ANALYTIC WAVELETS AND MULTiresolution Analysis
A NOTE ON CERTAIN ORTHOGONALITY CONDITIONS

Roza Aceska
Faculty of Mechanical Engineering, Ss Cyril and Methodius University,
P.O.Box 464, MK-1001 Skopje, Republic of Macedonia
aroza@mf.ukim.edu.mk

The main disadvantages of the Fourier series and transforms are left behind by a new tool: wavelets! The properties of wavelets are well presented, apart from continuity, by the oldest example – the Haar wavelets. This work deals with expanding wavelets on the complex plane using their analytic representations. Here are reviewed analytic wavelets and their basic properties. The corresponding multiresolution analysis, however, does not preserve the orthogonality it had on the real line. Here is given a consequence regarding the orthogonality of the basis generated by the scaling function. Under some conditions, the orthogonality is preserved, as seen with the Shannon wavelets.

Key words: wavelets; multiresolution analysis; scaling function; analytic wavelets; orthogonality

1. REVIEW ON WAVELETS

In the last twenty years the Fourier series and transforms are finally replaced by a new tool: wavelets! Wavelet expansions have quite a few properties not available in Fourier expansions (or any other expansions). To see this in the simplest context, consider a real-valued function \( f(x) \) on the interval \([0,1]\). Under some conditions, it can be expanded in a Fourier series

\[
f(x) = a_0 + \sum_{k=1}^{\infty} (a_k \cos 2\pi k x + b_k \sin 2\pi k x)
\]

or in a Haar function series

\[
f(x) = c_{00} + \sum_{j=0}^{\infty} \sum_{k=0}^{2^j-1} c_{j,k} \psi(2^j x - k),
\]

where

\[
\psi(x) = \begin{cases} 1, & 0 \leq x < 0.5, \\ -1, & 0.5 \leq x < 1, \\ 0, & \text{otherwise}. \end{cases}
\]

(see Fig. 1).

Both series are examples of expansions in terms of orthogonal functions in \( L^2([0,1]) \), the space of square integrable functions on the interval \([0,1]\). There are simple formulas for calculating the coefficients. But the Fourier series is not well localized in space; if you are interested in the behavior of \( f \) on a subinterval \([a,b]\) you need to involve all the Fourier coefficients. On the other hand, the Haar series is very well localized: to restrict the attention to the subinterval \([a,b]\) only take the sum in (1.2) over those indices for which the support (closure of
the nonzero area for a function) of $\psi_k \{j \cdots -k\}$ intersects $[a,b]$. Further more, the partial sum of the Haar series (summing $0 \leq j \leq N$) clearly represents an approximation to $f$ taking into account details on the order of magnitude $2^{-N}$ or greater. Anything smaller will not be considered. These two properties, localization and scaling, are the attributes of wavelet expansions. In addition, the Haar functions are created out of a single function $\psi$ by dyadic dilations and integer translations. The last property has to be included in the definition of any family of wavelets.

The wavelet expansions can be thought of as a generalizations of the Haar series, in which the function $\psi$ is replaced by smoother functions. Before turning to the exact properties these functions need to have and how to construct them, it is useful to backtrack and see exactly how the Haar functions arise. It will turn out to be easier if we consider the whole line as the domain of our functions consider, in view of (i). Of course, $V_0$ is not all of $L^2(R)$. One can get a larger space by scaling. Let $V_1$ be the space of all piecewise constant functions with jumps at $\frac{k}{2}$, $k \in Z$. It is clear that $f \in V_0$ iff $f(2 \cdot ) \in V_1$; the functions $2^{1/2} \psi (2x-k)$ form an orthonormal basis for $V_1$. The scaling identity (2.1) gives $V_0 \subset V_1$.

Now, by iterating up and down the dyadic scale, an increasing sequence of subspaces $V_j$, $j \in Z$, is produced. Of course: $\ldots \subset V_{j-1} \subset V_j \subset V_{j+1} \subset \ldots$ . The sequence $V_j$, $j \in Z$, is called multiresolution analysis. There are two interesting properties, namely

$$\bigcap_{j \in Z} V_j = \{0\} \quad (2.2)$$

and

$$\bigcup_{j \in Z} V_j = L^2(R). \quad (2.3)$$

In view of (2.3) it is tempting to combine all the orthonormal bases \{$2^{j/2} \psi (2^j x -k) | k \in Z$\}; but, although $V_j \subset V_{j+1}$, the basis of $V_j$ is not contained in the basis of $V_{j+1}$! This naïve attempt for obtaining an orthonormal basis for $L^2(R)$ has failed! This is obvious by just looking at the graph of $\psi(x) = 1_{[0,1]}(x)$ and comparing it to the graph of $\psi_{10}(x) = 2 \cdot 1_{[0,0.5]}(x)$, an element of the basis for $V_1$, obtained by simple dilation:

$$\psi_{10}(x) = 2 \cdot \psi(2x) = 2[0,0.5](x).$$

The scalar product of these two is not zero, so they are not orthogonal.

