Structural VARs and VECs

Lutkepohl Chapter 9

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Motivation

- Different sets of impulse responses can be computed from same underlying VAR or VECM (impulse responses not unique)

- use non-sample information to decide on the "proper" set of impulses for a given model
  $\longrightarrow$ impose "structural" restrictions (based on economic theory) to identify the relevant innovations and, thus, impulse responses

Outline

- **Structural VARs**

2. The A-Model
3. The B-Model
4. The AB-Model
5. Blanchard-Quah

- **Structural VECs**

1. Structural Vector Error Correction Model
2. Beveridge Nelson MA representation

- **Empirical Illustration**

1. Examples

(Structural) Vector Autoregressions

- K-dimensional stationary, stable VAR(p).

$$y_t = A_1 y_{t-1} + \ldots + A_p y_{t-p} + u_t \tag{1}$$

- We know that (1) has a Wold MA representation

$$y_t = u_t + \Phi_1 u_{t-1} + \Phi_2 u_{t-2} + \ldots \tag{2}$$

where

$$\Phi_s = \sum_{j=1}^{s} \Phi_{s-j} A_j \qquad s = 1, 2, \ldots \text{with} \quad \Phi_0 = I_K$$

- Choleski decomposition to orthogonalize innovations ($\Sigma_u = PP'$ with P lower-triangular matrix – Wold causal ordering). Unless there are "structural" reasons for the ordering of the variables (derived from economic theory) this approach is arbitrary.
- $\longrightarrow$ Use nonsample information to specify unique innovations

The A-Model

- Find a model with instantaneously uncorrelated residuals; i.e. find matrix A such that

$$\mathsf{A}y_t = A_1^* y_{t-1} + \ldots + A_p^* y_{t-p} + \epsilon_t \tag{3}$$

- is a *structural* model, where

$$A_j^* := \mathsf{A}\, A_j \qquad \text{and} \qquad \epsilon_t := \mathsf{A}u_t \sim (0, \Sigma_\epsilon = \mathsf{A}\Sigma_u \mathsf{A}')$$

  For a proper choice of A, ϵ_t will have a diagonal covariance matrix.

- MA representation based on ϵ_t

$$y_t = \Theta_0 \epsilon_t + \Theta_1 \epsilon_{t-1} + \Theta_2 \epsilon_{t-2} + \ldots,$$

  where $\Theta_j = \Phi_j \mathsf{A}^{-1}$ and the Θ are impulse responses to ϵ_t shocks.

A-Model – Restrictions

- From $\Sigma_\epsilon = \mathsf{A}\Sigma_u\mathsf{A}'$ and the assumption of a diagonal Σ_ϵ we get $K(K-1)/2$ independent equations (i.e., all $K(K-1)/2$) off-diagonal elements of $\mathsf{A}\Sigma_u\mathsf{A}'$ are zero)

- To solve uniquely for K^2 elements of A, we need another set of $K(K+1)/2$ restrictions
- Normalize diagonal elements of A to unity $\longrightarrow$ additional $K(K-1)/2$ restrictions **from nonsample information**

Restrictions, cont.

- If, for example, Wold causal ordering is possible, then

$$\mathsf{A} = \begin{pmatrix} 1 & 0 & \dots & 0 \\ a_{21} & 1 & & 0 \\ \vdots & & & \vdots \\ a_{K1} & a_{K2} & \dots & 1 \end{pmatrix}$$

- With A having a unit main diagonal, $K(K-1)/2$ restrictions for the off-diagonal elements of A ensure just-identified shocks ϵ_t and, hence, just-identified impulse responses

A-Model – Rewrite restrictions

- Restrictions must not be arbitrary; write them in the form

$$C_{\mathsf{A}}\ vec(\mathsf{A}) = c_{\mathsf{A}}$$

with selection matrix $C_{\mathsf{A}} = (\frac{1}{2}K(K+1) \times K^2)$ and a suitable fixed vector $c_{\mathsf{A}} = (\frac{1}{2}K(K+1) \times 1)$

- The restrictions have to be such that the system of equations

$$\mathsf{A}^{-1}\Sigma_\epsilon \mathsf{A}'^{-1} = \Sigma_u \qquad \text{and} \qquad C_{\mathsf{A}}\ vec(\mathsf{A}) = c_{\mathsf{A}}$$

has a unique solution, at least locally (remember: $\epsilon_t := \mathsf{A}u_t \sim (0, \Sigma\epsilon = \mathsf{A})$)

The B-Model

- Idea: think of the forecast errors (u_t) as linear functions of the structural errors (ϵ_t)
  $\longrightarrow$ Identify structural innovations ϵ_t directly from reduced form residuals u_t

$$u_t = \mathrm{B}\epsilon_t \qquad \text{and} \qquad \Sigma_u = \mathrm{B}\Sigma_\epsilon \mathrm{B}'$$

- Normalizing the variances of the structural innovations to one; i.e. assuming $\epsilon_t \sim (0, I_K)$, gives

$$\Sigma_u = \mathrm{B}\mathrm{B}'$$

- Choose B again by a Choleski decomposition
- Assumed symmetry of the covariance matrix specifies $K(K+1)/2$ restrictions; we need another $K(K-1)/2$ restrictions to identify all K^2 elements of B

B-Model – Restrictions

- Empirically most relevant: choose B to be lower triangular (in principle, other zero restrictions on B possible)
- **Structural** because now recursive structure is only chosen if it has theoretical justification
- If only zero restrictions

$$\mathsf{C}_B vec(\mathsf{B}) = 0$$

- B can be uniquely identified, at least locally

The AB Model

- Combine both types of restrictions $\longrightarrow$ the AB-model
- Idea: formulate relations (restrictions) for the innovations

$$\mathsf{A}u_t = \mathsf{B}\epsilon_t \quad \text{with} \quad \epsilon_t \sim (0, I_K)$$

- From $\epsilon_t \sim (0, I_K)$ we get $u_t = \mathsf{A}^{-1}\mathsf{B}\epsilon_t$ and, hence
- $\Sigma_u = \mathsf{A}^{-1}\mathsf{B}\mathsf{B}'\mathsf{A}^{-1'}$ with K^2 elements for each, A and B.
- Restrictions typically normalizations or zero restrictions; written in the form of linear equations:

$vec(\mathsf{A}) = R_\mathsf{A}\gamma_\mathsf{A} + r_\mathsf{A}$ and $vec(\mathsf{B}) = R_\mathsf{B}\gamma_\mathsf{B} + r_\mathsf{B}$
where R_A and R_B are suitable matrices of zeros and ones, γ_A and γ_B are vectors of free parameters, and r_A and r_B vectors of fixed parameters

Blanchard-Quah (1989)

- Alternative approach: consider accumulated ("long-run") effects of shocks to a system as in Blanchard & Quah (1989)
- Remember structural impulses of the form

$$y_t = \Theta_0 \epsilon_t + \Theta_1 \epsilon_{t-1} + \Theta_2 \epsilon_{t-2} + \dots$$

Blanchard & Quah (1989) derive a *total impact matrix*

$$\Xi_\infty = \sum_{i=0}^{\infty} \Theta_i = (I_K - A_1 - \dots - A_p)^{-1} \mathsf{A}^{-1} \mathsf{B} \tag{4}$$

- Identify structural innovations by placing zero restrictions on this Ξ matrix; i.e. assume that some innovations do not have any total long-run effects

Blanchard-Quah, cont.

