

# Structural VARs and VECs

## Lutkepohl Chapter 9

Florian Kaulich  
Nora Prean  
Catherine Prettner

Seminar on VARs, January 2012

# Motivation

- Different sets of impulse responses can be computed from same underlying VAR or VECM (impulse responses not unique)
- use non-sample information to decide on the "proper" set of impulses for a given model
  - impose "structural" restrictions (based on economic theory) to identify the relevant innovations and, thus, impulse responses

# Outline

- **Structural VARs**
- 2. The A-Model
- 3. The B-Model
- 4. The AB-Model
- 5. Blanchard-Quah
  - **Structural VECs**
  - 1. Structural Vector Error Correction Model
  - 2. Beveridge Nelson MA representation
  - **Empirical Illustration**
  - 1. Examples

## (Structural) Vector Autoregressions

- K-dimensional stationary, stable VAR(p).

$$y_t = A_1 y_{t-1} + \dots + A_p y_{t-p} + u_t \quad (1)$$

- We know that (1) has a Wold MA representation

$$y_t = u_t + \Phi_1 u_{t-1} + \Phi_2 u_{t-2} + \dots \quad (2)$$

where

$$\Phi_s = \sum_{j=1}^s \Phi_{s-j} A_j \quad s = 1, 2, \dots \text{ with } \Phi_0 = I_K$$

- Choleski decomposition to orthogonalize innovations ( $\Sigma_u = PP'$  with  $P$  lower-triangular matrix – Wold causal ordering). Unless there are "structural" reasons for the ordering of the variables (derived from economic theory) this approach is arbitrary.
- → Use nonsample information to specify unique innovations

## The A-Model

- Find a model with instantaneously uncorrelated residuals; i.e. find matrix  $A$  such that

$$A y_t = A_1^* y_{t-1} + \dots + A_p^* y_{t-p} + \epsilon_t \quad (3)$$

- is a *structural* model, where

$$A_j^* := A A_j \quad \text{and} \quad \epsilon_t := A u_t \sim (0, \Sigma_\epsilon = A \Sigma_u A')$$

For a proper choice of  $A$ ,  $\epsilon_t$  will have a diagonal covariance matrix.

- MA representation based on  $\epsilon_t$

$$y_t = \Theta_0 \epsilon_t + \Theta_1 \epsilon_{t-1} + \Theta_2 \epsilon_{t-2} + \dots,$$

where  $\Theta_j = \Phi_j A^{-1}$  and the  $\Theta$  are impulse responses to  $\epsilon_t$  shocks.

## A-Model – Restrictions

- From  $\Sigma_\epsilon = A\Sigma_u A'$  and the assumption of a diagonal  $\Sigma_\epsilon$  we get  $K(K - 1)/2$  independent equations (i.e., all  $K(K - 1)/2$  off-diagonal elements of  $A\Sigma_u A'$  are zero)
- To solve uniquely for  $K^2$  elements of  $A$ , we need another set of  $K(K + 1)/2$  restrictions
- Normalize diagonal elements of  $A$  to unity  $\rightarrow$  additional  $K(K - 1)/2$  restrictions **from nonsample information**

## Restrictions, cont.

- If, for example, Wold causal ordering is possible, then

$$A = \begin{pmatrix} 1 & 0 & \dots & 0 \\ a_{21} & 1 & & 0 \\ \vdots & & & \vdots \\ a_{K1} & a_{K2} & \dots & 1 \end{pmatrix}$$

- With A having a unit main diagonal,  $K(K - 1)/2$  restrictions for the off-diagonal elements of A ensure just-identified shocks  $\epsilon_t$  and, hence, just-identified impulse responses

## A-Model – Rewrite restrictions

- Restrictions must not be arbitrary; write them in the form

$$C_A \text{vec}(\mathbf{A}) = c_A$$

with selection matrix  $C_A = (\frac{1}{2}K(K+1) \times K^2)$  and a suitable fixed vector  $c_A = (\frac{1}{2}K(K+1) \times 1)$

- The restrictions have to be such that the system of equations

$$\mathbf{A}^{-1} \Sigma_\epsilon \mathbf{A}'^{-1} = \Sigma_u \quad \text{and} \quad C_A \text{vec}(\mathbf{A}) = c_A$$

has a unique solution, at least locally  
(remember:  $\epsilon_t := \mathbf{A}u_t \sim (0, \Sigma_\epsilon = \mathbf{A})$ )



