On the profitability of collusion in location games

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Abstract

In this note we take a first step towards the analysis of collusion in markets with spatial competition, focusing on the case of pure location choices. We find that collusion can only be profitable if a coalition contains more than half of all players. This result holds for location games played in k-dimensional Euclidean space as long as consumers are distributed via atomless density functions. For competition on the unit interval, unit circle, and unit square we also derive sufficient conditions for collusion to be profitable. The results have immediate implications for mergers in spatial markets.
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1. Introduction

While the economics literature has paid considerable attention to collusion in Bertrand and Cournot markets, collusion with different sorts of competition has been largely neglected. In this note we take a first step towards the analysis of collusion in pure location games as introduced by Hotelling [5]. Such models capture competition in many important

industries where price competition is not feasible, for example, because of regulation (as in the case of pharmacies) or vertical restraints (as in the case of book sellers).[1]

Our results are based on an approach that relies on rather weak rationality requirements. In particular, we do not solve the non-cooperative game in which some of the players can reach binding agreements. Instead, we simply require that players will only decide to collude if they can *guarantee* themselves a payoff better than the payoff expected "behind the veil of ignorance." The reason for this approach is simple: it is as we will see extremely difficult to find Nash equilibria for location games with collusion. We argue that in the absence of reliable non-cooperative solutions players should be conservative and only collude if they know for sure that this will be profitable. Accordingly, our definition of profitability relies on a maxmin approach. Nevertheless, we include one section on Nash equilibrium where we show that, in those few cases where we can find one, the non-cooperative solution coincides with ours.

For linear and circular cities with a uniform distribution of consumers we find that collusion is profitable if and only if more than half of the players collude. Part of this result can be generalized to location games in multi-dimensional spaces with arbitrary density functions: As long as the distribution of consumers is atomless, collusion can only be profitable if more than half of all firms cooperate. For competition on the unit interval, unit circle, and unit square we are also able to derive sufficient conditions for collusion to be profitable. These results are of considerable relevance for the topic of merger in markets with limited price competition.

The remainder of the paper is organized as follows. Section 2 introduces the general setup and notation. Section 3 deals with the simplest one-dimensional cases, i.e., with linear and circular cities with uniform consumer densities. Section 4 deals with the general multi-dimensional case and establishes the main theorem of the paper.[2] Section 5 adds sufficient conditions for collusion to be profitable in games on the unit line, unit circle and unit square. Section 6 discusses Nash equilibria for location games with collusion and Section 7 concludes.

2. Setup and definitions

Let $\Gamma(O, P)$ be a location game on the connected subset $O \subseteq \mathbb{R}^k$ with set of players P. Let $p^i \in P$ be player i with $i = 1, 2, \ldots, n$. Each player p^i chooses a location $x^i \in O$. Consumers are distributed over O via a Lebesgue measurable density function f with total mass 1. Let $d(o, o')$ be the distance between two points $o, o' \in O$. Each consumer is assumed to buy one unit of an unspecified good from the player closest to her. That is, a consumer at $o \in O$ buys from player p^i only if $d(o, x^i) = \min_j d(o, x^j)$. If there are more

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[1] They can also be applied to parliamentary elections that are not winner-take-all contests.

[2] Note that the pioneering work by Eaton and Lipsey [4], Okabe and Suzuki [9], Okabe and Aoyagi [8] and Aoyagi and Okabe [1] on the stability of configurations of firms in $\mathbb{R}^2$, while not used directly in the proof of our main result regarding the general multi-dimensional case, provide a base for the study of market sets in higher dimensions. Furthermore, Okabe and Aoyagi [8] was used more directly (by way of Knoblauch [7]) in our work on the unit square.

than one closest player then the consumer is assumed to buy from each closest player with the same probability. The price of the good is fixed at 1 and production costs are normalized to zero.