- Example: bivariate system $y_t = (q_t, ur_t)'$
- structural innovations represent supply and demand shocks; assume that demand shocks have only transitory effects on q_t (accumulated long-run effect of such shocks on q_t is zero)
- Place supply shocks first, demand shocks second ($\epsilon_t = (\epsilon_t^s, \epsilon_t^d)'$), then the upper right-hand corner element of equation (4); i.e. the Ξ matrix, is restricted to zero (no restrictions placed on the instantaneous effects of the observable variables)
- This corresponds to AB-model with A$= I_K$ (that is, the B-model) with restriction

$$(0,0,1,0)vec[(I_K - A1 - \ldots - Ap)^{-1}\mathrm{B}] = 0$$

Cointegration and Vector Error Correction form

- Definition of a cointegrated process $y_t \sim CI(d, b)$ with all its K variables being I(d)

> but there **exist linear combinations** between the variables $z_t = \beta y_t$ which are I(d-b). Most often CI (1,1).

- Such a process can be **written in Error Correction form**:

$$\Delta y_t = \underbrace{\alpha \beta' y_{t-1}} + \underbrace{\Gamma_1 \Delta y_{t-1} + \ldots + \Gamma_{p-1} \Delta y_{t-p+1}} + u_t \qquad (5)$$

- So far, the VEC model does not explicitly include assumptions from theory.
- We can interpret (5) as the **reduced form** of a **structural VEC model**, which incorporates results from theory.

Structural Vector Error Correction model

- A **structural VEC** without deterministic terms and exogenous variables has the form:

$$\mathsf{A}\Delta y_t = \Pi^* y_{t-1} + \Gamma_1^* \Delta y_{t-1} + \ldots + \Gamma_{p-1}^* \Delta y_{t-p+1} + B\epsilon_t \quad (6)$$

- The $(K \times K)$ **matrix A** allows incorporating a structure reflecting a theoretical model.
- The structural equation in (6) has the **reduced form** representation:

$$\Delta y_t = \Pi y_{t-1} + \Gamma_1 \Delta y_{t-1} + \ldots + \Gamma_{p-1} \Delta y_{t-p+1} + u_t \quad (7)$$

  where $\Pi = \mathsf{A}^{-1}\Pi^*$, $\Gamma_j = \mathsf{A}^{-1}\Gamma_j^*$, and $A_j = \mathsf{A}^{-1}\mathsf{A}_j^*$ and the reduced form disturbances u_t are related to the underlying structural shocks ϵ_t by $u_t = \mathsf{A}^{-1}B\epsilon_t$.

- In order **to identify the structural form parameters**, we must impose restrictions on the parameter matrices.

Beveridge Nelson MA representation

- The process given in (7) has the Beveridge Nelson MA representation:

$$y_t = \underbrace{\Xi \sum_{i=1}^{t} u_i}_{I(1)} + \underbrace{\sum_{j=0}^{\infty} \Xi^* u_{t-j}}_{I(0)} + y_0^* \tag{8}$$

- **I(0) part** The Ξ^* are absolutely summable so that the infinite sum is well defined. (converges $\rightarrow 0$ for $j \rightarrow \infty$.)
- **I(1) part** The long run effects of shocks are captured by the common trend term $\Xi \sum_{i=1}^{t} u_i$.
- The **matrix** $\Xi = \beta\perp[\alpha'_\perp(I_K - \sum \Gamma_i)\beta_\perp]^{-1}\alpha'_\perp$ has rank **K-r**.
- There are $(K-r)$ common trends, at **most** r **can have transitory effects.**

B-model setup for sVEC

- Focus of interest on the residuals - the **B-model** setup is typically used.
- Connection of reduced form and structural form errors:

$$u_t = B\epsilon_t \qquad \epsilon_t(0, I_K) \tag{9}$$

- Substituting this equation in the Beveridge Nelson MA representation gives

$$\Xi B \sum_{i=1}^{t} \epsilon_i \quad \text{for the I(1) part.} \tag{10}$$

- Hence, the long-run effects of the structural innovations are given by ΞB.

B-model setup for sVEC

- Because $\Sigma_u = BB'$, $rk(\Xi B) = K - r$ there can be at most r zero columns in this matrix.
- This means that, r of the **structural innovations** can have **transitory effects** and $K - r$ of them must have **permanent effects.**
- The matrix ΞB has reduced rank $rk(\Xi B) = K - r$,
  $\rightarrow$ therefore **each column of zeros stands for $K - r$ independent restrictions.**
- $\rightarrow$ The r transitory shocks represent $r(K - r)$ independent restrictions.

Local just-identification

- For local just-identification of the structural innovations in the B-model, **a total of $K(K-1)/2$ restrictions are required**.
  - **We have** already $r(K-r)$ restrictions from the cointegration structure of the model.
  - **We need** $\frac{1}{2}K(K-1) - r(K-r)$ further restrictions for just-identification of the structural innovations.
  - In fact, $r(r-1)/2$ additional contemporaneous restrictions are needed to disentangle the transitory shocks
  - and $r(K-r)((K-r-1)$ restrictions to identify the permanent shocks. (King et al. (1991), Gonzalo & Ng (2001)).

Restrictions

- The restrictions take the form

$$C_{\Xi B} vec(\Xi B) = c_l \text{ or } C_l vec(\Xi B) = c_l \quad \text{and} \quad C_s vec(B) = c_s$$

- $C_l := C_{\Xi B}(I_K \otimes \Xi)$ is a matrix of **long-run** restrictions, that is, $C_{\Xi B}$ is a suitable selection matrix such that $C_{\Xi B} vec(\Xi B) = c_l$.
- c_s specifies **short-run** or instantaneous contraints by restriction elements of B directly.
- c_l and c_s are vectors of suitable dimensions. In applied work, typically zero vectors.