## The B-Model

- Idea: think of the forecast errors ( $u_t$ ) as linear functions of the structural errors ( $\epsilon_t$ )  
→ Identify structural innovations  $\epsilon_t$  directly from reduced form residuals  $u_t$

$$u_t = B\epsilon_t \quad \text{and} \quad \Sigma_u = B\Sigma_\epsilon B'$$

- Normalizing the variances of the structural innovations to one; i.e. assuming  $\epsilon_t \sim (0, I_K)$ , gives

$$\Sigma_u = BB'$$

- Choose B again by a Choleski decomposition
- Assumed symmetry of the covariance matrix specifies  $K(K + 1)/2$  restrictions; we need another  $K(K - 1)/2$  restrictions to identify all  $K^2$  elements of B

## B-Model – Restrictions

- Empirically most relevant: choose B to be lower triangular (in principle, other zero restrictions on B possible)
- **Structural** because now recursive structure is only chosen if it has theoretical justification
- If only zero restrictions

$$C_{Bvec}(B) = 0$$

- B can be uniquely identified, at least locally

## The AB Model

- Combine both types of restrictions  $\rightarrow$  the AB-model
- Idea: formulate relations (restrictions) for the innovations

$$Au_t = B\epsilon_t \quad \text{with} \quad \epsilon_t \sim (0, I_K)$$

- From  $\epsilon_t \sim (0, I_K)$  we get  $u_t = A^{-1}B\epsilon_t$  and, hence
- $\Sigma_u = A^{-1}BB'A^{-1'}$  with  $K^2$  elements for each, A and B.
- Restrictions typically normalizations or zero restrictions; written in the form of linear equations:

$$\text{vec}(\mathbf{A}) = R_A\gamma_A + r_A \quad \text{and} \quad \text{vec}(\mathbf{B}) = R_B\gamma_B + r_B$$

where  $R_A$  and  $R_B$  are suitable matrices of zeros and ones,  $\gamma_A$  and  $\gamma_B$  are vectors of free parameters, and  $r_A$  and  $r_B$  vectors of fixed parameters

## Blanchard-Quah (1989)

- Alternative approach: consider accumulated ("long-run") effects of shocks to a system as in Blanchard & Quah (1989)
- Remember structural impulses of the form

$$y_t = \Theta_0 \epsilon_t + \Theta_1 \epsilon_{t-1} + \Theta_2 \epsilon_{t-2} + \dots$$

Blanchard & Quah (1989) derive a *total impact matrix*

$$\Xi_\infty = \sum_{i=0}^{\infty} \Theta_i = (I_K - A_1 - \dots - A_p)^{-1} A^{-1} B \quad (4)$$

- Identify structural innovations by placing zero restrictions on this  $\Xi$  matrix; i.e. assume that some innovations do not have any total long-run effects

## Blanchard-Quah, cont.

- Example: bivariate system  $y_t = (q_t, ur_t)'$
- structural innovations represent supply and demand shocks; assume that demand shocks have only transitory effects on  $q_t$  (accumulated long-run effect of such shocks on  $q_t$  is zero)
- Place supply shocks first, demand shocks second ( $\epsilon_t = (\epsilon_t^s, \epsilon_t^d)'$ ), then the upper right-hand corner element of equation (4); i.e. the  $\Xi$  matrix, is restricted to zero (no restrictions placed on the instantaneous effects of the observable variables)
- This corresponds to AB-model with  $A = I_K$  (that is, the B-model) with restriction

$$(0, 0, 1, 0) \text{vec}[(I_K - A1 - \dots - Ap)^{-1}B] = 0$$

# Cointegration and Vector Error Correction form

- Definition of a cointegrated process  $y_t \sim CI(d, b)$  with all its  $K$  variables being  $I(d)$
- > but there **exist linear combinations** between the variables  $z_t = \beta y_t$  which are  $I(d-b)$ . Most often  $CI(1,1)$ .
- Such a process can be **written in Error Correction form**:

$$\Delta y_t = \underbrace{\alpha \beta' y_{t-1}} + \underbrace{\Gamma_1 \Delta y_{t-1} + \dots + \Gamma_{p-1} \Delta y_{t-p+1}} + u_t \quad (5)$$

- So far, the VEC model does not explicitly include assumptions from theory.
- We can interpret (5) as the **reduced form** of a **structural VEC model**, which incorporates results from theory.

## Structural Vector Error Correction model

- A **structural VEC** without deterministic terms and exogenous variables has the form:

$$A\Delta y_t = \Pi^* y_{t-1} + \Gamma_1^* \Delta y_{t-1} + \dots + \Gamma_{p-1}^* \Delta y_{t-p+1} + B\epsilon_t \quad (6)$$

- The  $(K \times K)$  **matrix A** allows incorporating a structure reflecting a theoretical model.
- The structural equation in (6) has the **reduced form** representation:

$$\Delta y_t = \Pi y_{t-1} + \Gamma_1 \Delta y_{t-1} + \dots + \Gamma_{p-1} \Delta y_{t-p+1} + u_t \quad (7)$$

where  $\Pi = A^{-1}\Pi^*$ ,  $\Gamma_j = A^{-1}\Gamma_j^*$ , and  $A_j = A^{-1}A_j^*$  and the reduced form disturbances  $u_t$  are related to the underlying structural shocks  $\epsilon_t$  by  $u_t = A^{-1}B\epsilon_t$ .