### 2. THE SIMPLE HAAR WAVELETS

Let $\varphi(x) = 1_{[0,1]}(x)$. Surely this is one of the simplest functions one can imagine, but it is chosen because it has two important properties:

(i) the translates of $\varphi$ by integers, $\varphi(-k)$, form an orthonormal set of functions for $L^2(R)$;

(ii) $\varphi$ is self-similar, i.e. if you cut the graph in half you can use each half to recover the whole graph. Or, algebraically expressed:

$$\varphi(x) = \varphi(2x) + \varphi(2x-1) \quad (2.1)$$

which is called the scaling identity.

Note: Recall that the scalar product of two functions, $f$ and $g$, is defined by

$$\langle f, g \rangle = \int f(x)\overline{g(x)}dx$$

and these two functions are orthogonal if $\langle f, g \rangle = 0$. Specially, if all functions from the set $\{f_n\}$ satisfy

$$\langle f_k, f_n \rangle = \begin{cases} 0, & k \neq n \\ 1, & k = n \end{cases},$$

then the set of functions is called orthonormal. The energy, i.e. the norm of a function is defined by

$$\|f\|_2 = \langle f, f \rangle^{1/2}.$$

Back to the function $\varphi(x) = 1_{[0,1]}(x)$; it is called the scaling function, which is quite proper considering the scaling property (2.1). The significance of that property is the following: Let $V_0$ denote the linear span of $\varphi(-k)$, $k \in Z$, consisting of all piecewise constant functions with jump discontinuities at the integers. This is a natural space to
Since $V_0 \subset V_1$ and $\varphi(-k)$ is an orthonormal basis for $V_0$, let’s add up some elements $\psi(-k)$ (which function would $\psi$ be?), so that the union is an orthonormal basis for $V_1$! So, the orthogonal complement of $V_0$ in $V_1$, let it be denoted by $W_0$, would have an orthonormal basis $\psi(-k)$, $k \in \mathbb{Z}$.

Now we can write $V_1 = V_0 \oplus W_0$, where $\oplus$ denotes an orthogonal sum of the spaces.

The answer is the Haar function $\varphi$. Note that for the Haar function

$$\varphi(x) = \varphi(2x) - \varphi(2x-1).$$ (2.4)

Now we can rescale the space $W_0$, so

$$V_{j+1} = V_j \oplus W_j$$ (2.5)

and $2^{j/2}\psi(2^j x - k)$ is an orthonormal basis for $W_j$.

Combining (2.2), (2.3) and (2.5) we get

$$L^2(\mathbb{R}) = \bigoplus_{j \in \mathbb{Z}} W_j.$$ (2.6)

Now the problems with the orthonormality of two distinct bases for $W_j$ and $W_k$ are over. So we can gather all the elements $2^{j/2}\psi(2^j x - k)$ into one grand orthonormal basis for $L^2(\mathbb{R})$. To make it more clear, just observe the graphs of the dilated wavelet contained in $V_1$ that cover the interval $[0,1]$ and how they combine with the Haar function $\psi = \psi_{00}$. Its linear combinations give dilations and translations of the scaling function $\varphi(x) = 1_{[0,1]}(x)$.

3. MULTiresolution Analysis

Definition. A multiresolution analysis (MRA)

$\ldots \subset V_{-1} \subset V_0 \subset V_1 \subset \ldots$ associated with a scaling function $\varphi$, is an increasing sequence of subspaces of $L^2(\mathbb{R})$ satisfying the following conditions
order \( r \) and for each \( k = 0, 1, 2, ..., r \) there exists a constant \( c_{p,k} \) such that
\[
\left| \varphi^{(k)}(x) \right| \leq \frac{c_{p,k}}{(1 + |x|)^p}
\] (3.2)
for any \( x \in \mathbb{R}, \ p \in \mathbb{N} \).