Examples for SVAR and SVEC in JMulTi

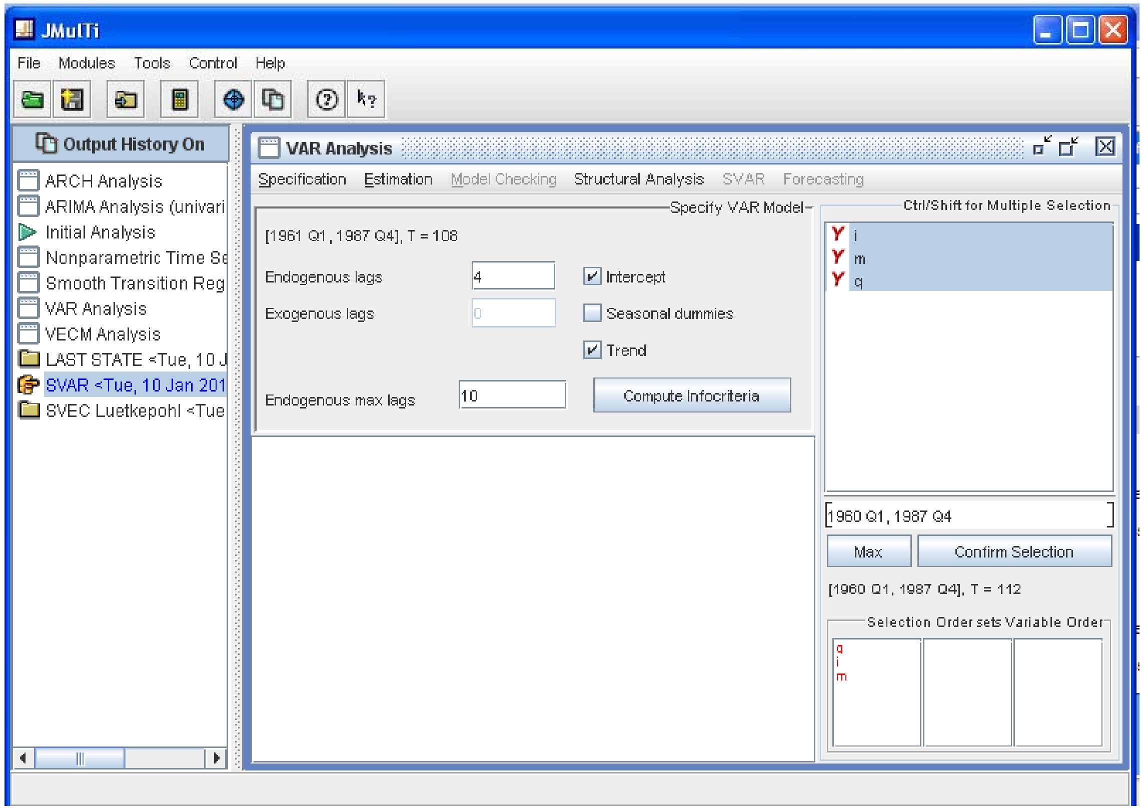
- JMulTi is an open-source interactive software for univariate and multivariate time series analysis
- The course textbook Lütkepohl (2006) as well as Lütkepohl & Krätzig (2004) refer to JMulTi
- Downloadable for free at www.jmulti.com
- Datasets from the Lütkepohl‘s textbooks can be downloaded here: www.jmulti.com/datasets.html

Example 1: SVAR

- Breitung, Brüggemann, Lütkepohl (2004), used in Lütkepohl (2006)
- Stylized IS-LM model
- Quarterly US data on
  - Real GDP
  - Three-month interbank interest rate
  - Real monetary base
- IS curve: $u_t q=-a_{12}u_t^i+b_{11}\varepsilon_t^{IS}$
- Inverse LM curve $u_t^i=-a_{21}u_t^q-a_{23}u_t^m+b_{22}\ \varepsilon_t^{LM}$
- Money supply rule: $u_t^m=b_{33}\ \varepsilon_t^m$