- In order to **identify the structural form parameters**, we must impose restrictions on the parameter matrices.

## Beveridge Nelson MA representation

- The process given in (7) has the Beveridge Nelson MA representation:

$$y_t = \underbrace{\Xi \sum_{i=1}^t u_i}_{I(1)} + \underbrace{\sum_{j=0}^{\infty} \Xi^* u_{t-j}}_{I(0)} + y_0^* \quad (8)$$

- **I(0) part** The  $\Xi^*$  are absolutely summable so that the infinite sum is well defined. (converges  $\rightarrow 0$  for  $j \rightarrow \infty$ .)
- **I(1) part** The long run effects of shocks are captured by the common trend term  $\Xi \sum_{i=1}^t u_i$ .
- The **matrix**  $\Xi = \beta_{\perp} [\alpha'_{\perp} (I_K - \sum \Gamma_i) \beta_{\perp}]^{-1} \alpha'_{\perp}$  has rank **K-r**.
- There are  $(K - r)$  common trends, at **most r can have transitory effects**.



## B-model setup for sVEC

- Focus of interest on the residuals - the **B-model** setup is typically used.
- Connection of reduced form and structural form errors:

$$u_t = B\epsilon_t \quad \epsilon_t(0, I_K) \quad (9)$$

- Substituting this equation in the Beveridge Nelson MA representation gives

$$\Xi B \sum_{i=1}^t \epsilon_i \quad \text{for the } I(1) \text{ part.} \quad (10)$$

- Hence, the long-run effects of the structural innovations are given by  $\Xi B$ .

## B-model setup for sVEC

- Because  $\Sigma_u = BB'$ ,  $rk(\Xi B) = K - r$  there can be at most  $r$  zero columns in this matrix.
- This means that,  $r$  of the **structural innovations** can have **transitory effects** and  $K - r$  of them must have **permanent effects**.
- The matrix  $\Xi B$  has reduced rank  $rk(\Xi B) = K - r$ ,  
→ therefore **each column of zeros stands for  $K - r$  independent restrictions**.
- → The  $r$  transitory shocks represent  $r(K - r)$  independent restrictions.

## Local just-identification

- For local just-identification of the structural innovations in the B-model, **a total of  $K(K - 1)/2$  restrictions are required.**
  - **We have** already  $r(K - r)$  restrictions from the cointegration structure of the model.
  - **We need**  $\frac{1}{2}K(K - 1) - r(K - r)$  further restrictions for just-identification of the structural innovations.
  - In fact,  $r(r - 1)/2$  additional contemporaneous restrictions are needed to disentangle the transitory shocks
  - and  $r(K - r)((K - r - 1))$  restrictions to identify the permanent shocks. (King et al. (1991), Gonzalo & Ng (2001)).

# Restrictions

- The restrictions take the form

$$C_{\Xi B} \text{vec}(\Xi B) = c_l \text{ or } C_l \text{vec}(\Xi B) = c_l \quad \text{and} \quad C_s \text{vec}(B) = c_s$$

- $C_l := C_{\Xi B}(I_K \otimes \Xi)$  is a matrix of **long-run** restrictions, that is,  $C_{\Xi B}$  is a suitable selection matrix such that  $C_{\Xi B} \text{vec}(\Xi B) = c_l$ .
- $c_s$  specifies **short-run** or instantaneous constraints by restriction elements of B directly.
- $c_l$  and  $c_s$  are vectors of suitable dimensions. In applied work, typically zero vectors.

# Examples for SVAR and SVEC in JMulTi

- JMulTi is an open-source interactive software for univariate and multivariate time series analysis
- The course textbook Lütkepohl (2006) as well as Lütkepohl & Krätzig (2004) refer to JMulTi
- Downloadable for free at [www.jmulti.com](http://www.jmulti.com)
- Datasets from the Lütkepohl's textbooks can be downloaded here:  
[www.jmulti.com/datasets.html](http://www.jmulti.com/datasets.html)

# Example 1: SVAR

- Breitung, Brüggemann, Lütkepohl (2004), used in Lütkepohl (2006)
- Stylized IS-LM model
- Quarterly US data on
  - Real GDP
  - Three-month interbank interest rate
  - Real monetary base
- IS curve:  $u_t^q = -a_{12} u_t^i + b_{11} \varepsilon_t^{IS}$
- Inverse LM curve  $u_t^i = -a_{21} u_t^q - a_{23} u_t^m + b_{22} \varepsilon_t^{LM}$
- Money supply rule:  $u_t^m = b_{33} \varepsilon_t^m$

