Manuscript submitted to
AIMS' Journals
Volume **X**, Number **0X**, XX **200X**

doi:10.3934/xx.xx.xx.xx

pp. **X–XX**

ON NONLINEAR CONSERVATION LAWS REGULARIZED BY A RIESZ-FELLER OPERATOR

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Abstract. Scalar one-dimensional conservation laws with nonlocal diffusion term are considered. The wellposedness result of the initial-value problem with essentially bounded initial data for scalar one-dimensional conservation laws with fractional Laplacian is extended to a family of Riesz-Feller operators.

The main interest of this work is the investigation of smooth traveling wave solutions. In case of a genuinely nonlinear smooth flux function we prove the existence of such traveling waves, which are monotone and satisfy the standard entropy condition. Moreover, the dynamic nonlinear stability of the traveling waves under small perturbations is proven, similarly to the case of the standard diffusive regularization, by constructing a Lyapunov functional.

Apart from summarizing our results in the article Achleitner et al. (2011), we provide the wellposedness of the initial-value problem for a larger class of Riesz-Feller operators.

1. Introduction.

We consider one-dimensional conservation laws with nonlocal diffusion term

$$\partial_t u + \partial_x f(u) = \partial_x \mathcal{D}^\alpha u \tag{1}$$

for a scalar quantity $u : \mathbb{R}_+ \times \mathbb{R}$, $(t, x) \mapsto u(t, x)$, a smooth flux function $f : \mathbb{R} \to \mathbb{R}$ and a non-local operator

$$(\mathcal{D}^\alpha u)(x) = \frac{1}{\Gamma(1-\alpha)} \int_{-\infty}^{x} \frac{u'(y)}{(x-y)^\alpha} dy\,, \tag{2}$$

with $0 < \alpha < 1$.

2010 *Mathematics Subject Classification.* Primary: 47J35; 26A33; 35C07.

Key words and phrases. nonlocal evolution equation, fractional derivative, traveling wave.

The authors acknowledge support by the Austrian Science Fund under grant numbers W8 and P18367.

1.1. **Motivation.** Conservation laws with nonlocal diffusion term of the form (1) appear in viscoelasticity - modeling the far-field behavior of uni-directional viscoelastic waves [11] - as well as in fluid mechanics - modeling the internal structure of hydraulic jumps in near-critical single-layer flows [9]. Moreover the nonlocal operator $\mathcal{D}^{1/3}$ appears in Fowler's equation

$$\partial_t u + \partial_x u^2 = \partial_x^2 u - \partial_x \mathcal{D}^{1/3} u \,, \tag{3}$$

which models the uni-directional evolution of sand dune profiles [7].

Equation (1) is closely related to

$$\partial_t u + \partial_x f(u) = D^{\alpha+1} u \tag{4}$$

with a fractional Laplacian $D^{\alpha+1} = (-\frac{\partial^2 u}{\partial x^2})^{(\alpha+1)/2}$, $0 < \alpha < 1$. This kind of nonlinear conservation law with nonlocal regularization has been studied e.g. in [3, 5].

Remark 1. The nonlocal operators $\partial_x \mathcal{D}^\alpha$, $0 < \alpha < 1$, and the fractional Laplacian $D^{\alpha+1}$, $0 < \alpha < 1$, are Fourier multiplier operators, i.e.

$$\mathcal{F}(\partial_x \mathcal{D}^\alpha u)(\xi) = -(\sin(\alpha\pi/2) - i\cos(\alpha\pi/2)\,\mathrm{sgn}(\xi))|\xi|^{\alpha+1}\mathcal{F}u(\xi)$$

and

$$\mathcal{F}(D^{\alpha+1}u)(\xi) = -|\xi|^{\alpha+1}\mathcal{F}u(\xi)\,,$$

whereat the Fourier transform $\mathcal{F}$ is defined as $\mathcal{F}\varphi(\xi) = \widehat{\varphi}(\xi) = \frac{1}{\sqrt{2\pi}}\int e^{-ix\xi}\varphi(x)dx$.

