Inequity-Averse Stochastic Decision Processes

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Abstract. The paper extends two alternative approaches in inequity-averse optimization under uncertainty, the ex-ante approach and the ex-post approach, from a static to a dynamic decision making context. This is done by developing a stochastic multistage optimization framework evaluating payoffs by an equitable aggregation function. It is shown that global optimization of strategies leads to time-consistent policies only in the ex-post case. For the ex-ante case, a variant of a policy for which time consistency holds is proposed. To illustrate the concepts, a two-stage stochastic location-allocation problem from humanitarian logistics is investigated. For this application, the general algorithmic approaches can be cast into mathematical programming formulations, which yields a two-stage stochastic program and a bilevel program in the ex-post and in the ex-ante case, respectively. The resulting models are solved to optimality for a set of randomly generated instances, and a comparison of the outcomes for ex-post and ex-ante, also in terms of the "Price of Fairness", is given.

Keywords: Decision processes, equity, fairness, stochastic optimization, humanitarian logistics

1 Introduction

Inequity-averse decision making has recently found much interest in the Operations Research literature [Karsu and Morton, 2015]. Whenever decisions affecting several people are to be made, the question is not only how to achieve "the greatest good for the greatest number", as it has been suggested by the philosophers of Utilitarianism, Bentham and Mill (cf. [Bentham, 1879, Mill, 1966]). Also the *distribution* of goods among the persons matters: numerous studies show that solutions are rejected if they are not considered as *fair*. Among the most convincing confirmations of this fact are empirical results of Behavioral Economics on the *Ultimatum Game* (see, e.g., [Nowak et al., 2000]). However, many practical experiences with quantitative decision making approaches in different fields such as staff scheduling, transportation logistics, health-care management, or resource sharing in computing systems, underline the importance of the fairness aspect as well.

The classical framework for equity considerations has been developed by John Rawls [Rawls, 1971]. His max-min principle, essentially based on the idea of reducing inequity aversion to risk aversion, prescribes to maximize the benefit of the worst-off individual. As alternatives

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to the Rawlsian measure of minimal benefit (or maximal cost), also more complex inequity measures have been elaborated in the economic literature, first and foremost the well-known *Gini coefficient* [Yitzhaki and Schechtman, 2013]. The survey [Marsh and Schilling, 1994] reviews no less than twenty inequity measures. Obviously, as soon as an inequity measure has been chosen, also optimization models taking the equity aspect into account can be formulated and solved.

While inequity-averse optimization in a deterministic context is conceptually still rather simple, though often computationally highly nontrivial and demanding, decisions under *uncertainty* – represented by suitable stochastic models – introduce additional challenges. The major difficulty in this area is the antagonism between two competing approaches, the *ex-ante* and the *ex-post* approach [Ben-Porath et al., 1997, Fleurbaey et al., 2015]. The former measures the quality of a solution by the *inequity of the expected outcomes*, while the latter measures it by the *expected inequity of the outcomes*. To put it at its simplest, the ex-ante approach strives for equal chances, whereas the ex-post approach strives for chances of equity. The two approaches and their comparison have found considerable interest in the literature on economics and on decision theory, but they have rarely been used in the context of computational *optimization*. An exception is [Mostajabdaveh et al., 2018], where a combination of the ex-ante and the ex-post objective function, both based on the Gini coefficient, has been applied to solve a location-allocation problem under uncertainty for the establishment of shelters in the preparation phase for a natural disaster.

A limitation of [Mostajabdaveh et al., 2018] is that the article assumes both the location and the allocation decision to be made already in the *first* decision stage, i.e., before uncertainty on demands is resolved by the occurrence of the disaster. One may wish to gain flexibility by deferring the allocation decision to the second stage so that it can exploit the knowledge of the post-disaster situation. If, however, the optimization model is extended in this direction, the static problem turns into a dynamic one, producing an inequity-averse *two-stage* stochastic optimization problem.

Multi-stage stochastic optimization problems with inequity-averse objective functions have, to the best of the author's knowledge, not yet been investigated with a view towards numerical solution. The present work is an attempt to work out mathematical problem representations and solution algorithms for this class of problems. This will be done mainly in a quite general framework. Nevertheless, the location-allocation problem outlined above will then be used to illustrate the concepts and to explore the possibility of a computational solution.

The paper is organized as follows: Section 2 gives a review of related publications. In

Section 3, basic concepts for multi-stage extensions of inequity-averse stochastic optimization are introduced. Section 4 is devoted to the issue of time (in)consistency. Section 5 presents a time-consistent variant of a policy following the ex-ante approach. In Section 6, we describe the shelter location-allocation problem and suggest a mathematical problem formulation as well as a solution approach. Section 7 deals with the determination of the "Price of Fairness". Section 8 presents experimental results and observations, while Section 9 concludes the paper.

2 Related Literature

For a recent survey on inequity-averse optimization, the reader is referred to [Karsu and Morton, 2015]. An example for an application in health economics is [Morton, 2014]. Applications in transportation logistics can be found in [Matl et al., 2017]. Several applications are related to location analysis or network problems, see, e.g., [Kostreva et al., 2004, Ogryczak et al., 2014].

Theoretical results on inequity under uncertainty, considered from an axiomatic point of view, are provided in [Chew and Sagi, 2012, Fleurbaey et al., 2015]. The issue of time consistency has been mainly studied in the literature on risk-averse optimization, see, e.g., [Rudloff et al., 2014, Pflug and Pichler, 2016]. Time inconsistency of ex-ante policies has been discussed in the economics literature early on (see, e.g., [Machina, 1989]), but a treatment in a context of computational multi-stage stochastic optimization seems to be missing up to now. The concept of the "Price of Fairness" goes back to [Bertsimas et al., 2011] and has been used in a number of recent investigations, e.g., [Nicosia et al., 2017].

Evacuation planning and shelter assignment problems have been studied by several authors, cf. the review in [Bayram, 2016]. [Bayram and Yaman, 2017] consider uncertainty on the road network and on demands by a scenario-based model and solve a location-allocation problem. [Li et al., 2011] propose a two-stage stochastic programming model for determining locations of permanent shelters and their capacities in a first decision stage, resource allocation and temporary shelter locations in a second decision stage. [Kulshrestha et al., 2011] introduce a bilevel optimization model for shelter location-allocation under demand uncertainty.

The last-mentioned works do not address the equity issue. However, part of the humanitarian logistics literature takes equity into account, though not in the form of dynamic extensions of ex-ante or ex-post policies. To give some examples, [Vitoriano et al., 2011] deal with equity in a multicriteria optimization context. [Huang et al., 2012] quantify inequity by three different measures. [Gutjahr and Fischer, 2018] show that the inequity of post-disaster relief good distribution can be considerably reduced at only negligible efficiency losses. [Eisenhandler and Tzur, 2018] incorporate equity aspects, quantified by means of a measure based on the Gini coefficient, into the optimization of the distribution of food by welfare agencies. By their two-stage programming model for last-mile relief distribution, [Noyan et al., 2015] come close to our present work, but they apply the equity measure in the constraints rather than in the objective function.

While in inequity-averse optimization, multistage stochastic decision processes seem to be an unexplored area, there is a huge body of literature on such processes in an inequity-neutral context, cf., e.g., multistage stochastic programming [Birge and Louveaux, 2011], or Markov Decision Processes [Puterman, 2014].

3 Problem Formulation

A sequence of decisions in T time stages $t = 1, \ldots, T$ have to be made. By x_t, we denote the decision in stage t ($1 \leq t \leq T$). Decision x_t is an element of a discrete finite set $\mathcal{X}_t$. Thus, the sequence $x = (x_1, \ldots, x_T)$ is an element of $\mathcal{X}_1 \times \ldots \times \mathcal{X}_T$.

Decisions made in previous stages may restrict the set of feasible decisions in some current stage t. To represent this, we consider a subset $\mathcal{Y} \subseteq \mathcal{X}$ of *feasible* decision sequences. From this set, the sets $\mathcal{Y}^t_{(x_1,\ldots,x_{t-1})}$ of feasible decisions in some stage t, given decisions $x_1, \ldots, x_{t-1}$ in the previous stages, can be immediately derived: $\mathcal{Y}^1 = \{x_1 \in \mathcal{X}_1 \mid \exists x_2 \in \mathcal{X}_2 \ldots \exists x_T \in \mathcal{X}_T : (x_1, \ldots, x_T) \in \mathcal{Y}\}$, and

$$\mathcal{Y}^t_{(x_1,\ldots,x_{t-1})} = \{x_t \in \mathcal{X}_t \mid \exists x_{t+1} \in \mathcal{X}_{t+1} \ldots \exists x_T \in \mathcal{X}_T : (x_1, \ldots, x_T) \in \mathcal{Y}\} \quad (t = 2, \ldots, T)$$

Consider a sequence of random events. It is supposed that the t-th event happens at some point in time between decision stages t and $t+1$ ($t = 1, \ldots, T-1$). Formally, we describe the t-th event by a random variable ζ_t with possible realizations $s_t \in S_t$ ($t = 1, \ldots, T-1$), where $S_1, \ldots, S_{T-1}$ are finite sets. Then, $(\zeta_1, \ldots, \zeta_{T-1})$ is a time- and state-discrete stochastic process. The sequence $(\zeta_1, \ldots, \zeta_{T-1})$ can have an arbitrary probability distribution on the set $S = S_1 \times \ldots \times S_{T-1}$. However, we assume that the random variables ζ_t representing the random events are independent from the decisions. Decisions, on the other hand, can depend on *previous* events. A sequence $s = (s_1, \ldots, s_{T-1})$ of realizations of $(\zeta_1, \ldots, \zeta_{T-1})$ will also be called a *scenario*.

For shortness of terminology, the realization s_t of random variable ζ_t will sometimes be identified with the "event in stage t". In this sense, scenario $s = (s_1, \dots, s_{T-1})$ will be conceived as an event sequence.

We obtain the following T-stage decision process: first, the decision maker (DM) chooses decision $x_1 \in \mathcal{Y}^1$. After that, the random event described by $\zeta_1 \in S_1$ happens. The DM can observe the realization s_1 of ζ_1 and, depending on s_1, make her next decision $x_2 \in \mathcal{Y}^2_{(x_1)}$. Now, the random event described by $\zeta_2 \in S_2$ occurs, etc. After the final decision x_T, the process terminates.

The probability distribution p on S is specified by probabilities p_s ($s \in S$), where p_s denotes the probability of scenario s. We assume $p_s > 0$ for all $s \in S$ and $\sum_{s \in S} p_s = 1$.

Next, we define *strategies* of the DM. Consider a function $Z : S \to \mathcal{Y}$ mapping each scenario $s \in S$ to a sequence $x \in \mathcal{Y}$ of feasible decisions. Note that Z is vector-valued: $Z(s) = (Z_1(s), \dots, Z_T(s))$ with $Z_t(s) \in \mathcal{X}_t$ ($t = 1, \dots, T$).

Definition 1. The function $Z : S \to \mathcal{Y}$ assigning decision sequences to event sequences is called *nonanticipative* (cf. [Shapiro et al., 2009]) if for each $t = 1, \dots, T-1$, the decision $Z_t(s)$ does not depend on the future events $s_t, s_{t+1}, \dots, s_{T-1}$, i.e., if $Z_1(s) = Z_1$, and $Z_t(s) = Z_t(s_1, \dots, s_{t-1})$ for $t = 2, \dots, T$. We shall consider only nonanticipative functions Z throughout the paper. In the following, a nonanticipative function $Z : S \to \mathcal{Y}$ will be called a *strategy*. By $\mathcal{Z} = \mathcal{Z}(\mathcal{Y}, S)$, we denote the set of all strategies Z for some given $\mathcal{Y}$ and S.