Let $O^i(\Gamma) = \{o \mid d(o, x^i) = \min_j d(o, x^j)\}$. Player p^i's market share and profit is then given by $\pi^i(\Gamma) = (1/r^i) \int_{O^i} f(o) \, do$ where r^i denotes the number of players located at x^i. By assumption, $\sum_i \pi^i = 1$. By virtue of this fact, we say that a player's expected payoff before the game is actually played ("behind the veil of ignorance") is $1/n$.[3]

Next we define for integer m with $1 \leqslant m < n$ a set $V(m)$ of reals with $v \in V(m)$ if there is a collusion strategy for a set $\mathcal{M} \subseteq P$ of m players that *guarantees* them a total payoff of at least v. Let $v(m) = \sup V(m)$.[4]

Definition 1. Collusion of a set of m players is profitable if $v(m) > m/n$.

Thus, collusion is profitable if the *maxmin payoff* of a coalition of size m is strictly greater than the expected non-cooperative equilibrium payoff when equilibrium positions are assigned randomly. This is a conservative approach that rules out any unpleasant surprises for the set of colluding players. In a framework where multiplicity of equilibria is to be expected and where computation of non-cooperative equilibria is very hard, this appears rather appealing. And it becomes even more appealing once we have shown that the maxmin strategy of a coalition (that is large enough) has a very simple and intuitive structure.

One implication of our definition is the following.

Remark 1. Collusion of all players (the "grand coalition") is never profitable.

This follows immediately from the observation that the location game is a constant-sum game. In contrast to Cournot or Bertrand games total industry profits cannot be manipulated by contracting output or raising price.

3. The one-dimensional case with uniform distributions

3.1. Linear cities

Let us first consider the standard textbook case of a "linear city" in which $O = [0, 1]$ and in which consumers are uniformly distributed. How can a set of m players guarantee itself a "high" payoff? Suppose $m > n - m$, i.e., suppose that more than half of all firms are in the set of colluding players. In that case the colluding players can minimize the payoff obtainable to a firm outside the coalition by "evenly spreading out." If f is uniform, the firms in the set can guarantee themselves a payoff of $(3m - n)/(2m)$ by locating themselves at $(k, 3k, 5k, \ldots, 1 - k)$ with $k = 1/(2m)$. To see this, note that in this case a firm outside

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[3] For example, a player could expect that every assignment of players to equilibrium locations is equally likely.

[4] In other words, $v(m)$ is the maxmin payoff of the coalition.

the set $\mathcal{M}$ is indifferent between all possible locations as each location yields a payoff of $1/(2m)$. Furthermore, the worst thing that can happen to the players in $\mathcal{M}$ is that the firms outside locate in different intervals, say, one between k and $3k$, one between $3k$ and $5k$ and so on. If they do, the players in $\mathcal{M}$ earn $1 - (n - m)/(2m) = (3m - n)/(2m)$. And as this is larger than m/n for $m > n/2$ collusion turns out to be profitable. Thus $m > n/2$ is sufficient for collusion to be profitable in linear cities with a uniform distribution of consumers. That it is also necessary in this case is stated in

Proposition 1. *In linear cities with a uniform distribution of consumers collusion is profitable if and only if $n > m > n/2$.*

Proof. The argument above shows that $m > n/2 \Rightarrow v(m) > m/n$. Next observe that, by definition,

$$v(m) + v(n - m) \leqslant 1. \tag{1}$$

Hence, $m = n/2 \Rightarrow m = n - m \Rightarrow v(m) \leqslant 1/2 = m/n$, i.e., collusion is not profitable if exactly half of all firms cooperate. The proof is completed by showing that collusion is also not profitable if $m < n/2$: If $1 \leqslant m < n/2$, then $n/2 < n - m \leqslant n - 1$ so that by the first part of the proof $v(n - m) > (n - m)/n$. Therefore, by Eq. (1), $v(m) < 1 - (n - m)/n = m/n$. □

3.2. Circular cities

A further popular space to study location games on is a circle. In contrast to the line a set of m colluding firms can divide a circle into at most m arcs as opposed to $m + 1$ segments on the line. Nevertheless, one obtains the identical condition for collusion to be profitable.