1.2. **Riesz-Feller operators.** Riesz-Feller operators [6, 13, 8] are Fourier multiplier operators

$$(\mathcal{F}D_{a,\theta}f)(\xi) = -\psi_{a,\theta}(-\xi)(\mathcal{F}f)(\xi)$$

whose multiplier $\psi_{a,\theta}(\xi) = |\xi|^a e^{(i\,\mathrm{sgn}(\xi)\theta\pi/2)}$ is the logarithm of the characteristic function of a general Lévy strictly stable probability density with *index of stability* $0 < a \le 2$ and asymmetry parameter $|\theta| \le \min(a, 2-a)$. The nonlocal operators $\partial_x \mathcal{D}^\alpha$, $0 < \alpha < 1$, and the fractional Laplacian $D^{\alpha+1}$, $0 < \alpha < 1$, are Riesz-Feller operators, see also Remark 1 and Figure 1.

Theorem 1.1. *For $1 < a \le 2$ and $|\theta| \le \min\{a, 2-a\}$, the Riesz-Feller operator $D_{a,\theta}$ generates a strongly continuous, convolution semigroup*

$$T(t) : L^p(\mathbb{R}) \to L^p(\mathbb{R})\,, \quad u_0 \mapsto T(t)u_0 = K(t,\cdot) * u_0\,,$$

with $1 \le p < \infty$ and a convolution kernel $K(t,x) = \mathcal{F}^{-1}\exp(-t\psi(-.))(x)$ satisfying - for all $x \in \mathbb{R}$, $t > 0$ and $m \in \mathbb{N}$ - the properties

- *(non-negative) $K(t,x) \ge 0$,*
- *(integrable) $\|K(t,.)\|_{L^1(\mathbb{R})} = 1$,*
- *(scaling) $K(t,x) = t^{-\frac{1}{a}}K(1, xt^{-\frac{1}{a}})$,*
- *(smooth) $K(t,x)$ is $\mathcal{C}^\infty$ smooth,*
- *(bounded) there exists $B_m \in \mathbb{R}_+$ such that*

$$\left|\frac{\partial^m K}{\partial x^m}\right|(t,x) \le t^{-\frac{1+m}{a}}\frac{B_m}{1 + t^{-\frac{2}{a}}|x|^2}\,.$$

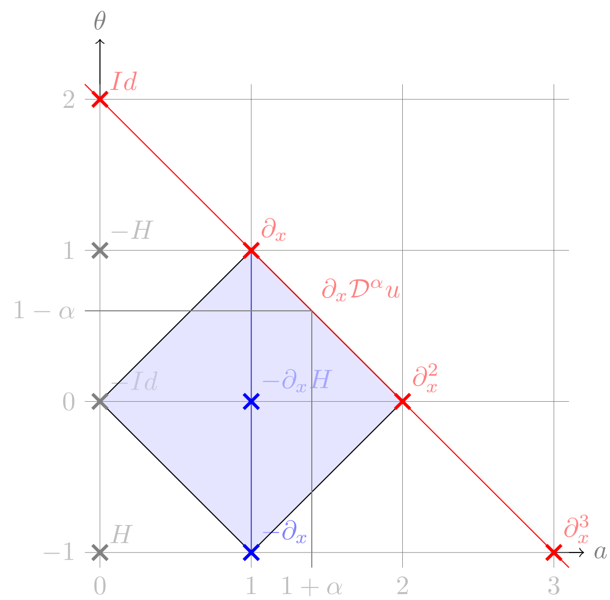

FIGURE 1. The family of Fourier multipliers $\psi_{a,\theta}(\xi) = |\xi|^a e^{(i\,\mathrm{sgn}(\xi)\theta\pi/2)}$ has two parameters a and θ. Some associated Fourier multiplier operators $(\mathcal{F}Tf)(\xi) = -\psi_{a,\theta}(-\xi)(\mathcal{F}f)(\xi)$ are displayed in the parameter space (a,θ). The Riesz-Feller operators $D_{a,\theta}$ are those operators, that take their parameters in the blue set, also known as Feller-Takayasu diamond. The family of operators $\partial_x\mathcal{D}^\alpha$, $0<\alpha<1$, interpolates formally between the first derivative ∂_x and second derivative ∂_x^2. Thus the limiting cases of equation (1) are a hyperbolic conservation law (for $\alpha = 0$) and a viscous conservation law (for $\alpha = 1$) [11].