Let N, indexed by $i = 1, \dots, n$, denote the set of *individuals* affected by the consequences of the decisions $x_1, \dots, x_T$. We assume that the DM is *not* contained in N, but represents "society", a public institution, or a non-governmental organization. For each $i \in N$, assume a *cost function* $f_i : \mathcal{Y} \times S \to \mathbb{R}$ to be given. Thus, $f_i(x, s)$ denotes the cost individual i has to face in case that the sequence $x \in \mathcal{Y}$ of decisions is made and the random events contained in $s \in S$ occur. To assess the "social cost" of the decision (the cost from the viewpoint of the DM), we use a (weakly) *equitable aggregation function* $\mathcal{I} : \mathbb{R}^n \to \mathbb{R}$ assigning to each cost vector $(f_1, \dots, f_n)$ (with f_i denoting the cost of individual i) an overall real-valued score $\mathcal{I}(f_1, \dots, f_n)$. By a weakly equitable aggregation function (short: equitable aggregation function), we understand a function $\mathcal{I} : \mathbb{R}^n \to \mathbb{R}$ that is nondecreasing and symmetric, and that satisfies the weak Pigou-Dalton principle of transfers: $f_j > f_i \Rightarrow \mathcal{I}(f) \le \mathcal{I}(f + \epsilon e_j - \epsilon e_i)$ for all $f \in \mathbb{R}^n$, where $\epsilon > 0$, and e_i, e_j denote the ith and jth unit vector in $\mathbb{R}^n$, respectively. (For details, see [Kostreva et al., 2004, Karsu and Morton, 2015].) Note that we do not require *strict* monotonicity nor the strict version of Pigou-Dalton.

Some candidates for equitable aggregation functions are the Rawlsian measure $\mathcal{I}(f_1, \ldots, f_n) = \max(f_1, \ldots, f_n)$, the conditional β-mean [Filippi et al., 2019], or a weighted sum of the average cost and Gini's absolute difference of costs [Ogryczak, 2000, Ogryczak, 2009, Gutjahr and Fischer, 2018].

For the application of a measure $\mathcal{I}$ to a cost vector $f = (f_i)_{i \in N} = (f_1, \ldots, f_n)$, we shall also write $\mathcal{I}(f_i : i \in N)$ instead of $\mathcal{I}(f_1, \ldots, f_n)$.

Definition 2. The *Global Ex-Ante* (GEA) Problem is the minimization problem

$$\min \; \mathcal{I}\left(\sum_{s \in S} p_s \, f_i(Z(s), s) : \; i \in N\right) \quad \text{s.t. } \; Z \in \mathcal{Z}. \tag{1}$$

Note that the sum in Eq. (1) is just the expected value of the cost resulting for individual $i \in N$ if strategy Z is chosen, where the expectation is taken with respect to the probability distribution on the scenarios $s \in S$. Thus, the objective function Eq. (1) represents the *inequity of expected costs*. By the word "global" in "global ex-ante" we refer to the property that Eq. (1) optimizes the ex-ante objective function without any restriction on the strategy.

Definition 3. The *Global Ex-Post* (GEP) Problem is the minimization problem

$$\min \; \sum_{s \in S} p_s \, \mathcal{I}\left(f_i(Z(s), s) : \; i \in N\right) \quad \text{s.t. } \; Z \in \mathcal{Z}. \tag{2}$$

Contrary to the GEA problem, the GEP problem minimizes the *expected inequity of costs*.

Observe that the terminological distinction between "ex-ante" and "ex-post" refers to the viewpoint from which decisions are to be judged as fair or not: from the viewpoint before the random event has occurred (ex-ante approach), or from the viewpoint after this has happened (ex-post approach).

In the special case $T = 1$, no random event occurs, and a strategy is of the form $Z_1(s) = Z_1 = x_1 \in \mathcal{Y}^1$, i.e., it reduces to a simple action x_1. In this case, $f_i(Z(s), s)$ reduces to $f_i(x_1)$, with the consequence that the problems GEA and GEP collapse to an identical static, deterministic problem, namely the problem $\min \{\mathcal{I}(f_i(x_1) : \; i \in N) \mid x_1 \in \mathcal{Y}^1\}$.

In the case $T > 1$, we may also consider the "smaller" decision problem that results after the first decision x_1 has been made and the subsequent random event s_1 has occurred. This new decision problem has only $T - 1$ decision stages, and compared to the original problem, the DM has now more information insofar as she already knows the value of s_1. For defining this "reduced problem" in formal terms, let us introduce a bit more of notation. We split the vector

$s = (s_1, s_2, \ldots, s_{T-1})$ into the parts s_1 and $s' = (s_2, \ldots, s_{T-1})$, writing $s = (s_1, s')$. Let

$$p_{s'|s_1} = p_{(s_1,s')}/p_{(s_1)} \quad \text{with} \quad p_{(s_1)} = \sum_{\bar{s}_j \in S_j\,(j=2,\ldots,T-1)} p_{(s_1,\bar{s}_2,\ldots,\bar{s}_{T-1})} \tag{3}$$

denote the conditional probability for s', given s_1. Obviously, $\sum_{s'} p_{s'|s_1} = 1$ for each $s_1 \in S_1$. In the special case $T = 2$, we set $s' = \emptyset$ and $p_{\emptyset|s_1} = 1$.

Definition 4. Let a problem instance $(\mathcal{X}, \mathcal{Y}, S, p, f)$ with $T \geq 2$ be given. Consider some fixed $x_1 \in \mathcal{Y}^1$ and $s_1 \in S_1$. The (x_1, s_1)-*reduced instance* derived from $(\mathcal{X}, \mathcal{Y}, S, p, f)$ is the problem instance $(\mathcal{X}', \mathcal{Y}', S', p', f')$ defined by

$$\mathcal{X}' = \mathcal{X}_2 \times \ldots \times \mathcal{X}_T, \quad \mathcal{Y}' = \{(x_2, \ldots, x_T) \mid (x_1, x_2, \ldots, x_T) \in \mathcal{Y}\},$$

$$S' = S_2 \times \ldots \times S_{T-1} \text{ for } T > 2, \text{ and } S' = \{0\} \text{ for } T = 2,$$

$$p'_{s'} = p_{s'|s_1}\ \forall s' \in S' \text{ for } T > 2, \text{ and } p'_0 = 1 \text{ for } T = 2,$$

$$f'(x', s') = f((x_1, x'), (s_1, s')) \text{ for } T > 2, \text{ and } f'(x') = f((x_1, x'), s_1) \text{ for } T = 2.$$

Next, we shall define the notion of a *policy*; this term will be used to refer to a *class* of strategies, applicable to different problem instances.

Definition 5. Let *Inst* and $\mathcal{Z}^{tot}$ denote the set of all problem instances and the set of all strategies, respectively, and let $\mathcal{P}(\mathcal{Z}^{tot})$ denote the set of subsets of $\mathcal{Z}^{tot}$. A *policy* π is a function $\pi : Inst \to \mathcal{P}(\mathcal{Z})$ with $\pi(\mathcal{X}, \mathcal{Y}, S, p, f) \subseteq \mathcal{Z}(\mathcal{Y}, S)$ and $\pi(\mathcal{X}, \mathcal{Y}, S, p, f) \neq \emptyset$ for all $(\mathcal{X}, \mathcal{Y}, S, p, f) \in \mathcal{Z}$. In less formal terms, a policy π assigns to each instance $(\mathcal{X}, \mathcal{Y}, S, p, f)$ a nonempty set of strategies $Z : S \to \mathcal{Y}$, the set of strategies "proposed" by π.

Obvious examples of policies are: (i) the *Global Ex-Ante Policy* (GEA policy) which assigns to each instance the set of optimal solutions Z of Eq. (1), and (ii) the *Global Ex-Post Policy* (GEP policy) which assigns to each instance the set of optimal solutions Z of Eq. (2).

We would like to ensure *time consistency* of a policy: decisions planned for the future should actually going to be implemented – also in view of new information – if the DM keeps following the chosen policy (cf. Rudloff et al. 2012). Let us define this notion in precise terms:

Definition 6. A policy π is called *time-consistent*, if for every instance $(\mathcal{X}, \mathcal{Y}, S, p, f)$ and for each $x_1 \in \mathcal{Y}^1$ and $s_1 \in S_1$, the derived (x_1, s_1)-reduced instance $(\mathcal{X}', \mathcal{Y}', S', p', f')$ satisfies the implication

$$Z \in \pi(\mathcal{X}, \mathcal{Y}, S, p, f),\ Z_1 = x_1 \ \Rightarrow\ \exists Z' \in \pi(\mathcal{X}', \mathcal{Y}', S', p', f') :\ Z_2(s_1) = Z'_2.$$

Therein, $Z = (Z_1, \dots, Z_T)$ and $Z' = (Z'_2, \dots, Z'_T)$.

In other words: π is time-consistent if in every case where it proposes some strategy Z with first-stage action x_1 and whatever the random event s_1 following action x_1 may be, the same policy π applied to the *remaining* stages of the decision process does not force us to deviate from the second-stage action $Z_2(s_1)$ as it has been pre-planned by Z.

Before starting the investigation of which policies are time-consistent and which are not, it is helpful to discuss different ways to represent a strategy Z.

(a) Tree Visualization. A better intuitive understanding can be obtained by using the well-known representation of a decision process as a tree. Fig. 1(a) shows an example for $T = 2$ decision stages. Here, $\mathcal{X}_1 = \mathcal{X}_2 = \{1, 2\}$, $\mathcal{Y} = \mathcal{X} = \mathcal{X}_1 \times \mathcal{X}_2$, and $S_1 = \{1, 2\}$. A set $N = \{1, 2\}$ of two individuals is assumed. To each $s_1 \in S_1$, a probability p_i is assigned, e.g., $p_1 = p_2 = 1/2$. Nodes $D1$ and $D2 - D5$ are decision nodes, whereas in nodes $R1 - R2$ random events determine the choice of the outgoing arc. The terminal nodes (leaves) $T1 - T8$ are labeled by the vectors $f(x, s) = (f_1(x, s), f_2(x, s))$ of costs assigned to the two individuals $i = 1, 2$.

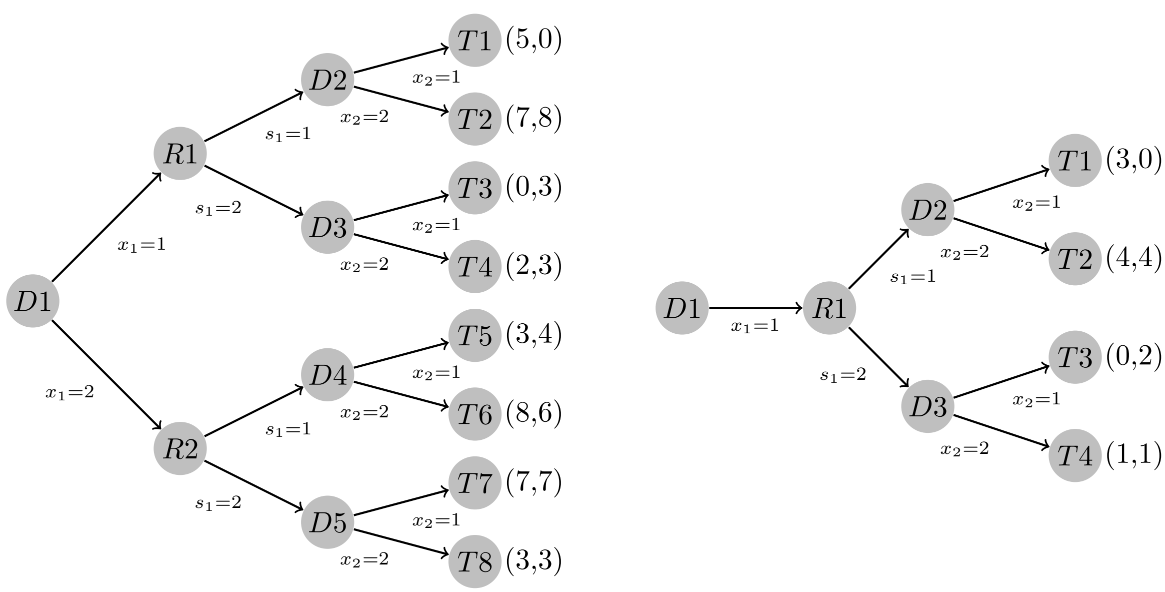

Figure 1: Two inequity-averse decision trees. (a) *Left:* Basic example. (b) *Right:* Counter-example in the proof of Prop. 2.

