+\epsilon \int \partial_\xi W_u P_{\mathcal{L}((v-s)g)} d\xi
\end{aligned}$$

The difference to the isentropic case appears only in the second line. Using the Young inequality for

$$\left|\int 2(-\partial_\xi K_1)W_\rho W_u d\xi\right| \le (1-\epsilon\sigma)\int(-\partial_\xi K_1)W_\rho^2 d\xi + \frac{1}{1-\epsilon\sigma}\int(-\partial_\xi K_1)W_u^2 d\xi,$$

the whole second line can be bounded from below by $\int \tilde\kappa(\phi)W_u^2 d\xi$, where $\tilde\kappa(\phi) = \frac{1}{\epsilon}[1-|u_\phi - u_l|/(1-\epsilon\sigma)]|\partial_\xi u_\phi| \ge 0$ for all ξ. The remaining estimates are analogous to the isentropic case.

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