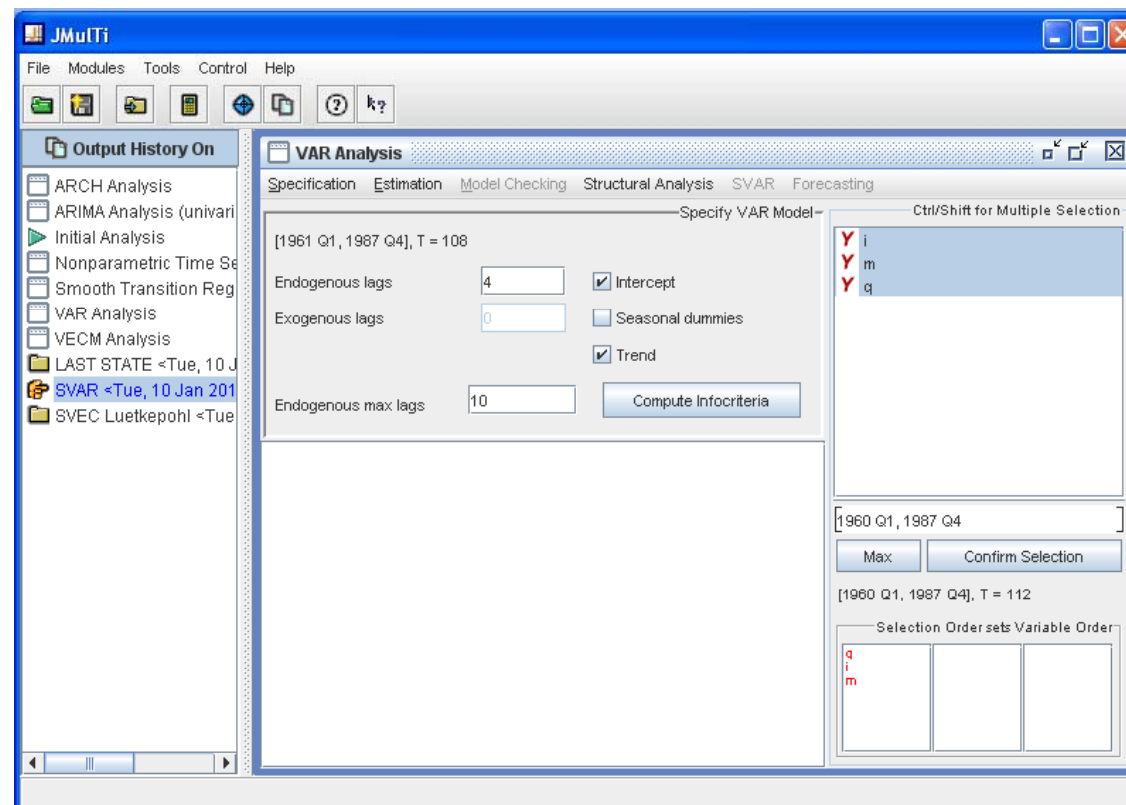
# VAR settings

Settings for reduced-form VAR with 4 lags, constant and trend

q: Output

i: interest rate

m: real monetary base



# VAR estimation

Estimated coefficients of the reduced-form VAR

The screenshot shows the JMulTi software interface for VAR Analysis. The main window displays the 'Output (save/print)' tab, showing the estimated model coefficients. The model is a VAR(4) with three variables: q(t), i(t), and m(t). The coefficients are estimated using OLS over the period [1961 Q1, 1987 Q4] with T = 108 observations.

The estimated model is represented by the following matrix equation:

$$\begin{bmatrix} q(t) \\ i(t) \\ m(t) \end{bmatrix} = \begin{bmatrix} 1.037 & -0.111 & -0.157 \\ 0.722 & 1.015 & 0.605 \\ -0.208 & -0.181 & 1.204 \end{bmatrix} \begin{bmatrix} q(t-1) \\ i(t-1) \\ m(t-1) \end{bmatrix} + \begin{bmatrix} -0.015 & -0.166 & -0.052 \\ -0.455 & -0.266 & -1.014 \\ 0.320 & 0.040 & -0.201 \end{bmatrix} \begin{bmatrix} q(t-2) \\ i(t-2) \\ m(t-2) \end{bmatrix} \\
 + \begin{bmatrix} -0.079 & 0.155 & 0.132 \\ -0.131 & 0.361 & 0.759 \\ -0.133 & -0.016 & 0.116 \end{bmatrix} \begin{bmatrix} q(t-3) \\ i(t-3) \\ m(t-3) \end{bmatrix} + \begin{bmatrix} -0.062 & -0.113 & 0.029 \\ 0.128 & 0.010 & -0.304 \\ -0.036 & 0.034 & -0.160 \end{bmatrix} \begin{bmatrix} q(t-4) \\ i(t-4) \\ m(t-4) \end{bmatrix} \\
 + \begin{bmatrix} 1.031 & 0.001 \\ -2.209 & -0.002 \\ 0.499 & -0.001 \end{bmatrix} \begin{bmatrix} \text{CONST} \\ \text{TREND}(t) \end{bmatrix} + \begin{bmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \end{bmatrix}$$