Proposition 2. *In circular cities with a uniform distribution of consumers collusion is profitable if and only if $n > m > n/2$.*

Proof. Position the colluding firms such that there are m arcs with mass $1/m$ each. If $m \geqslant n/2$, the maximum total payoff the non-colluding firms can obtain is $(n - m)/(2m)$, i.e., by using this strategy the colluding firms can ensure a payoff of $(3m - n)/(2m)$ which is greater than m/n if $m > n/2$. Using Eq. (1) again completes the proof. □

4. The multi-dimensional case

The following result is the main result of the paper. It generalizes one of the two insights gained above, namely that collusion in location games can only be profitable if more than half of all firms cooperate. This result holds for arbitrary bounded, connected and convex open subsets of $\mathbb{R}^k$ and for arbitrary bounded atomless density functions.

Theorem 1. *Suppose consumers are distributed over a bounded and convex open subset $O \subseteq \mathbb{R}^k$ via a bounded Lebesgue measurable density function f of total mass 1. For the*

n-player location game $\Gamma(O, P)$ it is not profitable for a set of m players to collude if $m \leqslant n/2$.

Proof. Suppose the m colluding players $p^1, p^2, \ldots, p^m$ locate at $x^1, x^2, \ldots, x^m \in O$, not necessarily distinct.

Case 1: $n - m \geqslant 2m$. Then for each i, $1 \leqslant i \leqslant m$, let p^{m+2i-1} and p^{m+2i} locate at x^{m+2i-1} and x^{m+2i}, two points ε units apart on a line through x^i, with x^i between x^{m+2i-1} and x^{m+2i} and ε chosen as follows. Let B be a k-dimensional ball containing O and let A be the $(k-1)$-dimensional volume of the $(k-1)$-dimensional disk formed by intersecting B with a hyperplane through its center. Choose ε such that $\varepsilon < 1/(nA \sup f)$ and such that ε is small enough to guarantee $x^{2m+i-1}, x^{2m+i} \in O$ for $1 \leqslant i \leqslant m$. Let the rest of the non-colluding players, $p^{3m+1}, p^{3m+2}, \ldots, p^n$ locate anywhere in O. Since the consumers won by p^i, $1 \leqslant i \leqslant m$, lie between two hyperplanes ε units apart, π^i is at most $\varepsilon A \sup f < 1/n$. Hence, $v(m) < m/n$.

Case 2: $m < n - m < 2m$. For $1 \leqslant i \leqslant m$, define the provisional market set $O_{\text{prov}}^i = O^i(\Gamma')$ with $\Gamma' = \Gamma(O, \mathcal{M})$, i.e., O_{prov}^i contains the points in O that are nearer to x^i than to any other $x^j \neq x^i$ with both $i, j \leqslant m$. Accordingly, define the provisional payoff $\pi_{\text{prov}}^i = \pi^i(\Gamma')$. Without loss of generality assume that the sequence $\pi_{\text{prov}}^1, \pi_{\text{prov}}^2, \ldots, \pi_{\text{prov}}^m$ is non-decreasing. Now locate $3m - n$ of the non-colluding players at $x^1, x^2, \ldots, x^{3m-n}$ and use the remaining $2n - 4m$ players to bracket $x^{3m-n+1}, x^{3m-n+2}, \ldots, x^m$ as in Case 1, but do net yet choose ε. Notice that

   (i) $3m - n > 0$;
   (ii) $2n - 4m > 0$;
   (iii) $(3m - n) + (2n - 4m) = n - m$; and
   (iv) $(3m - n) + (2n - 4m)/2 = m$.