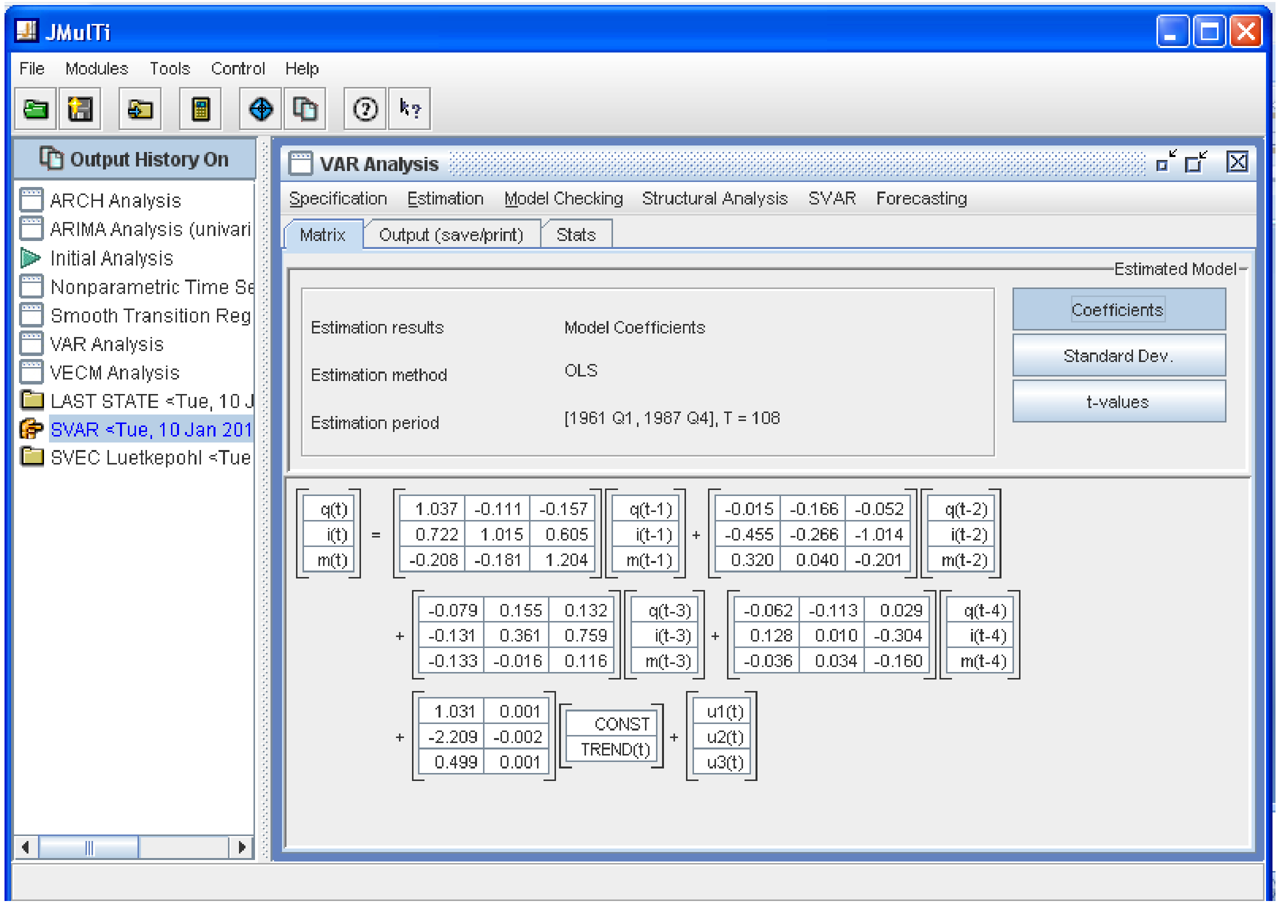
VAR settings

Settings for reduced-form VAR with 4 lags, constant and trend
q: Output
i: interest rate
m: real monetary base