(b) Standard representation. To distinguish it from an another representation introduced below, we shall call the representation $Z = (Z_1, \dots, Z_T)$ with $Z_t(s) = Z_t(s_1, \dots, s_{t-1})$ used in Def. 1 the *standard representation.* Expressed in terms of the decision tree, it lets a strategy Z determine, for each decision node of the tree that can be reached by Z, an action to be chosen in this decision node. An example for a strategy Z in Fig. 1(a) is $Z_1 \equiv 1$, $Z_2 \equiv (1,2)$, specifying that the first-stage decision is 1 and that the second-stage decision is 1 if $s_1 = 1$ and 2 if $s_1 = 2$ (observe that if the first-stage decision is 1, then only the two upper nodes D2 and D3 of the second decision level matter for the second-stage action). We shortly write this special strategy as $(1; 1, 2)$. For the Rawlsian measure $\mathcal{I} = \max$ and for $p_1 = p_2 = 1/2$, it is evaluated under the two objective functions (1) and (2) as follows: the GEA objective gives $\max\left(\frac{1}{2}\cdot 5 + \frac{1}{2}\cdot 2,\ \frac{1}{2}\cdot 0 + \frac{1}{2}\cdot 3\right) = \frac{7}{2}$, while the GEP objective gives $\frac{1}{2}\max(5,0) + \frac{1}{2}\max(2,3) = 4$.

In the general case, suppose $S_t = \{s_t^1, \ldots, s_t^{k_t}\}$, where the elements of S_t are pre-arranged in a fixed order ($t = 1, \ldots, T-1$). Then strategy Z is specified by the numbers

$$x = Z_1, \quad x^{(\kappa_1 \ldots \kappa_t)} = Z_{t+1}(s_1^{\kappa_1}, \ldots, s_t^{\kappa_t}) \quad (\kappa_\tau = 1, \ldots, k_\tau\ (1 \le \tau \le t);\ t = 1, \ldots, T-1)$$

which can be combined to a vector

$$(x;\, x^{(1)}, \ldots, x^{(k_1)};\, x^{(11)}, \ldots, x^{(1k_2)}, \ldots, x^{(k_1 1)}, \ldots, x^{(k_1 k_2)};\, \ldots\ldots;\, x^{(1\ldots 1)}, \ldots, x^{(k_1 \ldots k_{T-1})}). \quad (4)$$

(c) Nested representation. An alternative representation of a strategy Z is obtained as follows: Let $Z : S \to \mathcal{Y}$ with $Z = (Z_1, \ldots, Z_{T-1})$ be given, where $Z_t(s) = Z_t(s_1, \ldots, s_{t-1})$ ($t = 1, \ldots, T$) , and let again $s_1^1, \ldots, s_1^{k_1}$ be the elements of S_1, pre-arranged in a fixed order. We represent Z recursively by $\mathrm{nrep}(Z) = Z_1$ for $T = 1$ and

$$\mathrm{nrep}(Z) = [Z_1;\, \mathrm{nrep}(Z^{[s_1^1]}), \ldots, \mathrm{nrep}(Z^{[s_1^{k_1}]})] \quad \text{for } T > 1.$$

Therein, $Z^{[s_1]}$ is the strategy on stages $t = 2, \ldots, T$ given by

$$Z_t^{[s_1]}(s_2, \ldots, s_{T-1}) = Z_t(s_1, s_2, \ldots, s_{T-1}) \quad (t = 2, \ldots, T-1).$$

For example, let us extend the instance of Fig. 1(a) by a further stage $t = 3$ and binary alternatives both for the random events s_3 and the actions x_3, i.e., $S_3 = \{1, 2\}$ and $\mathcal{X}_3 = \{1, 2\}$. This gives a binary decision tree with 21 decision nodes, 10 random event nodes, and 32 terminal nodes. Consider the strategy that reads in standard form

$$Z \equiv (x;\, x^{(1)}, x^{(2)};\, x^{(11)}, x^{(12)}, x^{(21)}, x^{(22)}) = (2;\, 1, 2;\, 2, 1, 2, 2).$$

It says that always action 2 is chosen, except in the cases where $s_1 = 1$ has just been observed, or where first $s_1 = 1$ and then $s_2 = 2$ have been observed. The same strategy can then written in nested form as $\mathrm{nrep}(Z) = [2;\, [1;\, 2,\, 1],\, [2;\, 2,\, 2]]$. Therein, the first inner expression in brackets specifies the sub-strategy for the case $s_1 = 1$, whereas the second inner expression specifies the sub-strategy for the case $s_1 = 2$.

It is easy to see that each representation in nested form can be translated back to an expression in standard form, and that the two operations are inverse to each other. Therefore, both representations are equivalent. In the following, we shall simply write Z instead of $\mathrm{nrep}(Z)$ whenever the context makes it clear that Z is represented in nested form.

4 Time Consistency

We start with a positive result, which follows from time-consistency of *Markov Decision Processes* (MDPs).

Proposition 1. For any equitable aggregation function $\mathcal{I} : \mathbb{R}^n \to \mathbb{R}$, the Global Ex-Post Policy is time-consistent.

Proof. Denote by $\mathrm{GEP}(\mathcal{X}, \mathcal{Y}, S, p, f)$ the problem Eq. (2) applied to the instance $(\mathcal{X}, \mathcal{Y}, S, p, f)$. Suppose Z is a solution of $\mathrm{GEP}(\mathcal{X}, \mathcal{Y}, S, p, f)$, $x_1 = Z_1$, and $s_1 \in S_1$. Let $(\mathcal{X}', \mathcal{Y}', S', p', f')$ be the (x_1, s_1)-reduced instance derived from $(\mathcal{X}, \mathcal{Y}, S, p, f)$. We have to show that there exists at least one solution Z' of $\mathrm{GEP}(\mathcal{X}', \mathcal{Y}', S', p', f')$ such that $Z_2(s_1) = Z'_2$.

For the proof, it is convenient to use the tree representation of the problem. A leaf of the decision tree is specified by a pair (x, s), where $x = (x_1, \ldots, x_T) \in \mathcal{Y}$ is a sequence of actions, and $s = (s_1, \ldots, s_{T-1}) \in S$ is a sequence of random events. Similarly, a decision node at level $t > 1$ of the tree is specified by a pair $(\bar{x}, \bar{s})$ where $\bar{x} = (x_1, \ldots, x_{t-1})$ and $\bar{s} = (s_1, \ldots, s_{t-1})$ denote the sequence of previous decisions and the sequence of previous random events, respectively, that led to the considered decision node (cf. Fig. 1(a)).

To each leaf (x, s), a cost vector $f(x, s) = (f_i(x, s) : i \in N)$ is assigned. Let $\alpha(x, s) = \mathcal{I}(f_i(x, s) : i \in N)$. Then the numbers $\alpha(x, s)$ are *scalars* assigned to the leaves, and $\mathrm{GEP}(\mathcal{X}, \mathcal{Y}, S, p, f)$ can be written as $\min_{z \in \mathcal{Z}} \sum_{s \in S} p_s \, \alpha(Z(s), s)$. We reformulate this problem as an MDP with finite time horizon: The set of *states* σ is the set of all decision nodes. The set of feasible *actions* a in a given state (i.e., decision node) specified by the two sequences $(x_1, \ldots, x_{t-1})$ and $(s_1, \ldots, s_{t-1})$ is the set of decisions $x_t \in \mathcal{Y}^t_{(x_1, \ldots, x_{t-1})}$. The *transition probability* $\mathcal{T}(\sigma, a, \sigma')$ from state σ to state σ' under action a is determined by the property that action a in decision node σ leads to a random-event node with a well-specified probability distribution on the set of subsequent decision nodes. The *reward function* $R(\sigma, a)$ has the value zero, if σ does not lie at the deepest level T of decision nodes (immediately leading to leaves). Otherwise, for a decision node σ at level T, specified by the two sequences $\bar{x} = (x_1, \ldots, x_{T-1})$ and

$\bar{s} = (s_1, \ldots, s_{T-1})$, the reward function $R(\sigma, a)$ has the value $\alpha(x, \bar{s})$ with $x = (x_1, \ldots, x_{T-1}, a)$. (Note that $(x, \bar{s})$ is a leaf.) The total reward is the finite sum of (undiscounted) rewards over the time period $1, \ldots, T$. Obviously, this MDP is equivalent to the problem above.

By *Bellman's Principle of Optimality*, an optimal strategy in the MDP has the property that for each initial decision, the remaining decisions must constitute an optimal strategy with respect to the state following from the first decision. Therefore, after an initial decision for action x_1 has been made and the random event s_1 has occurred, the optimal decision $Z_2(s_1)$ of $\mathrm{GEP}(\mathcal{X}, \mathcal{Y}, S, p, f)$ must remain an optimal decision for the (x_1, s_1)-reduced instance and can thus constitute the first decision Z_2' of $\mathrm{GEP}(\mathcal{X}', \mathcal{Y}', S', p', f')$. $\square$

Let us now proceed to the Global Ex-Ante Policy. It will turn out that for most equitable aggregation functions, this policy lacks time-consistency.

Proposition 2. In the general case, the Global Ex-Ante Policy for the Rawlsian measure $\mathcal{I} = \max$ is *not* time-consistent.

Proof. We show the statement by the counter-example in Fig. 1(b). There are four strategies Z for this instance: in standard representation, they are given by $Z^{(1)} \equiv (1;\, 1, 1)$, $Z^{(2)} \equiv (1;\, 1, 2)$, $Z^{(3)} \equiv (1;\, 2, 1)$ and $Z^{(4)} \equiv (1;\, 2, 2)$, respectively. It is immediately seen that $Z^{(3)}$ is dominated by $Z^{(1)}$, and $Z^{(4)}$ is dominated by $Z^{(2)}$. The objective function values for $Z^{(1)}$ and $Z^{(2)}$ according to Eq. (1) result as follows: For $Z^{(1)}$, we get $\max\left(\frac{1}{2} \cdot 3 + \frac{1}{2} \cdot 0,\ \frac{1}{2} \cdot 0 + \frac{1}{2} \cdot 2\right) = \frac{3}{2}$, whereas for $Z^{(2)}$, we get $\max\left(\frac{1}{2} \cdot 3 + \frac{1}{2} \cdot 1,\ \frac{1}{2} \cdot 0 + \frac{1}{2} \cdot 1\right) = 2$. Thus, $Z = Z^{(1)}$ is the optimal GEA strategy. For $s_2 = 2$, it prescribes the action $Z_2(s_2) = 1$.