Since the sequence $\pi_{\text{prov}}^1, \pi_{\text{prov}}^2, \ldots, \pi_{\text{prov}}^m$ is non-decreasing, the sum of the provisional payoffs $\pi_{\text{prov}}^1 + \pi_{\text{prov}}^2 + \cdots + \pi_{\text{prov}}^{3m-n}$ is at most $(3m - n)/m$. Therefore, the final total payoffs to the colluding players $\sum_{i=1}^m \pi^i$ is at most $(3m - n)/2m + \varepsilon(n - 2m)A \sup f$. Now notice that $(3m - n)/(2m) < m/n$. Hence, it is possible to choose ε such that $m/n - \sum_{i=1}^m \pi^i > 0$. Collusion is not profitable.

Case 3: $m = n - m$. Nonprofitability follows from Eq. (1) as in the proof of Proposition 1. □

Thus, we know that collusion in location games (on bounded open subsets of $\mathbb{R}^k$ in which consumers are distributed via atomless density functions) can only be profitable if more than half of all firms join the set $\mathcal{M}$.

Remark 2. Note that neither the closed interval $[0, 1]$ nor a circle is an open subset of an Euclidean space. However, the conclusion of the theorem holds for location games on these sets, since the techniques of the proof apply. More particularly, it is possible to bracket colluding players as in the proofs. In fact, a colluding player at 0 or 1 in $[0, 1]$ can be bracketed by a single non-colluding player. Note, furthermore, that the proof of Theorem 1

would not work without requiring boundedness of the open subset $O \subseteq \mathbb{R}^k$. Without this assumption the volume A defined in the proof (and occurring in the definition of ε) would be infinite.

Remark 3. The theorem concerns location games defined using Euclidean distances, i.e., straight line distances. Implicitly, this means that consumers may travel along routes that do not belong to O. However, the theorem applies, for example, to a circle (or rather the conclusion of the theorem holds—see Remark 2) even when the distance between two points is the length of the arc joining them, since for a circle in $\mathbb{R}^2$ a consumer's nearest player is the same whether distance is defined as Euclidean distance or as arc length.

The theorem disallows atoms of consumers. The following example demonstrates the necessity of this assumption.

Example. Consider the 5-player location game on $[0, 1]$ with two consumers, one at $1/4$ and one at $2/3$. Suppose p^1 and p^2 collude by locating at $1/4$ and $2/3$, respectively. Their worst total payoff occurs when p^3 and p^4 locate at $1/4$ and p^5 locates at $2/3$. The total payoff of p^1 and p^2 is then $1/3 + 1/2 = 5/6$ which is greater than the veil of ignorance expected payoff of $2(2/5) = 4/5$. Collusion is profitable with $m = 2$ even though $m < n/2$. As in the proof of Proposition 1, where it is shown that the complement of a profitable set of colluding players is unprofitable, collusion is unprofitable for $m = 3$, even though in that case $m > n/2$.

5. Sufficient conditions for unit interval, unit circle, and unit square

The main theorem above showed that $m > n/2$ is necessary for collusion to be successful. In the following we will establish sufficient conditions for collusion to be profitable in a location game played on the unit interval, the unit circle, and the unit square. Notice that in each case the solution prescribes that the colluding players behave according to the above identified strategies, i.e., they will evenly spread out making other players indifferent between locations.

Proposition 3. *In linear cities, collusion is profitable if* $\sup f / \inf f < 2m/n$ *and* $m < n$.

Remark 4. Note that $\sup f / \inf f \geqslant 1$. Thus, the condition in Proposition 3 ensures that $m > n/2$.