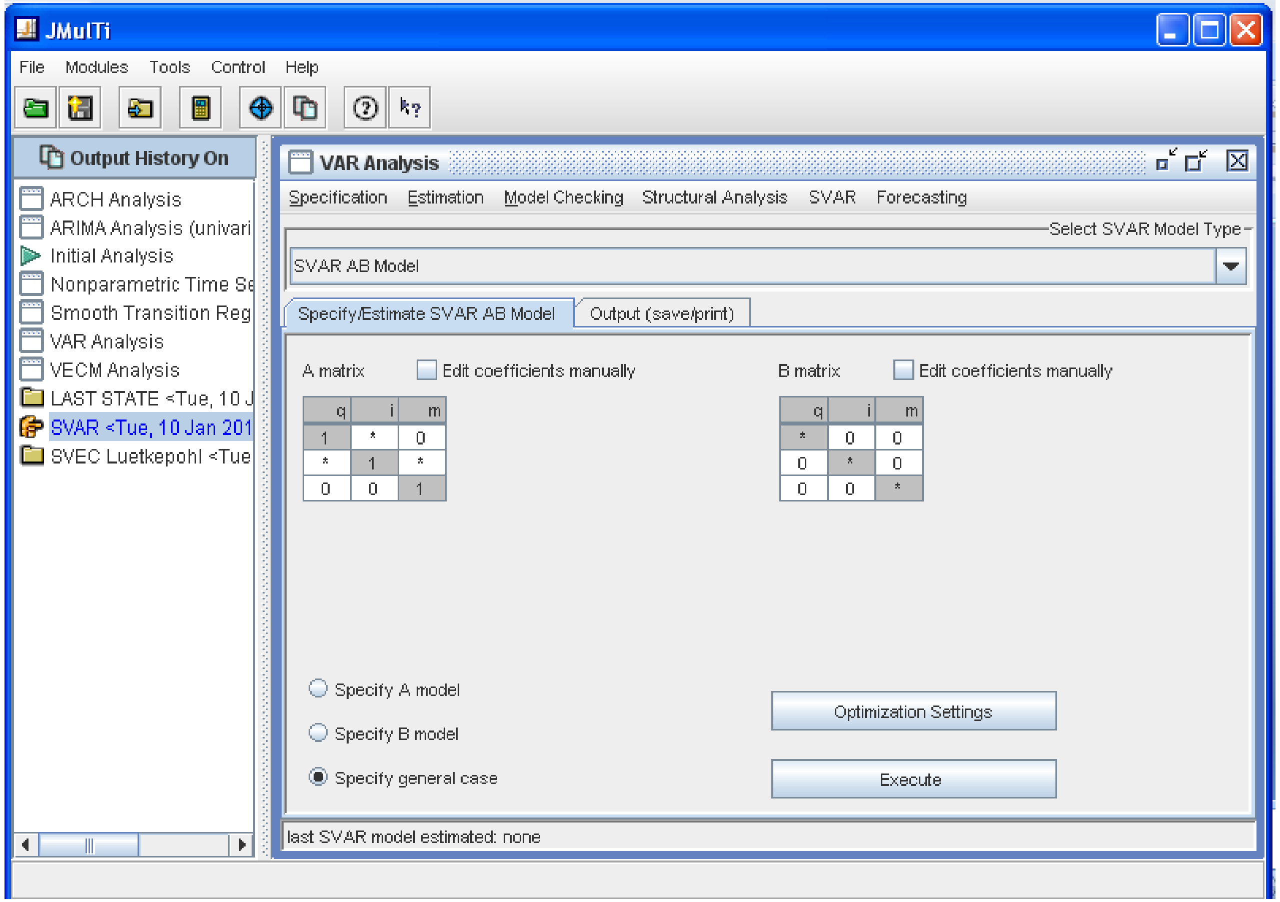
VAR estimation

Estimated coefficients of the reduced-form VAR



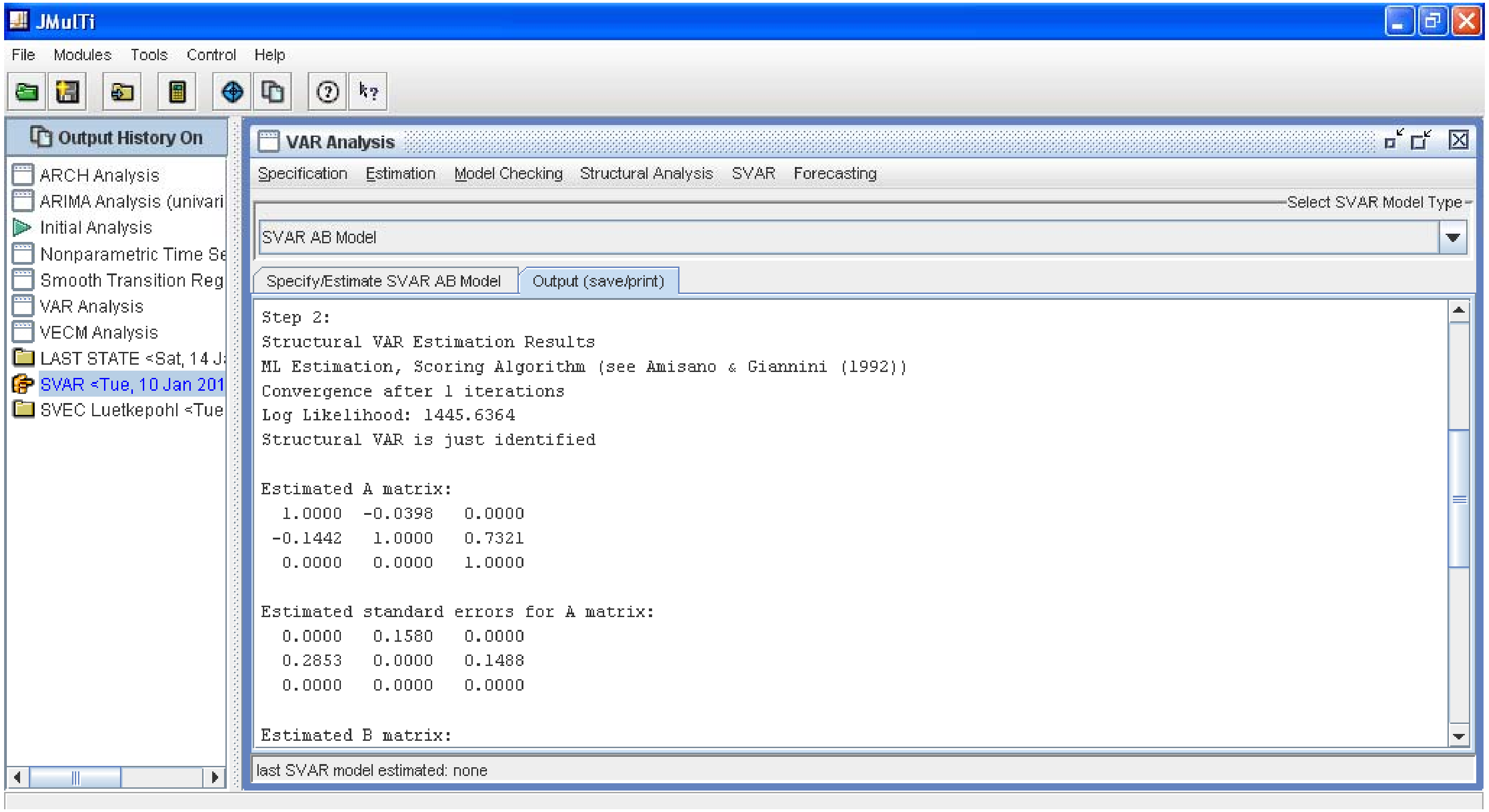
SVAR restrictions

Set structural restrictions for the A and the B matrix



SVAR results

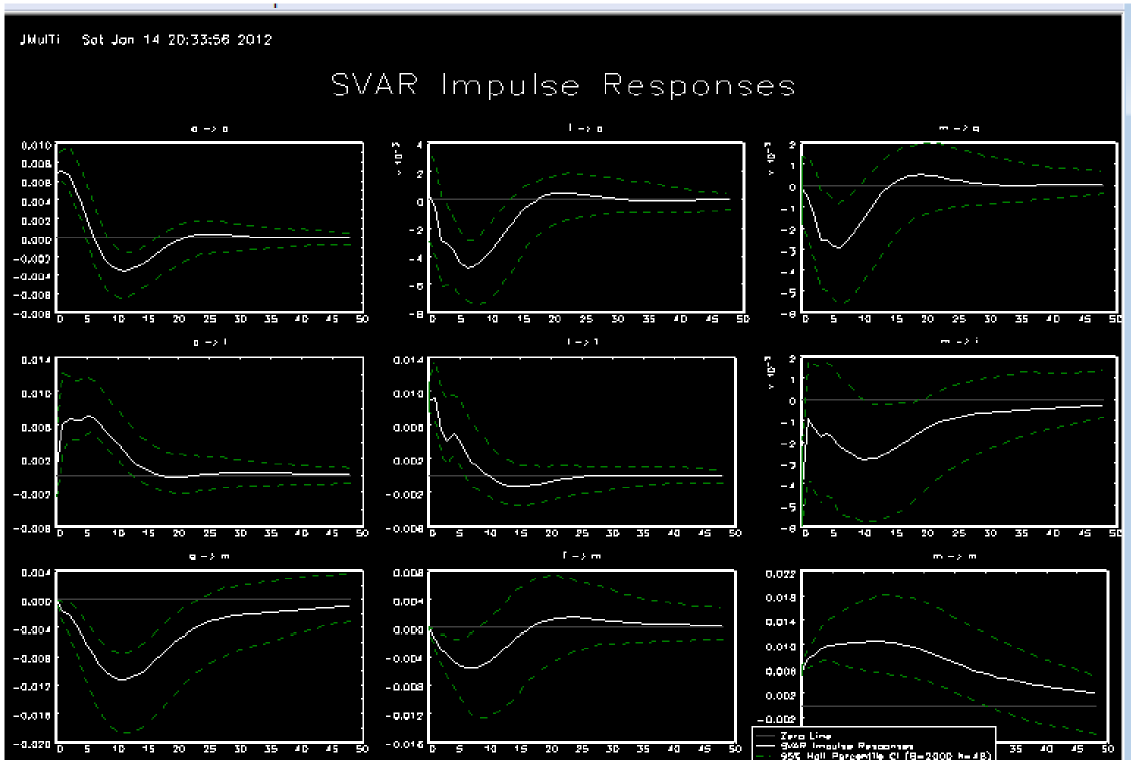
Estimation results for A and B



SVAR IRA

- •Impulse Response Analysis with 95% Hall bootstrap confidence intervals (2000 bootstrap replications)
- •Responses of q (upper row), i (middle row), m (bottom row) to three structural shocks
- •IS or spending shock in left column
- •LM shock in middle column
- •Money supply shock in right column

- •IS shock increases output immediately, increases interest rate (with maximum after 8 quarters), decreases real money holdings
- •LM shock increases interest rate and decreases output
- •Money supply shock decreases output (contrary to economic theory), decreases interest rate and increases real money holding