On the other hand, consider the $(1, 2)$-reduced instance, which consists only of the subtree rooted in decision node D3. The optimal GEA strategy for this reduced instance is to choose $x_2 = 2$, since $\max(1, 1) < \max(0, 2)$. That is, $Z_2' = 2$. We have $Z_2(s_2) \neq Z_2'$ for $s_2 = 2$ and hence a violation of time-consistency. $\square$

The counter-example in the proof of Prop. 2 was kept as small as possible; in particular, there is no proper decision to be made in stage 1. One might object that in this case, the decision maker might delay her choice on x_2 until the event s_1 has realized instead of making a decision on the strategy Z in advance. However, an extension of the decision tree by the addition of a subtree for a decision $x_1 = 2$ with the same structure as the subtree for $x_1 = 1$, but cost vectors $(\frac{7}{4}, \frac{7}{4})$ in all four leaves, would result in the same optimal solution for Z, since all strategies $Z \equiv (2;\, i, j)$ are dominated by $Z^{(1)}$. They are not dominated, however, by $Z^{(2)}$, so for the extended tree, it is important that the strategy Z is chosen already before the decision on x_1.

In several works, especially in the context of location problems, it has been proposed to use the weighted mean $\mathcal{I} = \mu + \lambda \cdot \Delta$ as the equitable aggregation function, where μ is the *utilitarian measure*

$$\mu = \mu(f_1, \ldots, f_n) = \frac{1}{n} \sum_{i=1}^{n} f_i, \tag{5}$$

and Δ is *Gini's mean absolute difference*

$$\Delta = \Delta(f_1, \ldots, f_n) = \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} |f_i - f_j|. \tag{6}$$

(See e.g. [Ogryczak, 2000, Ogryczak, 2009, Gutjahr and Fischer, 2018].) Gini's mean absolute difference Δ can also be represented as the double product $2\mu G$ of μ with the well-known *Gini coefficient* G. The weight factor λ controls the degree of inequity aversion: the more inequity-averse the decision maker is, the more emphasis she will give to the Gini term, i.e., the higher will be the value of λ. The boundary case $\lambda = 0$ produces the inequity-neutral utilitarian measure. It can be verified that for $0 \leq \lambda \leq 1/2$, the measure $\mathcal{I} = \mu + \lambda \cdot \Delta$ is monotonous (as required for an equitable aggregation function), contrary to the Gini coefficient itself.

Proposition 3. In the general case, for arbitrary $0 < \lambda \leq 1/2$, the Global Ex-Ante Policy for the equitable aggregation function $\mathcal{I} = \mu + \lambda \cdot \Delta$ is *not* time-consistent.

Proof. We use again the counter-example in Fig. 1(b). Let $Z^{(1)}$ to $Z^{(4)}$ be defined as in the proof of Prop. 2. $Z^{(3)}$ and $Z^{(4)}$ are dominated again by monotonicity. For $Z^{(1)}$ and $Z^{(2)}$, the following evaluations of the objective function are obtained:

$$Z^{(1)}: \quad \mathcal{I}\left(\frac{1}{2}(3,0) + \frac{1}{2}(0,2)\right) = \mathcal{I}\left(\frac{3}{2}, 1\right) = \mu\left(\frac{3}{2}, 1\right) + \lambda \cdot \Delta\left(\frac{3}{2}, 1\right) = \frac{5}{4} + \lambda \cdot \frac{1}{4}.$$

$$Z^{(2)}: \quad \mathcal{I}\left(\frac{1}{2}(3,0) + \frac{1}{2}(1,1)\right) = \mathcal{I}\left(2, \frac{1}{2}\right) = \mu\left(2, \frac{1}{2}\right) + \lambda \cdot \Delta\left(2, \frac{1}{2}\right) = \frac{5}{4} + \lambda \cdot \frac{3}{4}.$$

By the assumption $\lambda > 0$, the first objective value is strictly smaller than the second one, so $Z = Z^{(1)}$ is the unique optimal solution. For $s_2 = 2$, it prescribes the action $Z_2(s_2) = 1$.

On the other hand, consider the $(1,2)$-reduced instance (the subtree rooted in D3). For the two possible decision alternatives for Z'_2, we find $\mathcal{I}(0,2) = 1 + \lambda$ and $\mathcal{I}(1,1) = 1$, respectively. Again by $\lambda > 0$, the unique optimal decision is $Z'_2 = 2 \neq Z_2(2)$, which contradicts time-consistency. $\square$

Prop. 3 does not cover the boundary case $\lambda = 0$. Actually, in this case, the GEA policy *is* time-consistent:

Proposition 4. The Global Ex-Ante Policy for the utilitarian measure $\mathcal{I} = \mu$ is time-consistent.

Proof. The proof follows immediately from Prop. 1, since $\mu(f_1, \dots, f_n)$ is a linear function. Therefore, in Eq. (1), the function $\mathcal{I}$ can be interchanged with the summation over s and with the multiplication by p_i, which produces Eq. (2). As a consequence, the GEA problem and the GEP problem coincide in this case. As the policy solving the GEP problem is time-consistent, so is the policy solving the GEA problem. □

5 A Time-Consistent Ex-Ante Policy

The results of the previous section show that while global ex-post policies are time-consistent, global ex-ante policies typically lack this property. It does not make too much sense to propose an optimized policy if it is not time-consistent, since if the decision maker will probably deviate from the pre-specified strategy in later stages, the optimality property gets lost during practical execution.

On the other hand, one would be interested in having also ex-ante policies for inequity-averse decision processes. The reason is that the ex-ante consideration has advantages of its own. Fleurbaey et al. [Fleurbaey et al., 2015] outline two essential drawbacks of the ex-post consideration that can be overcome by the ex-ante view, namely (i) the inability to produce fairness by randomization, and (ii) social paternalism. The following example by Myerson [Myerson, 1981] illustrates both of these issues:

Suppose the parents of two twin daughters want to favor both of them in their careers. The financial resources of the family, however, allow only the choice between two alternatives: either to send both daughters to school (for four years each) to become teachers, or to send one daughter to university (for eight years) to become a doctor while the other daughter will become a clerk. Now suppose that both daughters prefer a fifty-fifty chance of becoming a doctor (and otherwise to become a clerk, which is the least preferred outcome) to a hundred percent chance of becoming a teacher. The parents want to be as fair as possible, so in the case they send one of the daughters to university, they would decide by tossing a fair coin which of the daughters gets this possibility. Assigning utilities 0, 1 and 4 to clerk, teacher and doctor, respectively, and applying the Rawls measure (here: minimum of the utilities), it is easy to see that the ex-post approach prescribes to let both daughters become teachers, while the ex-ante approach recommends to select one of the daughters by the coin toss for the university studies. Thus, the ex-post approach would deprive the two daughters of the more risky procedural alternative that is preferred by *both* of them. Acting in this way could be rightly considered as "paternalistic".

It should be noted, however, that also the ex-ante approach is based on the *equity* preferences of the *social decision maker* (in the example: the parents) rather than on those of the affected individuals (in the example: the daughters). Actually, the two daughters in the example may be inequity-neutral (i.e., they may not care about each other, but strive only for their own careers), they may be inequity-averse (i.e., interested also in a fair treatment of the sister), or they may even be inequity-seeking. In all the three cases, even in the last one, the parents may be concerned about an equitable treatment of both daughters, and this is what counts for the analysis.[1]

To design a *time-consistent* ex-ante policy, we draw from the concept of *consistent planning* which has been introduced in studies on consumption of a commodity over time under preferences favoring the presence over the future. Works on this subject go back to the seminal paper [Strotz, 1955]. The basic idea of this concept is "to choose the best plan among those that will actually be followed". In [Machina, 1989], consistent planning is called "folding back" and discussed also in relation to the fairness issue, but rejected in favor of what we call GEA.

In our problem context, "consistent planning" takes the following form: Consider a problem instance with $T = 1$. In this case, there is no influence of randomness. In particular, the GEA problem and the GEP problem coincide, and it is clear what to do: the leaf for which the application of $\mathcal{I}$ to the cost vector produces the best value has to be chosen. In this way, not only an action is determined, but also a corresponding *evaluation vector*, representing the (expected) costs of all individuals under the best-possible decision, is obtained.

On the other hand, suppose $T > 1$. Then a decision x_1 has to be made, which is followed by a random event s_1. By the above-mentioned key idea, we assume that the actual decision in decision stage 2 will be based on "consistent planning" again: if another choice would have been pre-planned, there would be an incentive to deviate from this choice as soon as decision stage 2 is reached. Now, if an *ex-ante* policy is to be chosen (i.e., the expectation operator is to be applied *before* the application of $\mathcal{I}$), what we have to do is to consider the evaluation vectors in each decision node of stage 2, to take their expected values with respect to the distribution

[1] Equity preferences of the individuals can possibly be incorporated into the quantitative formalism by modifications of the utilities. For example, one could imagine that the daughters originally assign utilities 1 and 5 to clerk and doctor, respectively, but both values are reduced then by one unit to express the discomfort of seeing either the sister or oneself disadvantaged, which finally results in the same numbers as above. We shall not further pursue here the topic of individual equity preferences, which requires a more detailed investigation in future research.

of the random event (which gives a vector in each random event node of stage 1), and finally to apply $\mathcal{I}$ to the resulting vectors to find the optimal decision in stage 1.

A technical difficulty arises when this concept is followed: the optimal decision in a considered decision node needs not to be unique. To cope with this difficulty, a tie-break order is required:

Definition 7. A *tie-break order* is a linear order $\preceq$ on the set $\mathbb{R}^n$ of possible cost vectors $f = (f_i)_{i \in N}$.

An example for a tie-break order is

$$f \preceq g \;\Leftrightarrow\; \left(\sum_{i\in N} f_i < \sum_{i\in N} g_i \text{ or } \left[\sum_{i\in N} f_i = \sum_{i\in N} g_i \text{ and } f \le g \text{ in lexicographical order}\right]\right). \tag{7}$$

It states that in cases where two solutions yield cost vectors f and g, respectively, that are equally good with respect to some chosen optimization criterion, the solution that is better in a utilitarian view is taken; if they are equally good even in this view, a decision is made simply based on lexicographical precedence. A limitation of this rule will be discussed below.

We are now in the position to formulate the algorithm outlined above in precise terms:

Recursive Ex-Ante (REA) Policy.

Input: A problem instance $(\mathcal{X}, \mathcal{Y}, S, p, f)$. *Output:* A set $\pi_{REA}(\mathcal{X}, \mathcal{Y}, S, p, f)$ of *solutions* $Z \in \mathcal{Z}$ to problem $(\mathcal{X}, \mathcal{Y}, S, p, f)$, and an assigned *evaluation vector* $\bar{f}$.

- **Case** $T = 1$**:** The instance is given by a single decision node, possible options $x_1^1, \dots, x_1^{r_1}$ for the action x_1, and r_1 leaves to which cost vectors $f(x_1^1), \dots, f(x_1^{r_1})$ are assigned, where $f(x_1^\rho) = (f_i(x_1^\rho))_{i\in N}$ $(\rho = 1, \dots, r_1)$. Let $\bar{\mathcal{U}}_1 = \arg\min_{x_1} \mathcal{I}(f_i(x_1) : \, i \in N)$ be the set of all decisions x_1 that lead to minimal evaluations of $\mathcal{I}$ on the set of leaves. Furthermore, let $\mathcal{U}_1 = \{x_1^* \in \bar{\mathcal{U}}_1 \,|\, f(x_1^*) \preceq f(x_1) \; \forall x_1 \in \bar{\mathcal{U}}_1\}$ be the set of minimal elements in $\mathcal{U}_1$. Just as $\bar{\mathcal{U}}_1$, the set $\mathcal{U}_1$ can contain more than one element. However, by the property that the linear order $\preceq$ is antisymmetric, $\mathcal{U}_1$ can only contains actions x_1^* to which the same cost vector $f(x_1^*)$ is assigned.