The initial-value problem

$$\partial_t u + \partial_x f(u) = D_{a,\theta} u\,, \quad u(0,x) = u_0(x)\,, \tag{5}$$

for Riesz-Feller operators $D_{a,\theta}$ with *index of stability* $1 < a \le 2$ and asymmetry parameter $a-2 \le \theta \le 2-a$ covers the special cases (1) and (4).

Theorem 1.2. *Suppose $1 < a \le 2$ and $a-2 \le \theta \le 2-a$. If $u_0 \in L^\infty$, then there exists a unique solution $u \in L^\infty((0,\infty)\times\mathbb{R})$ of (5) satisfying the mild formulation*

$$u(t,x) = K(t,.) * u_0(x) - \int_0^t \left[\frac{\partial K}{\partial x}(t-\tau,.) * f(u(\tau,.))\right](x)\,\mathrm{d}\tau \tag{6}$$

almost everywhere. In particular

$$\|u(t,.)\|_\infty \le \|u_0\|_\infty, \qquad \text{for } t > 0\,,$$

and, in fact, u takes its values between the essential lower and upper bounds of u_0. Moreover, the solution has the following properties:

(i) $u \in C^\infty((0,\infty)\times\mathbb{R})$ and $u \in C_b^\infty((t_0,\infty)\times\mathbb{R})$ for all $t_0 > 0$.
(ii) u satisfies equation (5) in the classical sense.
(iii) $u(t) \to u_0$, as $t \to 0$, in $L^\infty(\mathbb{R})$ weak-$*$ and in $L^p_{loc}(\mathbb{R})$ for all $p \in [1,\infty)$.

Sketch of proof. The analysis of the initial-value problem for (4) by Droniou, Gallouët and Vovelle [5] depends on the properties in Theorem 1.1 of the semigroup (and its convolution kernel $K(t,x)$) generated by the fractional Laplacian $D^{\alpha+1}$ for $0<\alpha<1$. However all Riesz-Feller operators $D_{a,\theta}$ with *index of stability* $1<a\le 2$ and asymmetry parameter $a-2\le\theta\le 2-a$ share these properties. Thus the analysis in [5] carries over to the initial-value problem (5). □

2. **Traveling wave solutions.**

Definition 2.1. Suppose $(u_-,u_+,s)\in\mathbb{R}^3$. A traveling wave solution of (1) is a solution of the form $u(t,x)=\bar{u}(\xi)$ with $\xi:=x-st$ and some function $\bar{u}:\mathbb{R}\to\mathbb{R}$ that connects the distinct endstates $\lim_{\xi\to\pm\infty}\bar{u}(\xi)=u_\pm$.

Inserting a traveling wave ansatz in (1) and integrating with respect to ξ yields the traveling wave equation

$$h(u):=f(u)-su-\big(f(u_-)-su_-\big)=\mathcal{D}^\alpha u=\frac{1}{\Gamma(1-\alpha)}\int_{-\infty}^{x}\frac{u'(y)}{(x-y)^\alpha}\;\mathrm{d}y\,, \qquad (7)$$

which is translation invariant.

If a smooth profile $\bar{u}$ approaches the endstates sufficiently fast, then the formal limit $\xi\to\infty$ in (7) leads to the Rankine-Hugoniot condition $f(u_+)-f(u_-)=s(u_+-u_-)$.

If f is a convex flux function, then the vector field h is non-positive for values between u_- and u_+. Thus and due to the right-hand side of (7), a monotone traveling wave solution has to be monotone decreasing and the standard entropy condition $u_->u_+$ has to hold.

The profile $\bar{u}$ of a traveling wave solution is governed by (7), whence its value at $\xi\in\mathbb{R}$ depends (only) on its values on the interval $(-\infty,\xi)$. Therefore, first the existence of a profile on an interval $(-\infty,\xi_\varepsilon]$ is established, subsequently its monotonicity and boundedness are verified and finally its global existence is deduced from an continuation argument.