# SVAR restrictions

Set structural restrictions for the A and the B matrix

The screenshot shows the JMulti software interface. The main window is titled "VAR Analysis" and has tabs for Specification, Estimation, Model Checking, Structural Analysis, SVAR, and Forecasting. The "SVAR" tab is active, and the "Specify/Estimate SVAR AB Model" sub-tab is selected. The "Output (save/print)" sub-tab is also visible. The "Specify/Estimate SVAR AB Model" sub-tab contains two matrices, A and B, with checkboxes for "Edit coefficients manually".

The A matrix is defined as:

q	i	m
1	*	0
*	1	*
0	0	1

The B matrix is defined as:

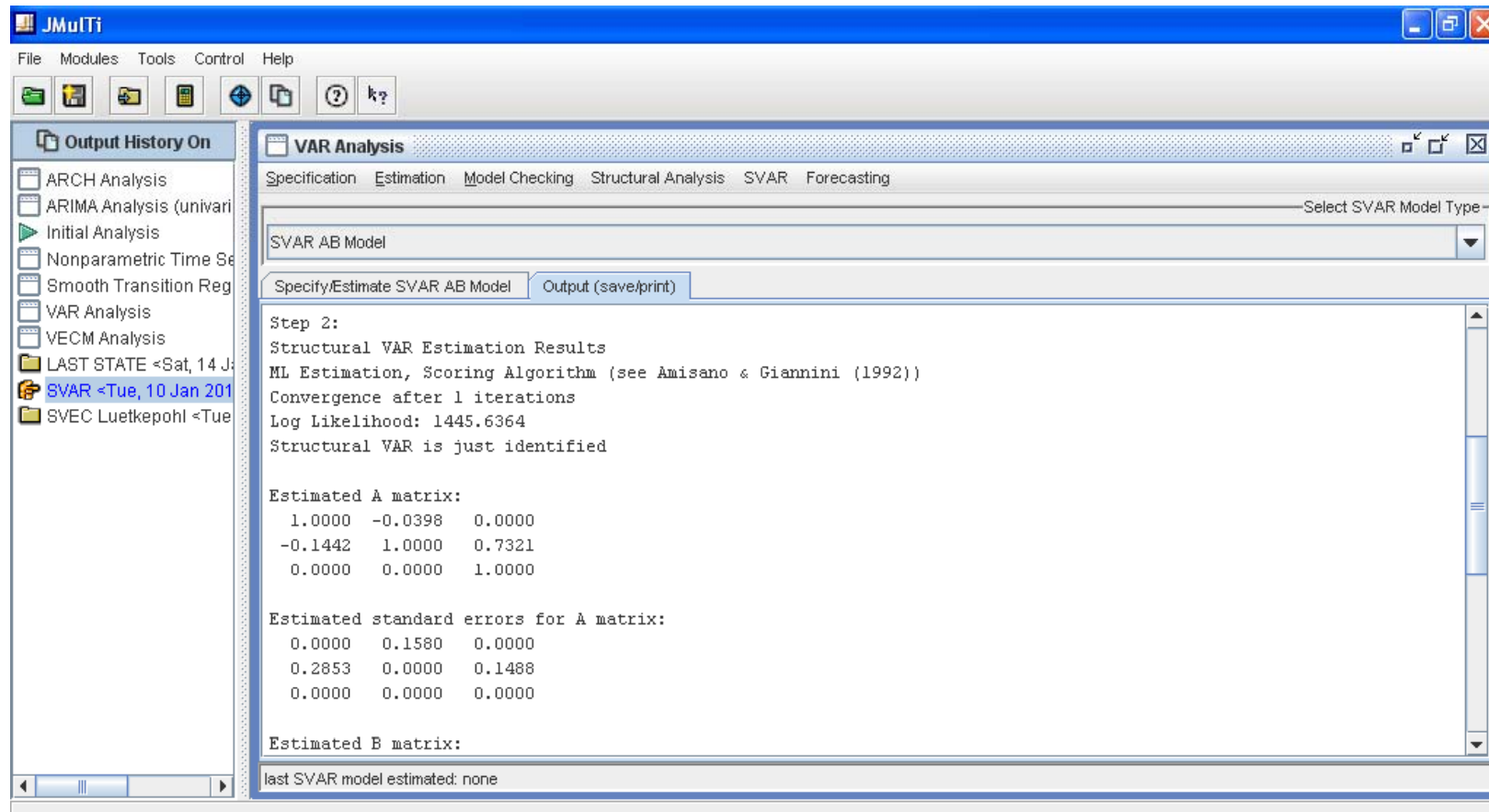
q	i	m
*	0	0
0	*	0
0	0	*

Below the matrices, there are three radio buttons for model specification: "Specify A model", "Specify B model", and "Specify general case". The "Specify general case" option is selected. There are also two buttons: "Optimization Settings" and "Execute".

The status bar at the bottom of the window displays "last SVAR model estimated: none".