Proof of Proposition 3. Without loss of generality, let $x^1 \leqslant x^2 \leqslant \cdots \leqslant x^m$ be the set of locations occupied by the colluding players chosen so that

$$\int_0^{x^1} f(o)\,\mathrm{d}o = \frac{1}{2}\int_{x^1}^{x^2} f(o)\,\mathrm{d}o = \frac{1}{2}\int_{x^2}^{x^3} f(o)\,\mathrm{d}o = \cdots = \int_{x^m}^{1} f(o)\,\mathrm{d}o = \frac{1}{2m}.$$

If a non-colluding player locates to the left of x^1 or to the right of x^m, his payoff is at most $1/(2m) < 1/n$. If a non-colluding player locates between x^i and x^{i+1}, his payoff is $\int_c^d f(o)\,do$ where $x^i < c < d < x^{i+1}$ and $d - c = (x^{i+1} - x^i)/2$. Then

$$
\int_c^d f(o)\,do \leqslant (d-c)\sup f = \frac{x^{i+1} - x^i}{2}\inf f\left(\frac{\sup f}{\inf f}\right) \leqslant \frac{1}{2}\int_{x^i}^{x^{i+1}} f(o)\,do\left(\frac{\sup f}{\inf f}\right)
$$

$$
< \frac{1}{2m}\left(\frac{2m}{n}\right) = \frac{1}{n}.
$$

If a non-colluding player locates at x^i, $1 \leqslant i \leqslant m$, then he shares the market set O^i with p^i. By the argument above, the portion of O^i to the left of x^i has consumer mass less than $1/n$, as does the portion of O^i to the right of x^i. Therefore, the payoff to each non-colluding player is less then $(1/n + 1/n)/2 = 1/n$. Since in all these cases the payoff to a non-colluding player is less than $1/n$, the total payoff to the colluding players is more than $1 - (n-m)/n = m/n$. Collusion is profitable. $\quad\square$

The sufficient condition in Proposition 3 is stronger than necessary. For instance, we used as an assumption on f only that

$$
\frac{\sup\{f(x):\ x^i < x < x^{i+1}\}}{\inf\{f(x):\ x^i < x < x^{i+1}\}} < \frac{2m}{n}.
$$

This allows any amount of variation to the left of x^1 and to the right of x^m and, if m is large, between x^1 and x^m.[5]

Proposition 4. *In circular cities, collusion is profitable if $n > m > n/2$ and $\sup f/\inf f < 2m/n$.*

Proof. Analogous to the proofs of Propositions 2 and 3. $\quad\square$

Finally, we look at location games played on the unit square with uniform consumer density.

Proposition 5. *For the n-player location game on the square $[0,1] \times [0,1]$ with consumers distributed uniformly, collusion is profitable if there is a positive integer h with $(2h+1)^2 - h^2 \leqslant m < n < (2h+1)^2$.*

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[5] Moreover, the firms located at x^1 and x^m could move further into the interior as the mass on the fringes has only to be smaller than $1/n$. Using this, one can increase the allowed variation between x^1 and x^m from $2m/n$ to $[2(m-1)/(n-2)]^{m-1} > 2m/n$. To see this, simply observe that the colluding players can position themselves so that the remaining mass between x^1 and x^m, $1 - 2/n$, is equally distributed over $m-1$ intervals. The proof then goes through with $\sup\{f(x):\ x^i < x < x^{i+1}\}/\inf\{f(x):\ x^i < x < x^{i+1}\} < 2(m-1)/(n-2)$. Therefore, $\sup\{f(x):\ x^1 < x < x^m\}/\inf\{f(x):\ x^1 < x < x^m\}$ can be as large as $[2(m-1)/(n-2)]^{m-1}$.

Proof. Suppose m, n and h satisfy the hypotheses of the theorem. Consider the set C of points in $[0, 1] \times [0, 1]$ of the form $((i - 1/2)/(2h + 1), (j - 1/2)/(2h + 1))$ where i and j are integers, $1 \leqslant i, j \leqslant 2h + 1$, and i and j are not both even. There are exactly $(2h + 1)^2 - h^2$ points in C. Locate the m colluding players so that there is at least one of them at each point of C (recall that $m \geqslant (2h + 1)^2 - h^2$). In the course of proving that an infinite square lattice is a Nash equilibrium for the location game in the plane with consumers distributed uniformly, Knoblauch [7] proved that in the location game on $[0, 1] \times [0, 1]$, any player with at least one opponent at each point of C earns a payoff of at most $1/(2h + 1)^2$ so that the non-colluding players' total payoff is at most