  (i) Define the set of solutions $\pi_{REA}(\mathcal{X}, \mathcal{Y}, S, p, f)$ as the set $\mathcal{U}_1$.

  (ii) Define the evaluation vector as the (unique) vector $\bar{f} = f(x_1^*)$, where $x_1^* \in \mathcal{U}_1$.

- **Case** $T > 1$**:** Denote, for each (x_1, s_1), the (x_1, s_1)-reduced instance by

$$(\mathcal{X}^{(x_1,s_1)}, \mathcal{Y}^{(x_1,s_1)}, S^{(x_1,s_1)}, p^{(x_1,s_1)}, f^{(x_1,s_1)}).$$

Each of these $|\mathcal{Y}^1| \cdot |S_1|$ instances has $T-1$ stages. Recursive calls of the procedure yield for each of these instances

- the set of solutions, $\mathcal{U}^{(x_1,s_1)} = \pi_{REA}(\mathcal{X}^{(x_1,s_1)}, \mathcal{Y}^{(x_1,s_1)}, S^{(x_1,s_1)}, p^{(x_1,s_1)}, f^{(x_1,s_1)})$,
- the assigned evaluation vector $\bar{f}^{(x_1,s_1)}$.

Compute for each $x_1 \in \mathcal{X}_1 = \{x_1^1, \ldots, x_1^{r_1}\}$:

$$\beta_i(x_1) = \sum_{s_1 \in S_1} p_{(s_1)} \, \bar{f}_i^{(x_1,s_1)} \;\; (i \in N), \quad \beta(x_1) = (\beta_i(x_1))_{i \in N}$$

where $p_{(s_1)}$ denotes the probability that the first random event is s_1 (see (3)). The vector $\beta(x_1)$ yields an evaluation of decision x_1 from the viewpoint of all individuals. Let $\bar{\mathcal{V}}_1 = \arg\min_{x_1} \mathcal{I}(\beta_i(x_i) : \; i \in N)$, and let $\mathcal{V}_1 = \{x_1^* \in \bar{\mathcal{V}}_1 \,|\, \beta(x_1^*) \preceq \beta(x_1) \; \forall x_1 \in \bar{\mathcal{V}}_1\}$ be the set of minimal elements in $\bar{\mathcal{V}}_1$. Again, by the antisymmetry of $\preceq$, the set $\mathcal{V}_1$ can only contain solutions x_1^* to which the same vector $\beta(x_1^*)$ is assigned.

(i) Let $s_1^1, \ldots, s_1^{k_1}$ be the elements of S_1, arranged in a fixed pre-specified order. Define the set of solutions $\pi_{REA}(\mathcal{X}, \mathcal{Y}, S, p, f)$ as the set of all strategies Z with a nested-form representation $Z = [x_1; \, Z^{(1)}, \ldots, Z^{(k_1)}]$ where $x_1 \in \mathcal{V}_1$ and $Z^{(\kappa)} \in \mathcal{U}^{(x_1, s_1^\kappa)}$ $(\kappa = 1, \ldots, k_1)$.

(ii) Define the evaluation vector as the (unique) vector $\bar{f} = \beta(x_1^*)$, where $x_1^* \in \mathcal{V}_1$.

end

An obvious limitation of the application of a tie-break order, say (7), is that it violates *impartiality*. Suppose the algorithm is indifferent between two solutions where the first implies costs $(1,2)$ for two individuals $i = 1,2$, while the second one implies costs $(2,1)$. Then Eq. (7) would prescribe to take the first solution. Thus, the payoffs of the individuals depend on the indices they get. However, three comments are in place: (i) In the context of *deterministic* solution algorithms and non-transferable utilities, the violation of impartiality is inescapable anyway, as for a problem instance with $T = 1$ and two leaves with assigned cost vectors $(1,2)$ and $(2,1)$, each deterministic algorithm is forced to make a decision for one of them, i.e., cannot be impartial. (ii) Omitting the constraint of being deterministic, our solution algorithm can easily be made impartial by a straightforward randomization, where before the algorithm is applied, the indices of the individuals $i \in N$ are shuffled by a random permutation. (iii) Generically, for real-valued costs, the situation $\sum_{i \in N} f_i = \sum_{i \in N} g_i$ leading to a decision that violates impartiality does not occur. If fixed-point numbers or floating-point numbers are used to represent real-valued costs, it can occur, but only as an exception and not as the regular case.

Evidently, the REA strategy for $(\mathcal{X}, \mathcal{Y}, S, p, f)$ can be computed by *backward induction*: start with the computation of the optimal actions at the nodes of decision level T, assign the sets $\mathcal{U}$ and the evaluation functions $\bar{f}$ to these nodes, go then back to decision level $T-1$ and repeat the procedure, etc.

Example 1. For the instance of Fig. 1(a) and the Rawlsian measure, the REA policy assigns the evaluation vectors $(5, 0)$, $(0, 3)$, $(3, 4)$ and $(3, 3)$ to the nodes D2 to D5, respectively, where for achieving the tie break in node D3, the order of Eq. (7) has been used. The optimal choices are going up in nodes D2 – D4, and down in node D5. By mixing under probabilities $(1/2, 1/2)$, the vectors $\beta(1)$ and $\beta(2)$ on decision level 1 compute as $(2.5, 1.5)$ and $(3, 3.5)$, respectively. Therefore, the evaluation vector in D1 becomes $(2.5, 1.5)$, and the optimal choice is going up. In total, the optimal strategy (written in nested form) is $[1; 1, 1]$. It is easy to verify that the GEP policy yields another solution, namely $[2; 1, 2]$. In particular, this shows that in general, the policies REA and GEP differ from each other. $\bowtie$

Proposition 5. For any equitable aggregation function $\mathcal{I} : \mathbb{R}^n \to \mathbb{R}$, the Recursive Ex-Ante Policy is time-consistent.

Proof. The proof is straightforward and omitted for the sake of brevity. $\square$

Finally, we would like to note that although the REA policy is typically easier to determine computationally than the GEA policy, the former should not be considered as a heuristic approximation to the later. A decision maker committed to consistent planning would argue that her/his recursive planning procedure is exactly what should be done, and that it is the GEA approach which is wrong by proposing decisions that, by the same GEA philosophy, will possibly be reversed already in the next decision stage. In game theory, a quite analogous consideration leads to the exclusion of Nash equilibria that are not "subgame-perfect".

6 An Application to Shelter Location-Allocation

In this section, the theory of the previous sections will be applied to deal with an extended version of a shelter location-allocation model published in [Mostajabdaveh et al., 2018]. In the original model of the latter paper, the task consists in the determination of the location of shelters to which people are to be evacuated after some natural disaster, as well as in the allocation of population nodes to the established shelters. The decision is subject to uncertainty on the demands (i.e., on geographical location and impact of the disaster), represented by a set of random scenarios with given probabilities. Both the location and the allocation decision are supposed to be made in the pre-disaster phase. The objective function relies on the distances each victim has to traverse to reach the assigned shelter, and uses the function $\mathcal{I} = \mu + \lambda \cdot \Delta$ (cf. Section 4 in the present paper) as the equitable aggregation function.

A drawback of the model in [Mostajabdaveh et al., 2018] is that it assumes that allocation decisions are already made *before* the disaster has stricken. There are cases where a pre-allocation of population nodes to shelters may be reasonable. In other cases, however, it may be seen as an unnecessary restriction of flexibility, considering that the assignment of victims to shelters could just as well be made dependent on the concrete circumstances of the situation as it can be observed *after* the disaster, in particular on the realized distribution of the demand. However, it is clear that as soon as allocation is made a *second-stage* decision, one obtains an inequity-averse two-stage stochastic optimization model. The concepts presented in [Mostajabdaveh et al., 2018] do not allow to cope with this situation, but those in Sections 3 – 5 of the present paper do.

In the following, we shall develop a model for shelter location and population-to-shelter allocation under (stochastically represented) uncertainty, where the location decision is a first-stage decision, and the allocation decision is a second-stage decision. This increases the computational complexity to a large degree. Contrary to [Mostajabdaveh et al., 2018], in our experiments, we even solve our model variants to optimality, which is desirable in order to produce reliable information on the differences between ex-ante and ex-post approaches. However, we deal with the simpler case of the Rawlsian measure instead of the Gini-type measure of [Mostajabdaveh et al., 2018], and we replace the chance constraints used in the last-mentioned work by their boundary case where the location-allocation has to be feasible under all scenarios. The latter replacement is not that restricting in our context than it would be in the context of [Mostajabdaveh et al., 2018], since the flexible allocation provides better chances to cover the total demand in every scenario.

6.1 The Model

A survey on parameters and variables is given in Table 1. It is assumed that there are m population nodes (PNs) $k \in K$. Population node k has w_k inhabitants. In decision stage 1, i.e., before the disaster, a decision has to be made on the locations at which shelters are to be established. The set of candidate locations is J. Opening a shelter at location $j \in J$ incurs opening costs of g_j and provides a capacity for c_j victims. The decision variable describing whether a shelter at location $j \in J$ is established is x_j; it is set to the value 1 if the shelter is opened and to the value 0 otherwise. There is a budget B for the overall opening costs.

The influence of randomness, which is caused by the unpredictable occurrence of the natural disaster under consideration, is represented by a set S of scenarios whose probabilities p_s ($s \in S$) are assumed to be known. Each scenario $s \in S$ contains the information on which PNs are affected by the disaster, and to which extent. The degree of affection of PN $k \in K$ in scenario $s \in S$ is expressed by the demand value b_k^s, where $0 \leq b_k^s \leq w_k$. The quantity b_k^s describes the number of individuals from PN k who need to be evacuated in scenario s.

The second-stage decision, to be made after the onset of the disaster, concerns the question of how to allocate PNs to opened CLs. This decision is described by variables y_{kj}^s ($s \in S$, $k \in K$, $j \in J$). We set $y_{kj}^s = 1$ if in scenario s, PN k is allocated to the shelter in CL j. Evacuation along a certain distance in a post-disaster environment is connected with discomfort and danger (subsumable by the technical term "deprivation cost" in the humanitarian operations literature) which we assume, in this work, to be proportional to the distance to be traversed. (This is a first approximation; also more complex dependencies could be modelled.) Let d_{kj}^s denote the distance between PK k and CL j in scenario s. We explicitly allow a dependence of the distance on the scenario, since in some scenarios, some links of the transportation network may be destroyed, such that the shortest paths between nodes vary with s. We look for solutions covering in each scenario the entire demand without violating a capacity constraint. It is clear that there are instances for which a feasible solution does not exist; however, on the assumption that there is at least one CL in the considered region that provides a practically unlimited capacity (e.g. by the possibility to set up an arbitrary number of rub halls), the demand can always be covered, probably at the price of very long distances to be traversed.

Linking the above-mentioned modelling components to concepts and notation of Section 3, we consider the distance to be traversed by each victim $i \in N$ as the cost $f_i(Z, s)$ assigned to this individual under solution Z and scenario s. A solution Z consists now of the pair (x, y), where x collects the first-stage decision variables x_k, and y collects the second-stage decision variables y_{kj}^s. Obviously, if individual i lives in PN $k \in K$ and is affected by the disaster, then $f_i(Z, s) = \sum_{j \in J} d_{kj}^s \, y_{kj}^s$. Actually, this distance depends only on y and not on x, but note that it is indirectly influenced by x as x restricts the set of feasible y.