The integral operator

$$\mathcal{D}^\alpha u(\xi)=\frac{1}{\Gamma(1-\alpha)}\int_{-\infty}^{\xi}\frac{u'(y)}{(\xi-y)^\alpha}\;\mathrm{d}y$$

is of Abel type and can be inverted by multiplying it with $(z-\xi)^{-(1-\alpha)}$ and integrating with respect to ξ from $-\infty$ to z. Thus the traveling wave problem

$$h(u)=\mathcal{D}^\alpha u\,,\qquad \lim_{\xi\to-\infty}\bar{u}(\xi)=u_-\,,\quad \lim_{\xi\to+\infty}\bar{u}(\xi)=u_+\,, \qquad (8)$$

and

$$u(\xi)-u_-=\mathcal{D}^{-\alpha}(h(u))(\xi):=\frac{1}{\Gamma(\alpha)}\int_{-\infty}^{\xi}\frac{h(u(y))}{(\xi-y)^{1-\alpha}}\;\mathrm{d}y \qquad (9)$$

are equivalent if $u\in C_b^1(\mathbb{R})$ and $u'\in L^1(\mathbb{R}_-)$, and in particular if $u\in C_b^1(\mathbb{R})$ is monotone. Equation (9) is a nonlinear Volterra integral equation with a locally integrable kernel, where a well developed theory exists for problems on bounded intervals.

The linearizations of (8) and (9) at $\xi=-\infty$ (or, equivalently, at $u=u_-$) are

$$h'(u_-)v=\mathcal{D}^\alpha v\quad\text{and}\quad v=h'(u_-)\mathcal{D}^{-\alpha}v\,, \qquad (10)$$

respectively. Both linearizations have solutions of the form $v(\xi)=be^{\lambda\xi}$ with $\lambda=h'(u_-)^{1/\alpha}$ and arbitrary $b\in\mathbb{R}$, see also [4]. We will need that these are the only non-trivial solutions of (10) in the space $H^2(-\infty,\xi_0]$ for some $\xi_0 \le 0$. In particular, we assume that

$$\mathcal{N}\big(id - h'(u_-)\mathcal{D}^{-\alpha}\big) = \operatorname{span}\{\exp(\lambda\xi)\} \qquad \text{with} \qquad \lambda = h'(u_-)^{1/\alpha}\,, \tag{11}$$

which is reasonable due to our analysis in [1, Appendix A].

In the existence result both formulations (8) and (9) will be used.

Theorem 2.2 ([1, Theorem 2])**.** *Suppose $f \in C^\infty(\mathbb{R})$ is a convex flux function, the shock triple (u_-, u_+, s) satisfies the Rankine-Hugoniot condition $f(u_+) - f(u_-) = s(u_+ - u_-)$ as well as the entropy condition $u_- > u_+$, and condition (11) holds. Then there exists a decreasing solution $u \in C_b^1(\mathbb{R})$ of the traveling wave problem (8). It is unique (up to a shift) among all $u \in u_- + H^2((-\infty,0)) \cap C_b^1(\mathbb{R})$.*

Remark 2 (Extensions)**.** In [1] we prove the result assuming only

$$\begin{aligned} &h \in C^\infty([u_+,u_-])\,, \quad h(u_+) = h(u_-) = 0\,, \quad h < 0 \text{ in } (u_+,u_-)\,, \\ &\quad \exists\, u_m \in (u_+,u_-) \text{ such that } \quad h' < 0 \text{ in } (u_+,u_m) \text{ and } h' > 0 \text{ in } (u_m,u_-]\,. \end{aligned} \tag{12}$$

This is a little less than asking for convexity of f and the Lax entropy condition, since it covers the case $f'(u_+) \le s < f'(u_-)$.

The case of an concave flux function f can be analyzed in a similar way.

Idea of proof. The nonlinear problem has, up to translations, only two nontrivial solutions u_{down} and u_{up}, which can be approximated for large negative ξ by $u_- - e^{\lambda\xi}$ and $u_- + e^{\lambda\xi}$, respectively. The choice 1 of the modulus of the coefficient of the exponential is irrelevant due to the translation invariance of the traveling wave equations (7) and (9).