$$\frac{(n - m)}{(2h + 1)^2} < \frac{(n - m)}{n}.$$

Hence, $v(m) > m/n$. $\square$

Proposition 5 tells us that there are profitable coalitions for the n-player location game on $[0, 1] \times [0, 1]$ if $n \in ((2h + 1)^2 - h^2, (2h + 1)^2)$ for some positive integer h, that is, if $n \in (21, 25) \cup (40, 49) \cup (65, 81) \cup (96, 121) \cup (133, 169) \cup (176, 225) \cup (225, +\infty)$. There are only seven intervals since for $h \geqslant 8$ the intervals $((2h + 1)^2 - h^2, (2h + 1)^2)$ and $((2(h + 1) + 1)^2 - (h + 1)^2, (2(h + 1) + 1)^2)$ overlap.

Proposition 5 implies a simpler but more restrictive sufficiency condition for profitability: $m \geqslant (225/226)n$. This condition is trivially sufficient for $n < 226$ and can be established as sufficient for $n \geqslant 226$ using the overlapping of the intervals. Furthermore, for large n, Proposition 5 says, roughly, that collusion is profitable if $m > 3n/4$. This interpretation follows from the fact that for large n there is an integer h such that $n < (2h + 1)^2$, $(2h + 1)^2/n \approx 1$, and $((2h + 1)^2 - h^2)/n \approx 3/4$. For example, if $n = 1,000,000$ choose $h = 500$. Then $(2h + 1)^2 = 1,002,001$ and $(2h + 1)^2 - h^2 = 752,001$. The proposition says collusion is profitable if $m/1,000,000 \geqslant 0.752001$.

To find profitable coalitions in two dimensions for some values of n not included in the hypotheses of Proposition 5, one can replace $[0, 1] \times [0, 1]$ with any figure constructed from finitely many copies of $[0, 3/(2h + 1)] \times [0, 3/(2h + 1)]$ with a coalition using the eight locations $((i - 1/2)/(2h + 1), (j - 1/2)/(2h + 1))$, $1 \leqslant i, j \leqslant 3$, i and j not both even. The only requirement is that these squares are assembled with an overlap of $1/(2h + 1)$ as in the proof of Proposition 5. For example for $[0, 1] \times [0, 3/(2h + 1)]$ made up of h squares in a line, mimicking the proof of Proposition 5 the sufficiency condition for profitability would be $3(2h + 1) - h \leqslant m < n < 3(2h + 1)$. Then an n-player location game allows profitable coalitions for some 1-by-h rectangle if $n \in (13, 15) \cup (18, 21) \cup (23, 27) \cup (28, 33) \cup (33, +\infty)$. The relation between m and n will always involve inequalities like those in Proposition 5, since the little that is known about location games in dimension 2 restricts us to working with figures constructed of finitely many squares of equal size on each of which eight locations are used.

6. Nash equilibria

It is natural to ask why we chose to use the definition of profitability introduced in Section 2 in our study of collusion in location games. Why did we not take a standard

approach and try to use the Nash equilibrium concept? We will see in this section that for a location game in which one player controls a coalition, establishing the existence of a Nash equilibrium is a difficult task. Fortunately, Proposition 6 below demonstrates that the profitability approach is a reasonable substitute for the Nash equilibrium concept: for the very limited conditions under which we were able to establish the existence of a Nash equilibrium, the configuration of coalition firms and the payoff to the coalition are the same as in the Section 3.1 demonstration that collusion is profitable on $[0, 1]$ for $n > m > n/2$.

Consider a location game on the unit interval $[0, 1]$ with consumers distributed uniformly, played by several independent firms and one player who controls a set of firms. Can we find location strategies for the independent firms so that these strategies together with the profitable collusion strategy identified above comprise a Nash equilibrium?

It is possible to answer this question in some special cases and we shall do this below. However, in general the problem is very difficult, perhaps intractable.