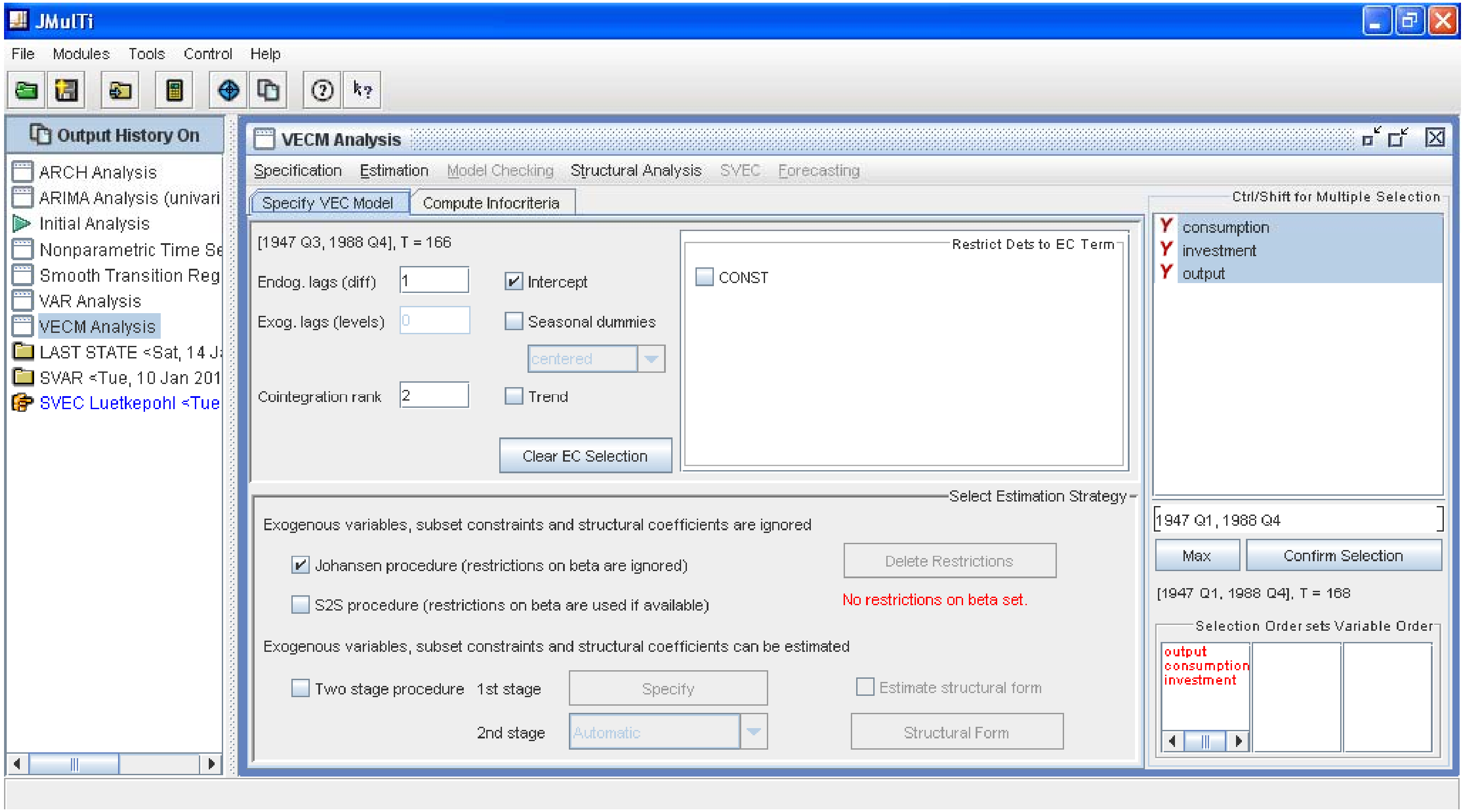


Example 2: SVEC

- Lütkepohl (2006)
- US quarterly data
- Output, consumption, investment
- All variables I(1), cointegrating rank=2,
- two transitory shocks, one permanent shock