The traveling wave equation (7) involves a causal integral operator, i.e. to evaluate $\mathcal{D}^\alpha \bar{u}(\xi)$ at a point ξ the profile $\bar{u}$ on the interval $(-\infty,\xi]$ is needed. Thus, for $\varepsilon > 0$ and $\xi_\varepsilon := \log\varepsilon/\lambda$, we investigate the existence of solution $u_{down} : I_\varepsilon \to \mathbb{R}$ of (7) on the interval $I_\varepsilon = (-\infty,\xi_\varepsilon]$

$$\lim_{\xi\to-\infty} u_{down}(\xi) = u_- \quad \text{and} \quad u_{down}(\xi_\varepsilon) = u_- - \varepsilon\,. \tag{13}$$

Due to the analysis of the linearized equation (10) and assumption (11), the solution is written as $u_{down}(\xi) = u_- - \exp(\lambda\xi) + v$. Thus the perturbation v satisfies a boundary value problem (BVP)

$$\big(\mathcal{D}^\alpha - h'(u_-)\big)v = h(u_- - \exp(\lambda\xi) + v) + h'(u_-)\big(\exp(\lambda\xi) - v\big)\,, \quad v(\xi_\varepsilon) = 0\,.$$

This can be formulated as a fixed point problem for a given right-hand side in $H^2(I_\varepsilon)$ and an application of Banach's fixed point theorem yields the existence of u_{down} which is unique among all functions u satisfying (13) and $\|u - u_-\|_{H^2(I_\varepsilon)} \le \delta$ for some sufficiently small δ, which is independent of ε. Moreover

$$\|u_{down} - u_- + e^{\lambda\xi}\|_{H^2(I_\varepsilon)} \le C\varepsilon^2 \tag{14}$$

for some ε-independent constant C. The boundedness and monotonicity of u_{down},

$$u_{down}(\xi) < u_- \quad \text{and} \quad u'_{down}(\xi) < 0 \qquad \forall \xi \in I_\varepsilon\,,$$

follows from (14), a Sobolev embedding $H^2(\mathbb{R}) \hookrightarrow C^1(\mathbb{R})$ and the properties of $u_- - \exp(\lambda\xi)$.

Next, the continuation of the solution $u_{down} : (-\infty,\xi_\varepsilon] \to \mathbb{R}$ is proven. The boundedness and monotonicity of u_{down} imply that u_{down} is also a solution of (9).

Due to the causality of the integral operator, (9) can be written as a Volterra integral equation on a bounded interval $[\xi_\varepsilon, \xi_\varepsilon + \delta)$ for some $\delta > 0$

$$u(\xi) = f(\xi) + \frac{1}{\Gamma(\alpha)} \int_{\xi_\varepsilon}^{\xi} \frac{h(u(y))}{(\xi - y)^{1-\alpha}} \, \mathrm{d}y \,.$$

with a well-defined inhomogeneity $f(\xi) = u_- + \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{\xi_\varepsilon} \frac{h(u(y))}{(\xi-y)^{1-\alpha}} \, \mathrm{d}y$. The (local) existence of a smooth solution for sufficiently small δ is a standard result in the theory of Volterra integral equations on bounded intervals, see e.g. Linz [10].

Then, the boundedness and monotonicity of these continued solutions is proven, such that the argument for local existence can be iterated to imply the existence of a solution

$$u_{down} \in C_b^1(\mathbb{R}) \quad \text{with} \quad \lim_{\xi \to \infty} u_{down}(\xi) = u_- \,.$$

Finally, the proof of Theorem 2.2 is completed by proving $\lim_{\xi\to\infty} u(\xi) = u_+$. Assuming to the contrary $\lim_{\xi\to\infty} u(\xi) > u_+$, would imply $\lim_{\xi\to\infty} h(u(\xi)) < 0$. Then, however, $-\mathcal{D}^{-\alpha} h(u) = u_- - u$ would increase above all bounds, which is impossible by the boundedness of the solution. □

Remark 3 (Discussion of previous results)**.** Sugimoto and Kakutani [11, 12] studied the existence of traveling wave solutions of (1). They prove that bounded continuous traveling wave solution may exist, but give no analytical proof of existence, instead they construct numerical solutions and study the asymptotic behavior analytically.