The difficulty arises from two sources. The first thing one discovers when working on the problem is that a Nash equilibrium requires mixed strategies for the independent firms. Unfortunately, due to the computational complexities, little is known about mixed strategy equilibria for location games on the unit interval. Shaked [12] constructed a mixed strategy Nash equilibrium for three firms locating on $[0, 1]$, and there are nonconstructive existence theorems by Dasgupta and Maskin [2,3] and Simon [13]. Second, the difficulty in finding mixed strategy equilibria is compounded when one player controls m locations, due to the added computational complexity.

Prospects are even bleaker for location games with collusion in dimension 2 and higher. Up to now, nothing has been published on location games in dimension 3 or above, and little on solutions for location games in dimension 2. Shaked [11] showed that there are no pure-strategy Nash equilibria for a wide variety of 3-player location games in the plane; Okabe and Suzuki [9] established the existence of a configuration of firms in a square that satisfy a weak stability condition; Okabe and Aoyagi [8] proved that an infinite square array of firms in the plane is a Nash equilibrium for a uniform distribution of consumers; Aoyagi and Okabe [1] studied the relationship between the shape of a 2-dimensional market and the configuration of an equilibrium in that market, and Knoblauch [6] catalogued all 3-player equilibria on the 2-sphere when consumers are distributed uniformly.

In summary, the difficulty of finding mixed strategy equilibria for location games translates into difficulty for our problem-finding equilibria for location games with collusion. It seems reasonable that firms that engage in games that game theorists are unable to solve should choose rather conservatively when it comes to making big decisions such as decisions about colluding or merging with others. We have therefore proposed profitability as a conservative criterion to be used by firms faced with collusion decisions or merger proposals.

Nevertheless, the following proposition answers the question posed at the beginning of this section in the affirmative in the special case that the number of locations controlled by the "big" player is an integral multiple of the number of independent firms.

Proposition 6. *Let G be a location game in which consumers are distributed uniformly on $[0, 1]$ with density 1, players $1, 2, \ldots, n - m$, are independent firms, player $n - m + 1$ controls m locations and $m = a(n - m)$ where a is a positive integer greater than 1. Let*

s_{n-m+1} *be player $n - m + 1$'s strategy from Section 3.1, which picks locations at each element of the set $\{k, 3k, 5k, \ldots, 1 - k\}$, where $k = 1/2m$. For $i = 1, 2, \ldots, n - m$, let s_i be player i's strategy that assigns probability $1/a$ to each of the a points $(2ai - 2a + 1)/2m, (2ai - 2a + 3)/2m, \ldots, (2ai - 1)/2m$.*[6] *Then $(s_1, s_2, \ldots, s_{n-m}, s_{n-m+1})$ is a Nash equilibrium of G.*

Proof. Fix $i \in \{1, 2, \ldots, n - m\}$. Let H be a game like G but with only two players, so that 1 is an independent firm and 2 controls m locations. For $x \in [0, 1]$,

$$\pi_i^G(s_1, \ldots, s_{i-1}, x, s_{i+1}, \ldots, s_{n-m+1}) \leqslant \pi_1^H(x, s_{n-m+1}) \leqslant 1/2m$$
$$= \pi_i^G(s_1, s_2, \ldots, s_{n-m+1}).$$

It remains to show that player $n - m + 1$ cannot improve his payoff by a unilateral strategy change.