Table 1: Parameters and Decision Variables

Notation	Description
J	set of candidate locations (CLs); $\lvert J\rvert = m$
K	set of population nodes (PNs); $\lvert K\rvert = n$
w_k	number of inhabitants of PN $k \in K$
S	set of scenarios; $\lvert S\rvert = M$
p_s	probability of scenario $s \in S$
b_k^s	demand in PN $k \in K$ under scenario $s \in S$
d_{kj}^s	distance between PN $k \in K$ and CL $j \in J$ in scenario $s \in S$
c_j	capacity of CL $j \in J$
g_j	opening cost for a shelter at CL $j \in J$
B	budget
x_j	1 if CL $j \in J$ is opened, 0 otherwise
y_{kj}^s	1 if in scenario $s \in S$, PN $k \in K$ is allocated to CL $j \in J$, 0 otherwise

6.1.1 Global Ex-Post Formulation

Let us start with the GEP approach. It turns out that by this approach, a problem representation by a *two-stage stochastic program* is obtained. (As we shall see later, this is not anymore the case for the REA approach.)

In the first decision stage, the location decision, i.e., that on the vector $x = (x_1, \ldots, x_m)$, has to be made. A strategy Z for the overall process is now given as $Z(s) = (Z_1, Z_2(s)) = (x, y^s)$, where y^s denotes the second-stage decision under scenario $s \in S$, i.e., the allocation decision. Recall that the Rawlsian measure $\mathcal{I} = \max$ is used. Thus, according to Eq. (2), the objective function is $\min \sum_{s\in S} p_s \max_{k\in K} \hat{f}_k(Z(s), s)$ where the cost function $\hat{f}_k(Z(s), s) = \hat{f}_k((x, y^s), s)$ is defined as

$$\hat{f}_k((x, y^s), s) = \begin{cases} \sum_{j\in J} d_{kj}^s\, y_{kj}^s, & \text{if } b_k^s > 0. \\ 0, & \text{otherwise.} \end{cases}$$

(We write now $\hat{f}_k$ instead of f_k, as the index k refers to entire PNs rather than to individuals.)

In the second decision stage, for given scenario s, the allocation y^s is optimized. After insertion of the *optimal* y^s, we get the value $R(x, s) = \max_{k\in K} \hat{f}_k((x, y^s), s)$ of the *recourse cost*. The recourse cost $R(x, s)$ expresses the maximum distance to be traversed from an affected PN to the allocated shelter, given location decision x and scenario s, if the allocations have been chosen in an optimal way. Although $\hat{f}$ does not depend on x directly, $R(x, s)$ depends on x since the variables y^s are linked to x by constraints $y_{kj}^s \leq x_j$ for all $k \in K$, $j \in J$ and $s \in S$.

The first-stage objective optimizes the expected recourse costs; the first-stage opening costs do not occur in the objective as a separate term, but influence the solution through the budget constraint.

In total, first-stage and second-stage program take the following form:

$$\textit{1st stage:} \qquad \min_{x} \sum_{s \in S} p_s \, R(x,s) \tag{8}$$

$$\text{s.t.} \sum_{j \in J} g_j x_j \leq B \tag{9}$$

$$x_j \in \{0,1\} \quad \forall j \in J \tag{10}$$

$$\textit{2nd stage:} \qquad R(x,s) = \min_{y^s} \sigma^s \tag{11}$$

$$\text{s.t. } \sigma^s \geq \sum_{j \in J} d^s_{kj} \, y^s_{kj} \quad \forall k \in K \text{ with } b^s_k > 0 \tag{12}$$

$$\sum_{j \in J} y^s_{kj} = 1 \quad \forall k \in K \tag{13}$$

$$y^s_{kj} \leq x_j \quad \forall k \in K, \, j \in J \tag{14}$$

$$\sum_{k \in K} b^s_k \, y^s_{kj} \leq c_j \quad \forall j \in J \tag{15}$$

$$y^s_{kj} \in \{0,1\} \quad \forall k \in K, \, j \in J \tag{16}$$

The objective function Eq. (8) represents the expected recourse cost. Eq. (9) is the budget constraint for the opening costs. Eq. (10) defines the opening decision variables as binaries. The objective Eq. (11) of the second-stage program minimizes the worst-case distance to be traversed by any affected individual to the allocated shelter. By Eq. (12), this distance is bounded from below by every distance from an affected PN to the allocated shelter. Note that for this bound, all we need to know about b^s_k is whether or not $b^s_k > 0$. This is a consequence of choice of the Rawls measure which only cares about the cost of the "worst off" individual. Constraint (13) requires that each PN is allocated to exactly one CL. By constraint (14), it is ensured that a CL j can only be assigned to a PN if at shelter is opened at CL j. Constraint (15) makes sure that the total number of victims assigned to CL j does not exceed the capacity of CL j. Eq. (16), finally, defines the second-stage decision variables as binaries. Observe that (11) – (16) is an ILP; relaxation of the integrality constraint (16) within a branch-and-bound (or branch-and-cut) framework yields an LP.

6.1.2 Recursive Ex-Ante Formulation

In a similar way as for the GEP policy, the REA policy for the considered location-allocation problem can be formulated as a mathematical program. However, it turns out that the resulting formulation does *not* have the structure of a two-stage stochastic program. Rather than that, a *bilevel program* is obtained, as it will be seen. Let

$$y^*(x,s) \in \arg\min_{y^s}\{\sigma^s \mid \text{Eqs. } (12) - (16)\} \tag{17}$$

be a solution of (11) – (16), that is, an optimal allocation $y^* = (y^*_{kj})_{j\in J}$ leading under first-stage decision x and under scenario s to a solution value of $R(x,s)$. If there are several such solutions, we select one of them that is minimal under the tie-break order (7). We assume that if in a PN k with w_k inhabitants, the number of affected individuals is b^s_k, then each individual i from PN k has the same probability of b^s_k/w_k of being affected by the disaster. Then if the allocations are defined by the second-stage solutions $y^*(x,s)$, the expected cost of an individual from PN k is

$$\varphi_k(x) = \sum_{s\in S} p_s \cdot \frac{b^s_k}{w_k} \cdot \sum_{j\in J} d^s_{kj}\, y^*_{kj}(x,s). \tag{18}$$

By construction, for $\mathcal{I} = \max$, the REA policy minimizes the maximum of the values $\varphi_k(x)$ over all individuals or, which is the same, over all PNs k.

The optimal first-stage decision is given as the solution of

$$\min_x \{\rho \mid \rho \geq \varphi_k(x)\ \forall k \in K,\ \text{Eqs. } (9) - (10)\}.$$

However, note that for determining this solution, the second-stage solutions $y^*(x,s)$ have already to be known. Thus, we obtain a *bilevel programming* structure: Imagine two decision makers, the *leader* who chooses the first-stage solution x, and the *follower* who makes the second-stage decision. When deciding on x, the leader has already to take into account that each possible choice of x will result in a corresponding choice of $y^*(x,s)$ for each $s \in S$, optimized with respect to the *follower's* objective function. This gives a bilevel program. The follower's problem decomposes into $M = |S|$ independent subproblems, one for each scenario. We get the formulation in Table 2.

Example 2. For an illustration of the difference between the GEP and the REA approach in the concrete framework of the shelter location application, consider Figure 2. Two earthquake scenarios $s^{(1)}$ and $s^{(2)}$ threaten an area where two cities lie. Earthquake $s^{(1)}$ would only hit city 1, while earthquake $s^{(2)}$ would hit both cities. It is estimated that the two scenarios are

Table 2: Bilevel program

$$\min_{x} \rho \tag{19}$$

$$\text{s.t. } \rho \geq \sum_{s \in S} p_s \cdot \frac{b_k^s}{w_k} \cdot \sum_{j \in J} d_{kj}^s \, y_{kj}^s \quad \forall k \in K \tag{20}$$

$$\text{Eqs. } (9) - (10) \tag{21}$$

$$\min_{\sigma^1} \sigma^1 \tag{22}$$

$$\text{s.t. } \sigma^1 \geq \sum_{j \in J} d_{kj}^1 \, y_{kj}^1 \quad \forall k \in K \text{ with } b_k^1 > 0 \tag{23}$$

$$\sum_{j \in J} y_{kj}^1 = 1 \quad \forall k \in K \tag{24}$$

$$y_{kj}^1 \leq x_j \quad \forall j \in J \tag{25}$$

$$\sum_{k \in K} b_k^1 \, y_{kj}^1 \leq c_j \quad \forall j \in J \tag{26}$$

$$y_{kj}^1 \in \{0, 1\} \quad \forall k \in K, \, j \in J \tag{27}$$

$$\ldots\ldots$$

$$\min_{\sigma^M} \sigma^M \tag{28}$$

$$\text{s.t. } \sigma^M \geq \sum_{j \in J} d_{kj}^M \, y_{kj}^M \quad \forall k \in K \text{ with } b_k^M > 0 \tag{29}$$

$$\sum_{j \in J} y_{kj}^M = 1 \quad \forall k \in K \tag{30}$$

$$y_{kj}^M \leq x_j \quad \forall j \in J \tag{31}$$

$$\sum_{k \in K} b_k^M \, y_{kj}^M \leq c_j \quad \forall j \in J \tag{32}$$

$$y_{kj}^M \in \{0, 1\} \quad \forall k \in K, \, j \in J \tag{33}$$

equally likely. One and only one shelter can opened, either at point a or at point b. The distances between cities and the potential locations of the shelter are shown in the figure. The inequity-averse measure is the Rawlsian measure $\mathcal{I} = \max$, that is, the maximum cost (i.e., distance) over all individuals.

In the second decision stage, both the GEP and the REA approach assign each affected city to the opened shelter. This results in the following costs (distances to be traversed) (f_1, f_2) for the inhabitants of city 1 and 2, respectively:

$$\begin{array}{rll} \text{Shelter in } a: & s^{(1)} \text{ (only city 1 affected)}: & (3,0) \\ & s^{(2)} \text{ (both cities affected)}: & (3,6) \\ \text{Shelter in } b: & s^{(1)} \text{ (only city 1 affected)}: & (4,0) \\ & s^{(2)} \text{ (both cities affected)}: & (4,1) \end{array}$$

The GEP approach minimizes the expected value of the inequity-averse measure (i.e., the maximum) of the costs. If location a is chosen, maximum costs of 3 and 6 result in the first and in the second scenario, respectively, so this choice is evaluated by the expected value $\frac{1}{2} \cdot 3 + \frac{1}{2} \cdot 6 = 4.5$. Analogously, the choice of location b is evaluated by the expected value $\frac{1}{2} \cdot 4 + \frac{1}{2} \cdot 4 = 4$, which is smaller than 4.5. Therefore, GEP selects location b in its first-stage decision.

Conversely, the REA approach minimizes the inequity-averse measure (i.e., the maximum) of the expected values of the costs. If location a is chosen, the expected costs of the individuals in city 1 and 2, respectively, are given by the vector $\frac{1}{2} \cdot (3,0) + \frac{1}{2} \cdot (3,6) = (3,3)$. The Rawls measure applied to $(3,3)$ is 3, so this is the evaluation of choice a by REA. If, on the other hand, location b is chosen, the expected costs of the individuals in city 1 and 2, respectively, are given by the vector $\frac{1}{2} \cdot (4,0) + \frac{1}{2} \cdot (4,1) = (4,0.5)$. The Rawls measure applied to $(4,0.5)$ is 4, which is larger than 3. Therefore, REA would (contrary to GEP) choose location a.

Intuitively, the difference can be outlined as follows: while GEP only looks at the fairness of the final states, REA also takes the chances or risks by which the individuals arrived at these states into account. Thus, GEP considers it as very undesirable that city 2 can end up with a large distance of 6 to be traversed. REA, on the other hand, sees this large value compensated to some extent by the possibility that city 2 might not be affected by the earthquake at all, and is more concerned about the disadvantage city 1 has to face by the position of b closer to city 2 than to city 1. $\bowtie$

6.2 Computational Solution

The solution of the two-stage stochastic program of Subsection 6.1.1 is standard. Note, however, that the computational complexity of this problem is considerably increased, compared to ordinary (continuous) stochastic programs, because of the integrality constraints on *both* the first-stage decision *and* the second-stage decision. For the numerical solution of two-stage stochastic programs with integer recourse, see [Carøe and Tind, 1998].