# SVAR results

Estimation results for A and B



The screenshot displays the JMulTi software interface. The main window is titled "VAR Analysis" and shows the "Output (save/print)" tab. The results are as follows:

Step 2:  
Structural VAR Estimation Results  
ML Estimation, Scoring Algorithm (see Amisano & Giannini (1992))  
Convergence after 1 iterations  
Log Likelihood: 1445.6364  
Structural VAR is just identified

Estimated A matrix:

1.0000	-0.0398	0.0000
-0.1442	1.0000	0.7321
0.0000	0.0000	1.0000

Estimated standard errors for A matrix:

0.0000	0.1580	0.0000
0.2853	0.0000	0.1488
0.0000	0.0000	0.0000

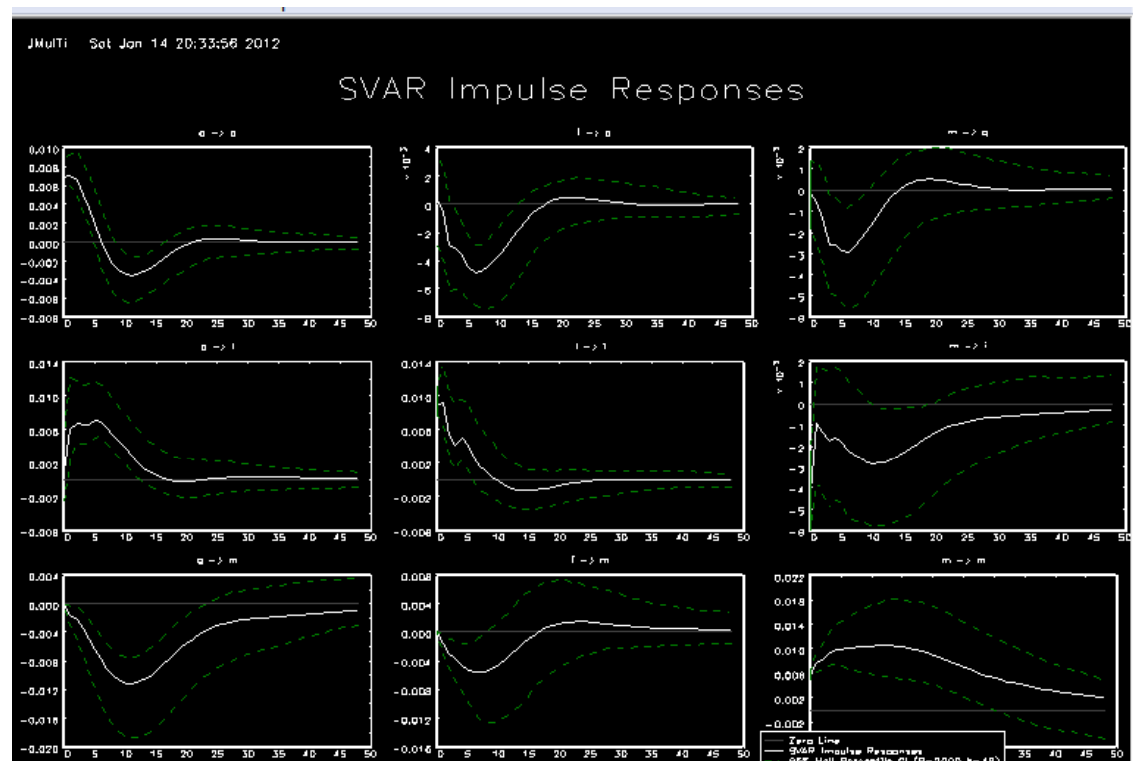
Estimated B matrix:

last SVAR model estimated: none

# SVAR IRA

- Impulse Response Analysis with 95% Hall bootstrap confidence intervals (2000 bootstrap replications)
- Responses of  $q$  (upper row),  $i$  (middle row),  $m$  (bottom row) to three structural shocks
- IS or spending shock in left column
- LM shock in middle column
- Money supply shock in right column