In case of Burgers' equation with fractional Laplacian (4), Biler et al. [3] showed that no continuous traveling wave solutions can exist for $\alpha \in (-1, 0]$, however they provide no existence result for the case $\alpha \in (0, 1)$.

Alvarez-Samaniego and Azerad [2] proved the existence of traveling wave solutions of (3) with perturbation methods.

Remark 4 (Comparison with previous results)**.** The dynamical systems approach to prove the existence of traveling wave solutions in [1, Theorem 2], parallels the one in case of viscous conservation laws. This approach is possible due to the causality of the operator $\mathcal{D}^\alpha$ in (7) and the monotonicity of the profiles.

In contrast in case of a conservation law with fractional Laplacian (4) the traveling wave equation for traveling wave solutions $u(t, x) = \bar{u}(\xi)$ with $\bar{u} \in C_b^2(\mathbb{R})$ can be written as

$$h(u) := f(u) - su - \big(f(u_-) - su_-\big) = \frac{1}{\Gamma(1-\alpha)} \int_{-\infty}^{\infty} \frac{u'(y)}{(x - y)^\alpha} \, \mathrm{d}y \,.$$

Thus the value of a profile $\bar{u}$ at $\xi \in \mathbb{R}$ depends on the entire profile $\bar{u}$, such that a different approach is needed.

Whereas in case of Fowler's equation (3) the profile of a traveling wave solution is not necessarily monotone, such that the boundedness of a profile is difficult to establish.

2.1. **Asymptotic stability of traveling wave solutions.** To study the asymptotic stability of traveling wave solutions ϕ of (1), equation (1) is cast in a moving coordinate frame $(t, x) \to (t, \xi = x - st)$,

$$\partial_t u + \partial_\xi (f(u) - su) = \partial_\xi \mathcal{D}^\alpha u \,, \tag{15}$$

such that a traveling wave solution becomes a stationary solution of (15). Analogous to viscous conservation laws asymptotic stability of ϕ is only to be expected for integrable zero-mass perturbations $U_0 := u_0 - \phi$, i.e.

$$\int_{\mathbb{R}} U_0(\xi)\ \mathrm{d}\xi = 0\,. \tag{16}$$

The evolution of a perturbation $U := u - \phi$ is governed by

$$\partial_t U + \partial_\xi (f(\phi + U) - f(\phi) - sU) = \partial_\xi \mathcal{D}^\alpha U\,. \tag{17}$$

However the L^2-norms of the perturbation U and its derivative are not enough to construct a Lyapunov functional. Therefore the primitive

$$W(t,\xi) = \int_{-\infty}^{\xi} U(t,\eta)\ \mathrm{d}\eta$$

of the perturbation U has to be considered.

The flux function will be assumed to be convex between the far-field values $u_\pm$ of the traveling wave solution ϕ, i.e.

$$f''(\phi(\xi)) \geq 0 \quad \text{for all} \quad \xi \in \mathbb{R}\,. \tag{18}$$

Theorem 2.3 ([1, Theorem 4]). *Suppose $f \in C^\infty(\mathbb{R})$, the conditions (12) and (18) hold and ϕ is a traveling wave solution of (1) as in Theorem 2.2. Let u_0 be such that $W_0(\xi) = \int_{-\infty}^{\xi} (u_0(\eta) - \phi(\eta))\ \mathrm{d}\eta$ satisfies $W_0 \in H^2(\mathbb{R})$. If $\|W_0\|_{H^2}$ is small enough, then the initial-value problem for equation (15) with initial datum u_0 has a unique global solution converging to the traveling wave solution ϕ in the sense that*

$$\lim_{t\to\infty} \int_t^\infty \|u(\tau,.) - \phi\|_{H^1}\ \mathrm{d}\tau = 0\,. \tag{19}$$

Proof. First, the local-in-time wellposedness of the initial-value problem

$$\partial_t W + (f(U+\phi) - f(\phi) - sU) = \partial_\xi \mathcal{D}^\alpha W\,, \quad W(0,x) = W_0(x)\,, \tag{20}$$

is established by an fixed point argument [1, Proposition 2].