Let t_{n-m+1} be any pure strategy of player $n - m + 1$. Let K be a game like G but with two players each of whom controls m locations. Then by the definitions of s_i,

$$\sum_{i=1}^{n-m} \pi_i^G(s_1, s_2, \ldots, s_{n-m}, t_{n-m+1})$$

$$= (1/a) \sum_{i=1}^{n-m} \sum_{j=1}^{a} \pi_i^G\left(s_1, \ldots, s_{i-1}, (2ai - 2a + 2j - 1)/2m, s_{i+1}, \ldots, s_{n-m}, t_{n-m+1}\right)$$

$$\geqslant \pi_1^K(s_{n-m+1}, t_{n-m+1})/a.$$

The inequality follows from the fact that any consumer (or fraction of a consumer) awarded to the first player in game K will contribute to one of the summands on the left side of the inequality. For example, suppose t_{n-m+1} assigns three locations to k, two locations to $5k + 1/2$ and no location to any point between. How does the consumer interval $(2k, 3k)$ contribute to the two sides of the inequality? Player 1 in game K wins all of $(2k, 3k)$. Player 1 in game G wins all of $(2k, 3k)$ in the summand $\pi_i^G(3k, s_2, \ldots, s_{n-m}, t_{n-m+1})$ *and* one quarter of $(2k, 3k)$ in the summand $\pi_i^G(k, s_2, \ldots, s_{n-m}, t_{n-m+1})$.

Next, $\pi_1^K(s_{n-m+1}, t_{n-m+1}) \geqslant (3m - n)/(2m) = 1/2$ by the profitability argument in Section 3.1. Combining the above inequalities,

$$\sum_{i=1}^{n-m} \pi_i^G(s_1, s_2, \ldots, s_{n-m}, t_{n-m+1}) \geqslant 1/2a = (n - m)/2m.$$

Therefore

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[6] To illustrate, consider the case in which there is only one independent player (i.e., $m = a = n - 1$). In this case the independent player chooses each of the m equidistant locations chosen by the firm controlling the coalition with probability $1/m$. Alternatively, consider the special case of $n = 6$ and $m = 4$ ($a = 2$), i.e. there are two independent players and one player controlling a 4-firm coalition. In this case the coalition firms will occupy locations $1/8$, $3/8$, $5/8$, and $7/8$ whereas the first (second) independent player chooses locations $1/8$ and $3/8$ ($5/8$ and $7/8$) with probability $1/2$ each.

$$\pi^G_{n-m+1}(s_1, s_2, \ldots, s_{n-m}, t_{n-m+1}) \leqslant 1 - \frac{n-m}{2m} = \frac{3m-n}{2m}$$
$$\leqslant \pi^G_{n-m+1}(s_1, s_2, \ldots, s_{n-m}, s_{n-m+1})$$

with the last inequality following again from the profitability argument of Section 3.1. □

7. Discussion

We find that collusion in location games only pays if the set of colluders is larger than the set of non-colluding competitors. Bilateral collusion, for example, can only pay if there are no more than three competitors. This result is based on an approach which relies on rather weak rationality requirements. It assumes that players discussing some binding agreements to collude will only go ahead if they can guarantee themselves a payoff better than the payoff expected "behind the veil of ignorance."

This maxmin approach prescribes that colluding players should spread themselves out, making players outside the colluding set indifferent between locations. This seems to be rather intuitive: One would expect that two colluding supermarkets (or supermarkets belonging to the same chain) locate in different parts of one city to avoid cannibalization. For a special case of competition on the unit interval, we show that the maxmin strategy is also used in a non-cooperative equilibrium.

Our results may have implications for the topic of mergers in markets with (pure) spatial competition as an example of which competition among big book retailers (where price competition is extremely limited) may serve. As merger in the traditional sense (see Salant et al. [10]) where firms simply "disappear" never pays in such location games, merger can only be profitable if the merging units are kept as separate units which are governed by central headquarters. This is identical to the case of collusion analysed above. However, the analysis reveals that with this kind of competition only "mega mergers" are likely to occur.[7]

Acknowledgments

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[7] Concerning the market for books such a mega merger has recently occurred in the UK where Waterstone's took over Dillon's. And, interestingly, the new Waterstone's branches in London are pretty much "spread out." In particular, Waterstone's two flagship stores are not at Charing Cross Road, the traditional spot for large book stores but rather "to the left and to the right" of the competitors' big stores, namely at Piccadilly and UCL.

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