Even more challenging is the solution of the bilevel program of Subsection 6.1.2. It is known that already in the case where both the upper level and the lover level problem of a bilevel

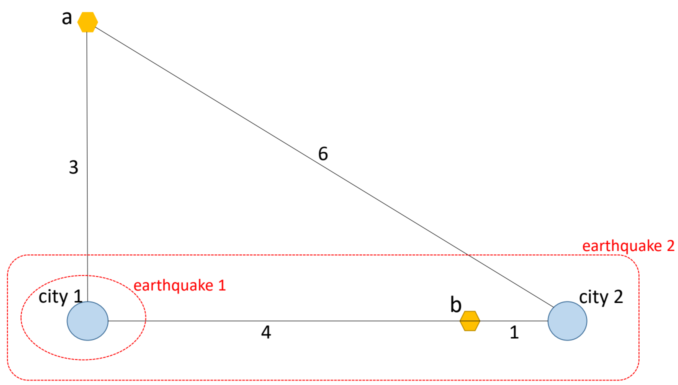

Figure 2: Example for showing the difference between GEP and REA in the shelter location application.

problem are ordinary LPs, the bilevel problem is in general NP-hard (see [Colson et al., 2005]). In our case, we have integer problems in both levels, and the lover level contains M subproblems to be solved, one for each scenario.

To be able to solve (not too large) instances to optimality nevertheless, we choose the following approach. The upper-level problem (the choice of x) is solved by complete enumeration. The lower-level problems, each of the structure of (22) – (27), are solved by CPLEX. These lower level subproblems of REA are identical to the second-stage subproblems (11) – (16) of GEP, so their solutions can be used for both approaches. However, while the GEP solution procedure only needs the solution values, the REA solution procedure needs also the solutions y^s themselves, more precisely: in case of ambiguity, it needs those solutions y^s that are minimal under the tie-break order (7). The latter requirement can be ensured by the following extension of the CPLEX calls: First of all, solve (11) – (16), which gives an optimal solution value of $(\sigma^s)^*$. After that, solve the program

$$\min_{y^s} \sum_{k \in K} \sum_{j \in J} b_k^s \, d_{kj}^s \, y_{kj}^s \tag{34}$$

$$\text{s.t. } (\sigma^s)^* \geq \sum_{j \in J} d_{kj}^s \, y_{kj}^s \quad \forall k \in K \text{ with } b_k^s > 0 \tag{35}$$

$$\text{Eqs. (13)} - \text{(16)} \tag{36}$$

The objective function Eq. (34) is the utilitarian objective function, multiplied by the number of individuals. (In case of further ambiguity, take the lexicographically smaller solution.) This produces the solution y^s required in the upper level problem of the bilevel formulation.

7 Price of Fairness of the Allocation

Following [Bertsimas et al., 2011], it would be desirable to determine the *price of fairness* (PoF) of the GEP solution and of the REA solution. The PoF of solution x^π provided by a given policy π for a certain instance is defined as

$$\mathrm{PoF} = \frac{\mathrm{cost}(x^\pi) - \mathrm{cost}(x^u)}{\mathrm{cost}(x^u)} \tag{37}$$

where cost is expressed in the utilitarian measure, i.e., as *average* cost over all individuals, and x^u is the utilitarian solution: the solution minimizing the utilitarian measure for the given instance. It is clear that for positive costs, the PoF has always a nonnegative value.

Analytical results on the PoF of the policies GEP and REA for the overall location-allocation problem of Section 6 may be difficult to obtain. The current section derives, for both approaches, a tight bound for the PoF of the *allocation subproblem* (i.e., the second-stage subproblem) of the location-allocation problem. Recall that in the second stage, the two approaches coincide. In the next section, we shall provide *numerical* results on the PoF for special instances of the *overall* problem.

For fixed location decision x, consider two solutions y^r and y^u, where y^u is the utilitarian solution of the allocation subproblem, and y^r is the Rawlsian solution of the allocation subproblem with tie-break order (7), i.e., the solution used in the second stage of GEP or REA. In particular, y^r has the properties that (i) it minimizes the maximal distance between PNs and assigned shelters, and (ii) among the solutions that do the same, it solves the tie-break condition (34) – (36). Obviously, for expressing "cost" in Eq. (37), we can replace "average cost" by "total cost" without changing the quotient. Let $\bar{\mu}$ denote the total cost of all individuals, i.e., average cost μ multiplied by the number of individuals, and let $K_r = \bar{\mu}(y^r)$ and $K_u = \bar{\mu}(y^u)$. Observe that K_r and K_u are uniquely defined, since for two different solutions y^r and y'^r of (34) – (36), we have $\bar{\mu}(y^r) = \bar{\mu}(y'^r)$, and for two different utilitarian solutions y^u and y'^u, the equality $\bar{\mu}(y^u) = \bar{\mu}(y'^u)$ holds as well. The quantity of interest is $\mathrm{PoF} = K_r/K_u - 1$.

Let $w_{max} = \max_{k\in K} w_k$, $w_{min} = \min_{k\in K} w_k$, and suppose that all distances d_{kj} $(k \in K, j \in J)$ are sorted in ascending order as $d^{(1)} \leq d^{(2)} \leq \ldots \leq d^{(mn)}$.

Proposition 6. With the notation above and with $x^+ = \max(x, 0)$, the price of fairness of the

GEP or the REA approach for the allocation part of the location-allocation problem satisfies

$$\text{PoF} \;\leq\; \left[\frac{w_{max}}{w_{min}} \cdot \max_{n+1 \leq i \leq mn} \frac{\sum_{\ell=i-n}^{i-1} d^{(\ell)}}{\sum_{\ell=1}^{n-1} d^{(\ell)} + d^{(i)}} - 1\right]^{+}$$

Proof. Consider a specific Rawlsian allocation y^r, and a specific utilitarian allocation y^u. Each allocation y defines a subset of distances $d_{kj} = d^{(\ell)}$ which are "used" in the allocation, i.e., get the value $y_{kj} = 1$. The allocation y^r minimizes, among all feasible allocations, the value of the longest used distance; the allocation y^u minimizes, among all feasible allocations, the average value of the used distances. Two cases can be distinguished:

Case (a): Also the allocation y^u minimizes the value of the longest used distance among all feasible allocations. Then y^u also satisfies both conditions (i) and (ii) stated above for y^r. That is, y^u is equivalent to y^r, and therefore $K_r = K_u$ and PoF $= 0$.

Case (b): The longest used distance in y^u is strictly larger than the longest used distance in y^r. Let $i = \max\{\ell \,|\, \text{allocation } y^u \text{ uses } d^{(\ell)}\}$. Then y^r does not use distance $d^{(i)}$ anymore, but only distances $d^{(\ell)}$ with $\ell < i$. Obviously, $n+1 \leq i \leq mn$, since $i < n$ is impossible, as n distances have to be used, and if $i = n$, then y^u would use just the distances $d^{(1)}, \ldots, d^{(n)}$ and would therefore provide a minimal possible maximum distance, contrary to the assumption. For each allocation y, the total cost can be represented as $\bar{\mu}(y) = \sum w^{(i)}\, d^{(i)}$, where the sum is over all distances used by y, and $w^{(i)} = w_k$ for that PN k to which distance $d^{(i)} = d_{kj}$ pertains.

Let us now subdivide case (b) by distinguishing all possibilities $i = n+1, \ldots, mn$. Denote by I_r the set of indices ℓ of the distances used by y_r, and by I_u the set of indices ℓ of the distances used by y_u. We have $|I_r| = |I_u| = n$. Furthermore, $I_r \subseteq \{1, \ldots, i-1\}$ and $I_u = I'_u \cup \{i\}$ with $I'_u \subseteq \{1, \ldots, i-1\}$. Then

$$K_r = \sum_{\ell \in I_r} w^{(\ell)}\, d^{(\ell)} \leq w_{max} \sum_{\ell \in I_r} d^{(\ell)} \leq w_{max} \sum_{\ell=i-n}^{i-1} d^{(\ell)}$$

and

$$K_u = \sum_{\ell \in I_u} w^{(\ell)}\, d^{(\ell)} = \sum_{\ell \in I'_u} w^{(\ell)}\, d^{(\ell)} + w^{(i)}\, d^{(i)} \geq w_{min} \sum_{\ell \in I'_u} d^{(\ell)} + w_{min}\, d^{(i)} \geq w_{min} \left(\sum_{\ell=1}^{n-1} d^{(\ell)} + d^{(i)}\right).$$

Hence, for that $i \in \{n+1, \ldots, nm\}$ that is defined by the given y^u,

$$PoF \leq \frac{w_{max}}{w_{min}} \cdot \frac{\sum_{\ell=i-n}^{i-1} d^{(\ell)}}{\sum_{\ell=1}^{n-1} d^{(\ell)} + d^{(i)}} - 1.$$

By taking all possibilities for i and by including case (a), we get the result. □

Corollary. On the conditions of Prop. 6, the following upper bound on the PoF, expressed by the number n of PNs and the maximum and minimum PN sizes w_{max} and w_{min}, respectively, is valid:

$$\text{PoF} \;\leq\; (w_{max}/w_{min}) \cdot n - 1. \tag{38}$$

Proof. The statement follows immediately from Prop. 6 by $\sum_{\ell=i-n}^{i-1} d^{(\ell)} \,/\, \left(\sum_{\ell=1}^{n-1} d^{(\ell)} + d^{(i)}\right)$ $\leq\; n \cdot d^{(i-1)} \,/\, d^{(i)} \leq n.$ □

The bound given by the Corollary above may seem weak, but the following result shows that at least if cases where the triangle inequality is not fulfilled are included, the order $O(n)$ of the bound is best-possible:

Proposition 7. For distances that need not satisfy the triangle inequality, the order $O(n)$ of the upper bound in Eq. (38) is tight.

Proof. Since we are only interested in the dependence on n, it suffices to show tightness for the special case $w_{max} = w_{min}$. For arbitrary n, set $m = n$, $K = \{1, \ldots, n\}$, $J = \{n+1, \ldots, 2n\}$, and $c_j = 1$ for all $j \in J$. Assume that the location decision x opens all shelters, i.e., $x_k = 1$ for all k. An allocation y is then given by a permutation matrix $(y_{kj})_{k \in K, j \in J}$, containing binary values where all lines sum up to 1 and all columns sum up to 1 as well. Consider now the distance matrix $D = (d_{kj})$ with the following entries:

$$d_{kj} = \begin{cases} \epsilon, & \text{if } 2 \leq k \leq n \text{ and } j = n + k, \\ 1, & \text{if } ((1 \leq k \leq n-1 \text{ and } j = n+k+1) \text{ or } (k = n \text{ and } j = n+1)), \\ 1+\epsilon, & \text{for all other pairs } (k, j) \in \{1, \ldots, n\} \times \{n+1, \ldots, 2n\}. \end{cases}$$

Therein, $0 < \epsilon < 1/n$. The matrix D is not yet a complete distance matrix between all $2n$ considered nodes; it would have to be extended by choosing distances between PNs and PNs on the one hand, and between CLs and CLs on the other hand, and it is not guaranteed that such an extension can be performed without violating the triangle inequality. However, the distance values added by this extension do not play a role for the following consideration.