- IS shock increases output immediately, increases interest rate (with maximum after 8 quarters), decreases real money holdings
- LM shock increases interest rate and decreases output
- Money supply shock decreases output (contrary to economic theory), decreases interest rate and increases real money holding



## Example 2: SVEC

- Lütkepohl (2006)
- US quarterly data
- Output, consumption, investment
- All variables  $I(1)$ , cointegrating rank=2,
- two transitory shocks, one permanent shock

# VEC settings

Settings for reduced-form VEC with 1 lags, constant and 2 cointegration relations

The screenshot displays the JMulTi software interface for VECM Analysis. The window title is "JMulTi" and the menu bar includes "File", "Modules", "Tools", "Control", and "Help". The toolbar contains icons for file operations and help. The left sidebar shows a list of analysis modules, with "VECM Analysis" selected. The main window is titled "VECM Analysis" and has tabs for "Specification", "Estimation", "Model Checking", "Structural Analysis", "SVEC", and "Forecasting". The "Specify VEC Model" tab is active, showing the following settings:

- Time period: [1947 Q3, 1988 Q4], T = 166
- Endog. lags (diff): 1
- Exog. lags (levels): 0
- Cointegration rank: 2
- Intercept:
- Seasonal dummies:  (centered)
- Trend:
- CONST:  (under "Restrict Dets to EC Term")

The "Select Estimation Strategy" section shows:

- Exogenous variables, subset constraints and structural coefficients are ignored
- Johansen procedure (restrictions on beta are ignored)
- S2S procedure (restrictions on beta are used if available)
- Exogenous variables, subset constraints and structural coefficients can be estimated
- Two stage procedure: 1st stage (Specify), 2nd stage (Automatic)
- Estimate structural form:  (Structural Form)

On the right side, there is a list of variables: consumption, investment, and output, each with a "Y" in a red box. Below this, the time period is set to [1947 Q1, 1988 Q4], T = 168. The "Selection Order sets Variable Order" section shows the variables in the order: output, consumption, investment.

# VEC results

Estimated coefficients of reduced-form VEC

The screenshot shows the JMulTi software interface for VECM Analysis. The main window displays the following information:

**VECM Estimation Results**

Estimation results	Model Coefficients
Estimation method	One stage, Johansen approach
Estimation period	[1947 Q3, 1988 Q4], T = 166

Buttons for Coefficients, Standard Dev., and t-values are available on the right side of the results panel.

The reduced-form VEC model is displayed as a matrix equation:

$$\begin{bmatrix} d(\text{output})(t) \\ d(\text{consumption})(t) \\ d(\text{investment})(t) \end{bmatrix} = \begin{bmatrix} -0.225 & 0.204 \\ -0.062 & 0.072 \\ -0.112 & 0.255 \end{bmatrix} \begin{bmatrix} 1.000 & \dots & -1.020 \\ \dots & 1.000 & -1.099 \end{bmatrix} \begin{bmatrix} \text{output}(t-1) \\ \text{consumption}(t-1) \\ \text{investment}(t-1) \end{bmatrix} + \begin{bmatrix} 0.123 & 0.090 & 0.159 \\ 0.208 & -0.207 & 0.025 \\ 0.703 & -0.169 & 0.331 \end{bmatrix} \begin{bmatrix} d(\text{output})(t-1) \\ d(\text{consumption})(t-1) \\ d(\text{investment})(t-1) \end{bmatrix} + \begin{bmatrix} -0.877 \\ -2.835 \\ -30.065 \end{bmatrix} \text{CONST} + \begin{bmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \end{bmatrix}$$

# SVEC restrictions

- Restrictions on  $B$  (short-run) and  $\Xi B$  (long-run)
- One permanent shock that can have effects on all three variables
- Two transitory shocks, with the first one allowed to have effects on all three variables, and the second one not to be allowed to affect the second variable (0 restriction in  $B$ ). By that, the two transitory shocks are disentangled

The screenshot displays the JMulti software interface for SVEC Luetkepohl analysis. The main window is titled "VECM Analysis" and shows the "Specify/Estimate SVEC Model" tab. The "Output (save/print)" sub-tab is active. The interface displays the B matrix and the Identified long-run impact matrix.

**B matrix:**

	output	consumption	investment
output	*	*	*
consumption	*	*	0
investment	*	*	*

**Identified long-run impact matrix:**

	output	consumption	investment
output	*	0	0
consumption	*	0	0
investment	*	0	0

Number of bootstrap replications:

Use this seed (0 < s < 2,147,483,647)

Buttons: Optimization Settings, Point Estimates Only, Estimate with Boot. Std. Err.