Then a (Lyapunov) functional

$$J(t) = \frac{1}{2}(\|W\|_{L^2}^2 + \gamma_1 \|U\|_{L^2}^2 + \gamma_2 \|\partial_\xi U\|_{L^2}^2)$$

is defined with positive constants $\gamma_1, \gamma_2 > 0$. The functional $J : H^2(\mathbb{R}) \to \mathbb{R}$, $W(t) \mapsto J(t)$, is equivalent to $\|W(t)\|_{H^2}^2$, since $\gamma_* \|W(t)\|_{H^2}^2 \leq 2J(t) \leq \gamma^* \|W(t)\|_{H^2}^2$ with $\gamma_* = \min\{1, \gamma_1, \gamma_2\}$ and $\gamma^* = \max\{1, \gamma_1, \gamma_2\}$. Combining the energy estimates of the perturbation U, its primitive W and its derivative $\partial_\xi U$, and using a Gagliardo-Nirenberg inequality yields

$$\begin{aligned} \frac{d}{dt} J + a_\alpha \left(\|W\|_{\dot H^{(1+\alpha)/2}}^2 + \gamma_1 \|W\|_{\dot H^{(3+\alpha)/2}}^2 + \gamma_2 \|W\|_{\dot H^{(5+\alpha)/2}}^2 \right) \\ - \gamma_1 C_0 \|U\|_{L^2}^2 - \gamma_2 C_1 \|U\|_{H^1}^2 - L(\|W\|_{H^2}) \|W\|_{H^2} \|U\|_{H^{(5+\alpha)/4}}^2 \leq 0\,, \end{aligned}$$

where $a_\alpha = \sin(\alpha\pi/2) > 0$ and $\dot H^s$ denotes the homogeneous Sobolev space of order s. Finally, the constants $\gamma_1, \gamma_2 > 0$ are chosen such that

$$\begin{aligned} \gamma_1 C_0 \|U\|_{L^2}^2 + \gamma_2 C_1 \|U\|_{H^1}^2 \\ \leq \frac{a_\alpha}{2} \left(\|W\|_{\dot H^{(1+\alpha)/2}}^2 + \gamma_1 \|W\|_{\dot H^{(3+\alpha)/2}}^2 + \gamma_2 \|W\|_{\dot H^{(5+\alpha)/2}}^2 \right)\,, \end{aligned}$$

which implies the final estimate

$$\frac{d}{dt}J + \left(\frac{a_\alpha}{2} - \frac{1}{\gamma_*}L(\|W\|_{H^2})\|W\|_{H^2}\right)\left(\|W\|^2_{\dot{H}^{(1+\alpha)/2}} + \gamma_1\|W\|^2_{\dot{H}^{(3+\alpha)/2}}\right)$$
$$+ \gamma_2\left(\frac{a_\alpha}{2} - \frac{1}{\gamma_*}L(\|W\|_{H^2})\|W\|_{H^2}\right)\|W\|^2_{\dot{H}^{(5+\alpha)/2}} \ \le\ 0\,.$$

For initial data such that $J(0)$ is sufficiently small, the functional $J(t)$ - being equivalent to $\|W(t)\|^2_{H^2}$ - is non-increasing for all times. This implies the global-in-time existence of $W(t)$ as a solution of (20) and moreover (19). □

Remark 5. In case of Burgers' flux $f(u) = u^2$ and $\alpha > 1/2$, asymptotic stability of a traveling wave solution ϕ is established in case of $W_0 \in H^1(\mathbb{R})$, see also [1, Theorem 3].

Due to a Sobolev embedding $H^1(\mathbb{R}) \hookrightarrow C_b(\mathbb{R})$, the asymptotic stability result $\lim_{t\to\infty}\|U(t)\|_{H^1} = 0$ implies also $\lim_{t\to\infty}\|U(t)\|_{L^\infty} = 0$.

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Received December 2012; revised October 2013.

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