The utilitarian solution y^u assigns, for all k, the k-th CL (index $n+k$ in the overall node representation) to the kth PN, since using all $n-1$ available epsilon entries in the lines 2 to n is the only possibility to push down the total cost (the sum of the distances) to a value smaller than 2. This gives $K_u = 1 + n\epsilon$. On the other hand, the Rawlsian solution y^r is the matrix obtained from the matrix y^u by shifting the 1-elements cyclically to the right by one position, i.e., it assigns, for all $k = 1, \ldots, n-1$, the $(k+1)$th CL (index $n+k+1$ in the overall node representation) to PN k, and the first CL (index $n+1$ in the overall node representation) to PN n: It is easy to see that while this yields a maximal distance of 1, for each other allocation, a maximal distance of $1+\epsilon$ results. This gives $K_r = n$.

We obtain PoF $= K_r/K_u - 1 = n/(1+n\epsilon) - 1$. For $\epsilon = \delta/n$ with sufficiently small δ, this value comes arbitrarily close to the bound $n-1$ given by (38) for equal PN sizes. □

Of course, Propositions 6 – 7 do not give sufficient information about the PoF in the *average case*. We investigate the question of the "typical size" of the PoF for the *overall* two-stage location-allocation problem (with Euclidean distances) in the next section, using numerical experiments.

8 Numerical Results

Computational experiments were carried out to test the approaches of the previous sections, and, in particular, to compare the solutions produced by the GEP policy and the REA policy, respectively. We generated random test instances according to the procedure below.

(a) A total of n population nodes and m candidate locations were selected uniformly at random in the unit square $[0,1]^2$. This gave the sets K and J, respectively. From the obtained points in the square, the distances d_{kj} ($k \in K$, $j \in J$) were computed as Euclidean distances. In the tests, we did not make the distances scenario-dependent.

(b) For each PN $k \in K$, the numbers w_k of inhabitants were chosen uniformly at random from the interval $[w_{min}, w_{max}]$.

(c) Scenarios $s \in S$ were generated as follows: Each scenario represents the area affected by the disaster under consideration. We focused on the case of earthquakes. Therefore, the affected area was assumed to be given by an *epicenter* $(\text{epi}_1^s, \text{epi}_2^s)$ of the disaster and by a *radius* rad^s. The epicenter was selected uniformly at random from $[0,1]^2$, while the radius was selected uniformly at random from $[\text{rad}_{min}, \text{rad}_{max}] = [0.1, 0.7]$. The PNs $k \in K$ within a distance smaller or equal to rad^s were assumed to be hit by the disaster. To determine the number of affected people within such a PN, a fraction $\text{frac}_k^s \in [0.5, 1]$, selected uniformly at random from this interval, was chosen, and b_k^s was set to the value $\text{frac}_k^s \cdot w_k$. For PNs k with a distance larger than rad^s to the epicenter, b_k^s was set to 0. The number of scenarios was chosen as $M = 5$.

(d) The opening costs g_j were assumed as identical for all shelters. As a consequence, the budget constraint is equivalent to the constraint that a fixed number q of shelters can be opened.

(e) The capacities c_j of the shelters $j \in J$ were chosen uniformly at random from the interval $[c_{min},\, c_{max}] = [1000,\, 1500]$.

We tested all combinations of the following choices: (i) $n = 10,\, 20$, (ii) $(m, q) = (7, 3)$, $(8, 4)$, $(10, 5)$, $(12, 6)$, $(15, 4)$, $(20, 3)$, (iii) $[w_{min}, w_{max}] = [50,\, 1000]$, $[400,\, 600]$.

This gives 24 instance types of different sizes, the largest of which (e.g., those with 15 CLs, out of which 4 shelters have to be selected) are already of realistic size for some practical application cases. For each of the 24 instance types, 5 instances were randomly generated. In total, this produces 100 test instances.

We applied the solution method of Subsection 6.2 which uses CPLEX for the solution of the scenario-specific subproblems. The resulting computation times were very favorable: Even the larger instances needed only up to 2 minutes of computation time on an ordinary PC. Note, however, that we used only $M = 5$ scenarios; the computation time scales linearly in the number of scenarios, so that for 1000 scenarios, runtimes in the order of 7 hours are to be expected. This is still feasible, considering that the choice of shelter locations is a strategic decision which does not have to be made under time pressure, and the (second-stage) allocation decision is fast.

Table 3 shows some numerical results for the 100 test instances. Outcome measures have been averaged over each set of 5 instances belonging to the same instance type, so that each line contains the aggregated results for an instance type. The first 5 columns of the table define the instance type. In the last 4 columns, the following outcome measures are reported:

(i) GEP-by-REA: This is the quotient $\text{ante}(x_{GEP})/\text{ante}(x_{REA})$. Therein, x_{GEP} and x_{REA} denote the optimal location decisions provided by the GEP policy and the REA policy, respectively, and $\text{ante}(x)$ is the ex-ante objective function applied to x, i.e., the value $\max_k \sum_s p_s\, \hat{f}_k(x, s)$, where $\hat{f}_k(x, s)$ represents the cost of PN k in scenario s. Thus, GEP-by-REA expresses how good the GEP solution would be evaluated from the viewpoint of the REA solution. Lower values are better. If the GEP solution is optimal also with respect to the criterion applied by REA, then GEP-by-REA takes the minimal value of 1.

(ii) REA-by-GEP: This is the quotient $\text{post}(x_{REA})/\text{post}(x_{GEP})$. Therein, x_{GEP} and x_{REA} are as above, and $\text{post}(x)$ is the ex-post objective function applied to x, i.e., the value $\sum_s p_s \max_k\, \hat{f}_k(x, s)$, with $\hat{f}_k(x, s)$ as above. Thus, REA-by-GEP expresses how good the REA solution would be evaluated from the viewpoint of the GEP solution. Again, the minimal possible value is 1.

(iii) PoF(REA): This is the price-of-fairness (see Eq. (37)) of the REA solution.

(iv) PoF(GEP): This is the price-of-fairness of the GEP solution.

Table 3: Comparison REA vs. GEP

n	m	q	w_{min}	w_{max}	GEP-by-REA	REA-by-GEP	PoF(REA)	PoF(GEP)
10	7	3	50	1000	1.0115	1.1046	0.4385	**0.1575**
10	8	4	50	1000	1.0138	1.1190	0.1561	**0.1235**
10	10	5	50	1000	1.0035	1.1324	0.3576	**0.1741**
10	12	6	50	1000	1.0000	1.0921	0.1167	**0.0702**
10	15	4	50	1000	1.1006	1.0746	0.2925	**0.2441**
10	20	3	50	1000	1.2157	1.1274	**0.2196**	0.2816
20	7	3	50	1000	1.0000	1.0275	**0.0581**	0.0705
20	8	4	50	1000	1.1906	1.0492	**0.0772**	0.1695
20	10	5	50	1000	1.0726	1.0376	**0.0253**	0.0561
20	12	6	50	1000	1.1121	1.1673	0.1930	**0.0875**
20	15	4	50	1000	1.2013	1.1359	**0.1647**	0.1779
20	20	3	50	1000	1.2383	1.0385	0.1134	**0.0671**
10	7	3	400	600	1.0192	1.0877	0.1783	**0.0638**
10	8	4	400	600	1.0124	1.1017	0.1528	**0.1290**
10	10	5	400	600	1.0035	1.1001	0.2200	**0.1349**
10	12	6	400	600	1.0000	1.0421	0.1356	**0.0744**
10	15	4	400	600	1.0103	1.1044	0.2177	**0.0839**
10	20	3	400	600	1.2223	1.1154	**0.1699**	0.2269
20	7	3	400	600	1.0593	1.0378	0.0614	**0.0410**
20	8	4	400	600	1.1154	1.0547	**0.0664**	0.0859
20	10	5	400	600	1.1227	1.0787	**0.0962**	0.1156
20	12	6	400	600	1.1104	1.1557	0.2499	**0.0619**
20	15	4	400	600	1.2842	1.1421	0.1891	**0.1723**
20	20	3	400	600	1.0497	1.0072	0.0586	**0.0435**

Observations:

(a) The quotients GEP-by-REA and REA-by-GEP do not exceed the value 1.3; their average values are 1.090 and 1.089, respectively. This means that in mutual cross-evaluations, the two solutions produced REA and GAP were never worse than the "own" optimum by more than 30 % from the viewpoint of the respective other approach. In the average, each of the two approaches "wastes" about 10 % from the perspective of the other approach. Even if this is not a too large number, it should be noted that computational optimization approaches to facility location problems are usually already regarded as worthwhile if they improve manually found solutions by a few percent. Compared to that magnitude, the distinction between ex-ante and ex-post is certainly not an issue of negligible impact.

(b) In 12 out of the 24 cases (instance types), GEP-by-REA was larger than REA-by-GEP, and vice versa in the other 12 cases. In other words, GEP sees the REA solution neither more nor less favorable than REA sees the GEP solution.

(c) All observed PoF values are smaller than 50 %. The average value of the PoF for REA and GEP over all instances is 16.7 % and 12.1 %, respectively. In 16 out of the 24 cases (instance types), PoF(REA) was larger than PoF(GEP). In a statistical sign test at significance level $\alpha = 0.05$, this confirms the hypothesis that PoF(REA) − PoF(GEP) has a median larger than zero, compared to the null hypothesis that the median is zero. In less formal terms, the price of fairness is significantly higher for the ex-ante solution than for the ex-post solution, which seems to be an interesting, nontrivial result that deserves further investigation.

(d) In a pairwise comparison of the smaller ($n = 10$) to the larger ($n = 20$) test cases, it turns out that PoF(REA) is significantly larger for $n = 10$ than for $n = 20$ ($\alpha = 0.05$). The same holds for PoF(GEP). In other words, both for REA and GEP, the PoF decreases with the instance size.

(e) Whether the PNs are of comparable size ($[w_{min}, w_{max}] = [400, 600]$) or of largely varying size ($[w_{min}, w_{max}] = [50, 1000]$) does not have a significant influence on PoF(REA). However, PoF(GEP) is significantly higher for the case of largely varying sizes ($\alpha = 0.05$).

The numerical results above should be interpreted with caution as they rely on synthetically generated instances; future research should test the made observations on real-life instances.

9 Conclusions

Inequity-averse optimization has a broad range of applications, but current methods in this field seem to be largely restricted to a static decision making context. The results in this work are a first attempt to extend methods of inequity-averse decision making to a context that is both stochastic and dynamic. It turns out that this is possible without conceptual difficulties for the *ex-post* approach of dealing with inequity under uncertainty. For the *ex-ante* approach, however, the issue of time inconsistency has shown to be a major obstacle. In order to overcome this problem, we proposed a time-consistent version of an ex-ante policy.

The introduced concepts have been applied to a shelter location-allocation problem under uncertainty, and it has been demonstrated that they allow exact computational solutions for not too large problem instances. In particular, experimental comparisons of the ex-post and the ex-ante approach provide interesting insights.

The present investigation is only a first step and suggests further research in a rich variety of directions. Let us mention a few. (i) An extension of the approach to continuous decision sets and continuous probability spaces would be highly desirable. (ii) Infinite time horizons could be dealt with by methods similar to those applied for Markov Decision Processes. (iii) An extension of the solution approach from the max-min measure to Gini-based measures or the conditional β-mean would be a very interesting next step. (iv) Whereas the computational results in this paper restrict themselves to a particular two-stage model, models for three or more decision stages could be addressed by specific techniques of scenario-tree analysis, e.g., progressive hedging. (v) For larger instances of the considered location-allocation problem, more sophisticated solution methods, exact as well as heuristic ones, should be developed. (vi) Applications in diverse fields, as health, workforce scheduling or resource sharing might be explored.

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