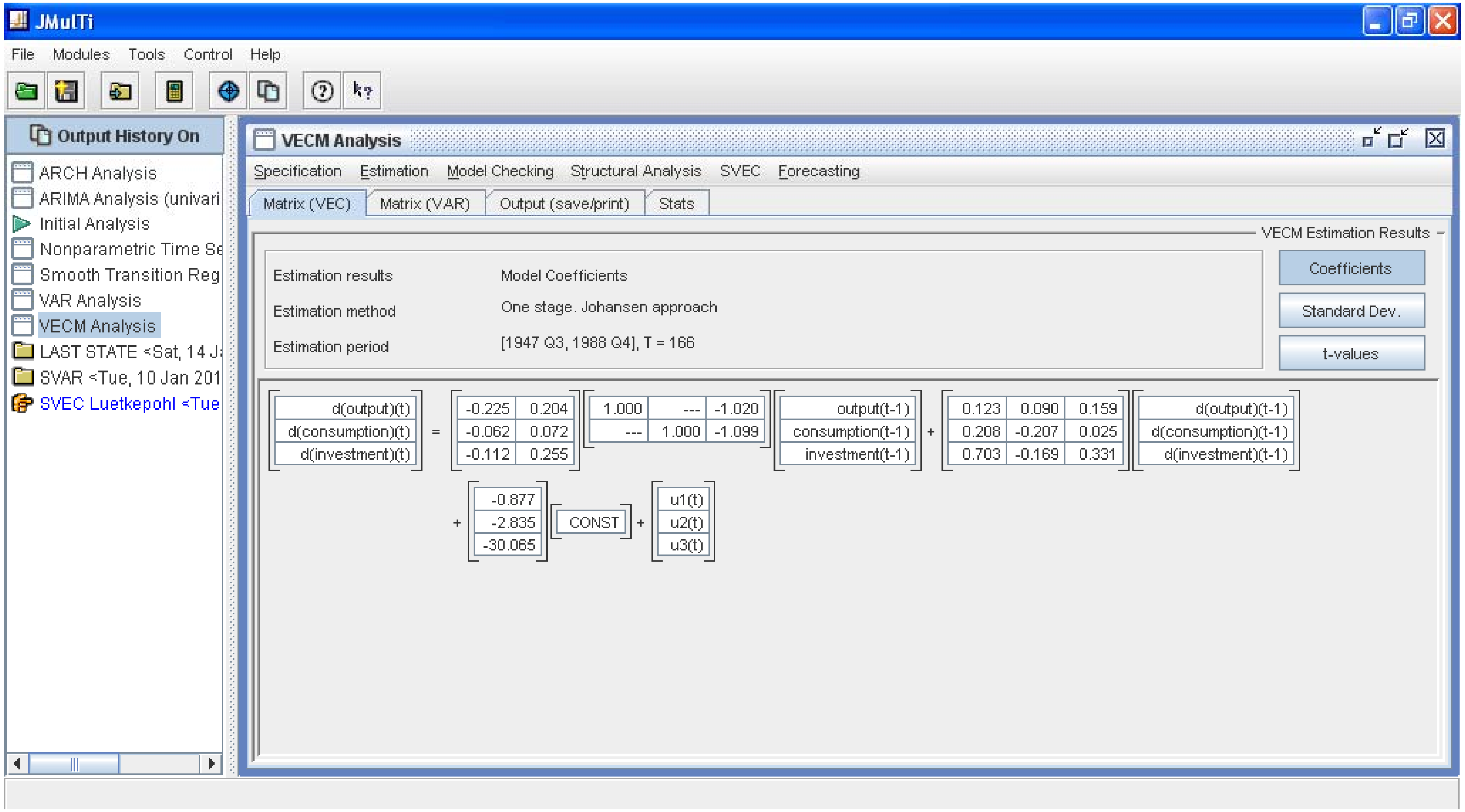
VEC settings

Settings for reduced-form VEC with 1 lags, constant and 2 cointegration relations