Just like in (2.4), any wavelet arising from a MRA is derived from the scaling function:
\[
\varphi(x) = 2 \sum_{k \in \mathbb{Z}} (-1)^k \overline{a_k} \varphi(2x - (k - 1)).
\] (3.3)

Furthermore, one can provide an orthonormal wavelet with compact support with any order of continuous differentiability that arises from a MRA, a 1980’s result, given by Ingrid Daubechies.

4. ANALYTIC WAVELETS

Let the scaling function \( \varphi \) be a rapidly decreasing function; then \( \varphi \) has a Cauchy analytic representation defined by
\[
\varphi^\pm(z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\varphi(x) dx}{x - z}, \ \text{Im} \ z \neq 0. \quad (4.1)
\]

It is an analytic function (i.e. has continuous derivatives of any order) on the upper/lower complex half plane \( \mathbb{C}^\pm \), corresponding to the index +/− . We call the function \( \varphi^\pm \) an analytic scaling function on \( \mathbb{C}^\pm \); it preserves most of the properties of \( \varphi \). Being rapidly decreasing, the function \( \varphi \) is square integrable. Its analytic continuation is a Hardy function.

Note: A function \( f(x+iy) \) is a Hardy function, if the integral
\[
\int_{\mathbb{R}} |f(x+iy)|^2 dx
\]
exists for any \( y \) and
\[
\sup_{y>0} \left( \int_{\mathbb{R}} |f(x+iy)|^2 dx \right)^{1/2} < \infty. \quad (4.2)
\]
The last inequality defines the norm, i.e., the energy of \( f \). Denote the space of these functions by \( H^2(\mathbb{C}^+) \).

The scalar product of two Hardy functions \( F,G \) is defined by
\[
\langle F,G \rangle := \lim_{y \to 0 - \infty} \int_{\mathbb{R}} F(x+iy) \overline{G(x+iy)} \, dx
\]

The scaling function \( \varphi \) forms multiresolution analysis \( \{V_n\}_{n \in \mathbb{Z}} \) on \( L^2(\mathbb{R}) \), defined in the previous section. It is interesting to form multiresolution analysis on the complex domain. To keep it simple, the discussion is restricted to the upper half plane. The results on the lower half plane are obtained by analogy.

Let \( V_0^+ \) denote the set of analytic representations of the functions in \( V_0 \). For the analytic representation \( f^+(z) \) of \( f(t) \) (where \( z = x+iy \)), applying the Cauchy inequality, one can derive
\[
|f^+(z)| \leq \frac{1}{2y} \|f\|_{L^2}. \quad (*)
\]
So, if \( \|f\| = 0 \), then \( |f^+(z)| = 0 \), for any \( \text{Im} \ z = y > 0 \).

Let \( f \in V_0 \) and observe the scaling representation
\[
f(x) = \sum_{n \in \mathbb{Z}} a_n \varphi(x - n);
\]
also, let \( f^+ \) be the analytic representation of \( f \) on \( \mathbb{C}^+ \). Then
\[
|f^+(z) - \sum n \varphi^+(z - n)| \leq \frac{1}{2y} \left\| f - \sum n \varphi(-n) \right\|^2. \quad (4.3)
\]
The last inequality follows from (*). So, if
\[ f(x) = \sum_{n \in \mathbb{Z}} a_n \varphi(x-n), \]
then
\[ \sum_{n \in \mathbb{Z}} a_n \varphi^+(z-n) = f^+(z). \]

**Property 1:** Therefore, the operator \( A^+ \) maps \( V_0 \) onto \( V_0^+ \). \( V_0^+ \) is a linear span of \( \{ \varphi^+(-n) \} \), i.e. \( \{ \varphi^+(-n) \} \) is a generator set for \( V_0^+ \). Recall that if the Fourier transform
\[ \widehat{\varphi}(\xi) = \int_{\mathbb{R}} \varphi(x) e^{-i\xi x} dx \]
transfers the function \( \varphi \) on the frequency domain and exists for any square integrable function. For any rapidly decreasing \( \varphi \) the analytic representation is
\[ \varphi^+(z) = \frac{1}{2\pi} \int_{0}^{\infty} \varphi(w) e^{-iwz} dw \]
(see [2], [3]).