# SVEC results

Estimated B and  $\Xi$ B matrices

The screenshot displays the JMulti software interface. The main window is titled "VECM Analysis" and shows the results of a Structural VAR Estimation. The output text is as follows:

```
*** Sat, 14 Jan 2012 14:09:51 ***  
  
This is a B-model with long run restrictions  
  
Long run restrictions provide(s) 2 independent restriction(s).  
Contemporaneous restrictions provide(s) 1 additional restriction(s).  
  
Structural VAR Estimation Results  
ML Estimation, Scoring Algorithm (see Amisano & Giannini (1992))  
Convergence after 10 iterations  
Log Likelihood: -283.1023  
Structural VAR is just identified  
  
Estimated B matrix  
0.0751  1.0265  -0.4467  
-0.6028  0.4286  0.0000  
0.2573  1.9565  1.0016  
  
Bootstrap standard errors:  
0.1866  0.2626  0.5929  
0.8413  0.1017  0.0000  
0.4191  0.3790  0.5081
```

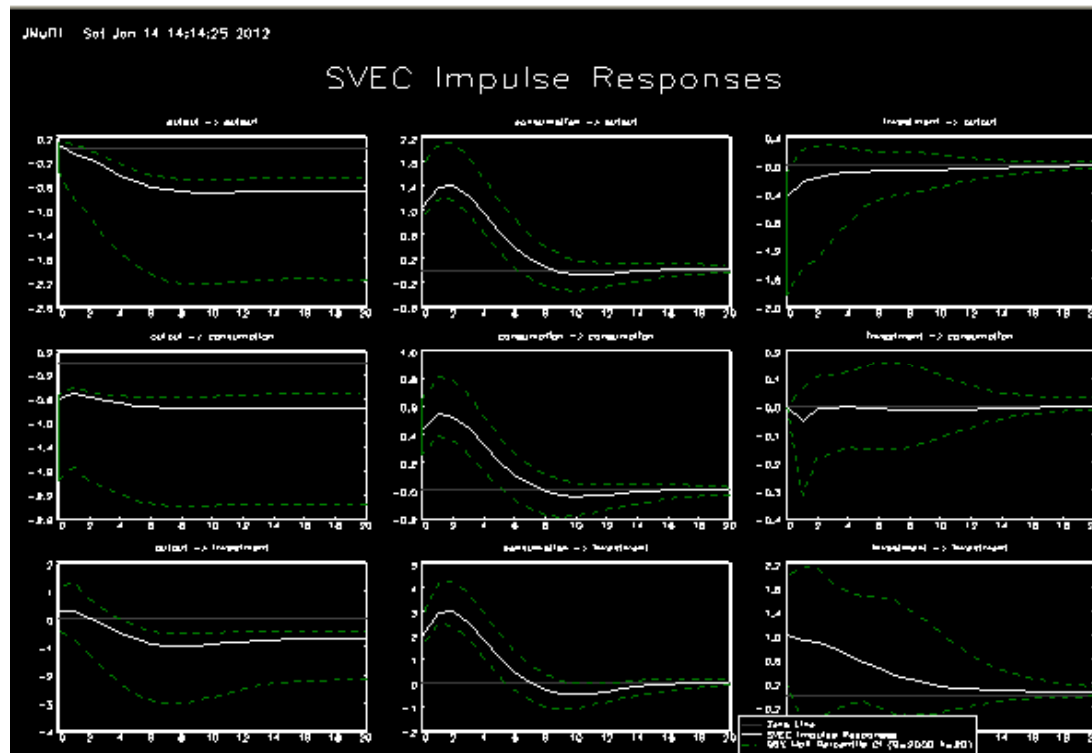
The interface also includes a menu bar (File, Modules, Tools, Control, Help), a toolbar with various icons, and a left-hand "Output History On" pane listing several analysis modules, with "SVEC Luetkepohl <Tue" selected.



# SVEC IRA

- IRA with 95% Hall bootstrap confidence intervals (2000 bootstrap replications)
- Responses of output (upper row), consumption (middle row), investment (bottom row) to three structural shocks
- Permanent shock in left column
- Transitory shocks in middle and right column

- Effects of the long-run shock are all negative in the long run. (To see the effect of an impulse which leads to positive long-run effects, just reverse the sign of the impulse responses)
- The transitory shocks indeed fade out quickly.
- The single 0 restriction in B can be seen in the right column, second row.



# SVEC FEVD

- Forecast error variance decomposition
- Permanent shock (light blue part) has increasing importance with higher forecast horizon
- The importance of the transitory shocks (green and dark blue) are decreasing with higher forecast horizon