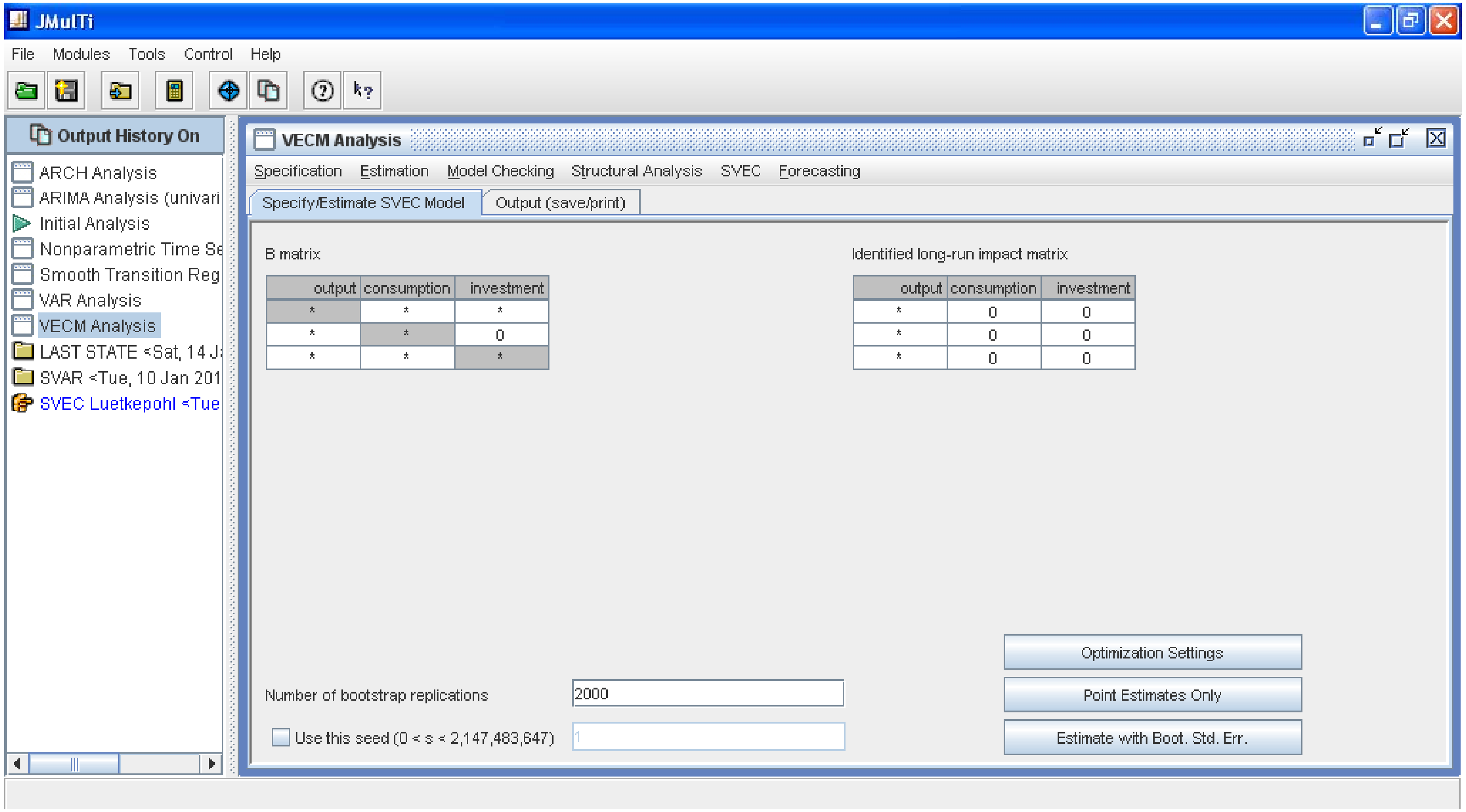
VEC results

Estimated coefficients of reduced-form VEC



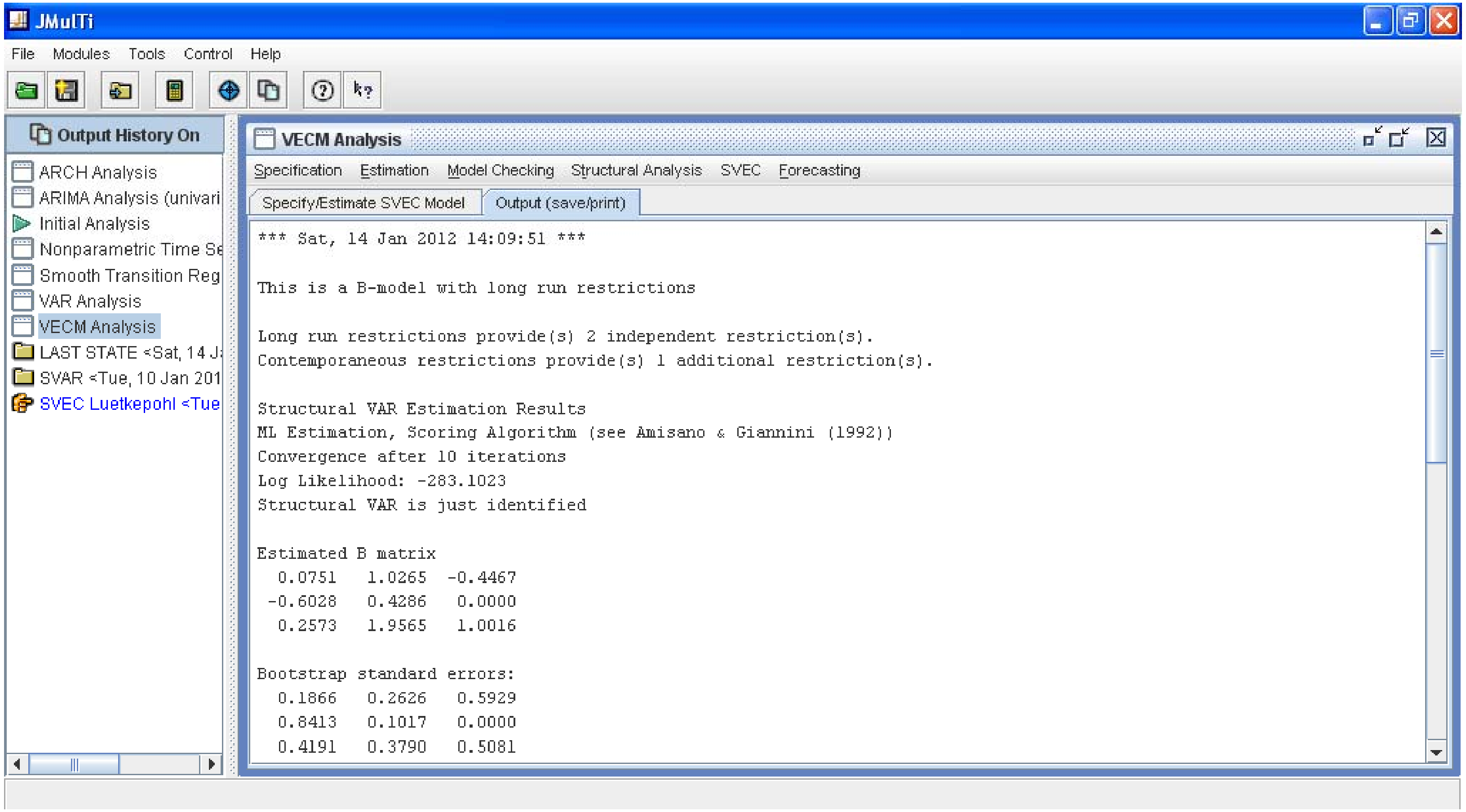
SVEC restrictions

- Restrictions on B (short-run) and ΞB (long-run)
- One permanent shock that can have effects on all three variables
- Two transitory shocks, with the first one allowed to have effects on all three variables, and the second one not to be allowed to affect the second variable (0 restriction in B). By that, the two transitory shocks are disentagled



SVEC results

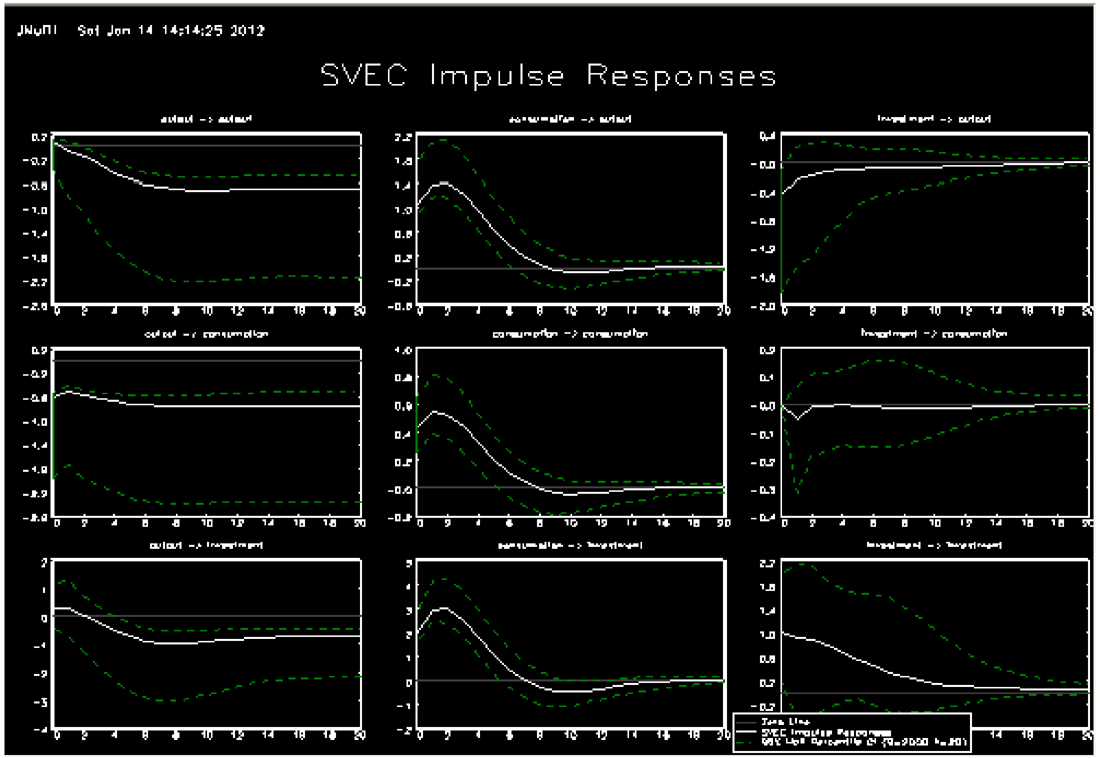
Estimated B and ΞB matrices



SVEC IRA

- IRA with 95% Hall bootstrap confidence intervals (2000 bootstrap replications)
- Responses of output (upper row), consumption (middle row), investment (bottom row) to three structural shocks
- Permanend shock in left column
- Transitory shocks in middle and right column

- Effects of the long-run shock are all negative in the long run. (To see the effect of an impulse which leads to positive long-run effects, just reverse the sign of the impulse responses)
- The transitory shocks indeed fade out quickly.
- The single 0 restriction in B can be seen in the right column, second row.



SVEC FEVD

•Forecast error variance decomposition
•Permanent shock (light blue part) has increasing importance with higher forecast horizon
•The importance of the transitory shocks (green and dark blue) are decresasing with higher forecast horizon