The set \( \{ \varphi^+(-n) \} \) is not always orthogonal. For example, for \( n \neq 0 \), the value of the scalar product is
\[ \int_{-\infty}^{\infty} \varphi^+(x+i0-n) \overline{\varphi^+(x+i0)} dx = \frac{1}{2\pi} \int_{0}^{\infty} |\varphi(w)|^2 e^{-iwn} dw, \]
which could have a nonzero value. Still, \( \{ \varphi^+(-n) \} \) is a basis on many occasions. The following property gives the conditions for a basis:

**Property 2:** \( \{ \varphi^+(-n) \} \) is a basis for \( V_0^+ \), if \( \widehat{\varphi}(w) \neq 0 \) almost everywhere on \( \mathbb{R} \). By dilation and translation, one can obtain the elements of the basis of all \( V_j^+ \):
\[ \varphi^+(2^{j}z-k) = \frac{1}{2^{j/2}} \int_{-\infty}^{\infty} \varphi^+(2^{j}t-k) \overline{\varphi^+(t)} dt, \quad k \in \mathbb{Z}. \]

As conclusion follows:

5. **NOTE ON SOME EXTRA CONDITIONS FOR ORTHOGONALITY**

**Example:** Let us look at the Shannon wavelets, defined by \( \psi(\xi) = e^{i\xi/2}1_{I}(\xi) \), where \( I = [-2\pi,-\pi] \cup (\pi,2\pi] \).

The Fourier transform of the scaling function is \( \varphi(\xi) = 1_{[-\pi,\pi]}(\xi) \). Looking from the space domain, the Shannon wavelet and its scaling function
are \( \psi(x) = -\frac{\sin(2\pi x) + \cos(\pi x)}{\pi(2x+1)} \), \( \varphi(x) = \frac{\sin(\pi x)}{\pi x} \) (see figures 4 and 5).

\[
\varphi_a * (z) := \frac{1}{2\pi} \int_{-a}^{a} \varphi(w)e^{inz} \, dw = \frac{1}{2\pi} \int_{-a}^{a} e^{inz} \, dw = \frac{1}{2\pi} \left( e^{iz\pi} - e^{-iz\pi} \right)
\]

Consequence 2: If \( \varphi(w) \) is bounded on its compact support in \([-a, a]\), where \( a = k\pi \) and \( \varphi(\pm a) \neq 0 \), then

\( \varphi_a * (z) = \frac{1}{2\pi} \int_{-a}^{a} \varphi(w)e^{inz} \, dw \quad (5.1) \)

provides an orthogonal basis for \( V_0^+ \). By analogy, \( \{\varphi_j * (z-n)\}_{j \in \mathbb{Z}} \) is an orthogonal basis for \( V_j^+ \).

Proof: The orthogonality of \( \{\varphi_a * (z-n)\} \) is ensured, because

\[
\langle \varphi_a * (-n), \varphi_a * (\cdot) \rangle = \int_{-\infty}^{+\infty} \varphi_a * (x+i0-n)\varphi_a * (x+i0) \, dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} |\varphi(w)|^2 e^{-iwn} \, dw = \frac{1}{2\pi} \int_{-\pi}^{\pi} |\varphi(w)|^2 e^{-iwn} \, dw \leq M^2 \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-iwn} \, dw = M^2 \frac{1}{2\pi} \left( e^{ina} - e^{-ina} \right) = 0 \quad n \neq 0.
\]

It is sufficient to prove that the zero function has unique representation. Let \( \varphi(w) \) have compact support as in the conditions of the consequence and let \( \sum_{n \in \mathbb{Z}} a_n \varphi_a * (z-n) = 0 \). Then

\[
0 = \sum_{n \in \mathbb{Z}} a_n \int_{-a}^{a} \varphi(w)e^{-izn} \, dw = \frac{1}{2\pi} \int_{-a}^{a} \sum_{n \in \mathbb{Z}} a_n \varphi(w)e^{izn} \, dw.
\]
Then \( \sum_n a_n \cdot e^{iwn} \cdot \varphi(w) \cdot \chi_{[-a,\infty)}(w) = 0 \) almost everywhere. By definition, \( \varphi(w) \cdot \chi_{[-a,\infty)}(w) = 0 \) out of the support of \( \varphi(w) \). So, \( \sum_n a_n e^{iwn} = 0 \) almost everywhere. Therefore, \( a_n = 0, \quad n \in \mathbb{Z} \).

**CONCLUSION**

This work deals with expanding wavelets on the complex plane using their analytic Cauchy representations. Such an expansion of wavelets on the complex plane exists. The compact support of the Fourier transformed scaling function will provide orthogonality on the complex plane – on the n-th level, though. It seems there is no reason against orthogonality on various levels. However, a conclusion on this matter has not been achieved.

**REFERENCES**
