A Locally Optimal Seasonal Unit–Root Test
paper by Mehmet Caner

Monika Turyna and Ziad Al Hajeb
Department of Economics, University of Vienna

15th of June

Introduction

- Null hypothesis of stationary seasonality is tested against the alternative of seasonal nonstationarity
- Generalization of the Leybourne and McCabe (1994) test adapted to seasonality
- Differences between Caner test and Canova–Hansen test:

Caner	CH
autocorrelation in a parametric way	non–parametric correction
test statistic consistent at a rate $O_p(N)$	test statistic consistent at a rate $O_p(N/z)$
	poor finite–sample performance with large AR component

The Model

- $\Phi(L)y_t = \mu + S_t + e_t$
  where $t = 1, 2, \ldots, N$ – a linear model with stationary seasonality
- $\Phi(L) = 1 - \phi_1 L - \phi_2 L^2 - \cdots - \phi_p L^p$ is a p-th order AR polynomial in the lag operator with roots outside the unit circle
- S_t is a real–valued deterministic seasonal process of period s
- e_t iid $(0, \sigma_e^2)$
- y_t does not have unit roots at zero frequency

Trigonometric representation

- $S_t = \sum_{j=1}^{q} f'_{jt}\gamma_j, \qquad q = \frac{s}{2}$ and
  1. for $j < q$ $f'_{jt} = \left[\cos\left(\frac{j}{q}\pi t\right), \sin\left(\frac{j}{q}\pi t\right)\right]$
  2. for $j = q$ $f_{qt} = \cos(\pi t)$
- Vector representation:

$$S_t = f'_t\gamma, \qquad \gamma = \begin{pmatrix} \gamma_1 \\ \vdots \\ \gamma_q \end{pmatrix} \text{ and } f_t = \begin{pmatrix} f_{1t} \\ \vdots \\ f_{qt} \end{pmatrix}$$

- $\Phi(L)y_t = \mu + f'_t\gamma + e_t, \qquad t = 1, 2, \ldots, N$
  - Seasonality as a cyclical process
  - At the seasonal frequency $\frac{j\pi}{q}$, the cyclical processes are elements of f_t
  - f_t is a zero–mean process whenever N is a multiple of s
  - Coefficients γ_j represent the effect of each cycle on the seasonal component S_t

Structural and Reduced Form Model I

- A form of alternative hypothesis is to allow a unit root in γ_t
- The structural model:

$$\Phi(L)y_t = \mu + f_t'\gamma_t + e_t, \qquad t = 1, 2, \ldots, N$$

and

$$\gamma_t = \gamma_{t-1} + u_t$$

- u_t is iid mean 0, independent of e_t and f_t and with covariance matrix

$$Eu_t u_t' = \left[\sigma_u^2 G\right]_{(s-1)\times(s-1)}$$

- Whenever $\sigma_u^2 \neq 0$ there will be seasonal unit roots

Structural and Reduced Form Model II

- Reduced–form model:

$$\Phi(L)S(L)y_t = \mu' + \Theta(L)\zeta_t$$

  where

$$\zeta_t \sim (0, \sigma^2_\zeta),$$

$$S(L) = \sum_{j=0}^{s-1} L^j$$

  is a seasonal filter, and

$$\mu' = s\mu.$$

- $\Theta(L)$ is an $MA(s-1)$ polynomial

Test Statistic I

- Testing hypothesis of a stationary $AR(p)$ process against nonstationary seasonality
- $H_0 : \rho = 0$ against $H_1 : \rho > 0$ where $\rho = \frac{\sigma_u^2}{\sigma_e^2}$
- The locally best invariant test statistic for H_0 is

$$D = \hat{\sigma}_e^{-2} N^{-2} \sum_{t=1}^{N} \hat{F}_t' G \hat{F}_t$$

where $\hat{F}_t = \sum_{i=1}^{t} f_i \hat{e}_i$ and
$\hat{\sigma}_e^2 = \frac{\hat{e}'\hat{e}}{N}$ is a consistent estimator of σ_e^2

Test Statistic II

- Residuals $\hat{e}_t$ are obtained via:
  1. ML estimates of (ϕ) from the model

$$y_t^* = \mu' + \sum_{l=1}^{p} \phi_l y_{t-l}^* + \Theta(L)\zeta_t \qquad \text{where} \quad y_t^* = S(L)y_t$$

  2. Construct the series

$$\bar{y}_t = y_t - \sum_{l=1}^{p} \phi_l^* y_{t-l}$$

  3. Regress $\bar{y}_t$ on an intercept and seasonal dummies to obtain $\hat{e}_t$. (e_t assumed normal)

Test Statistic III

- Use of MLE rather than OLS to obtain consistent estimates under both H_0 and H_1
- Properties of G:
  1. When H_1 is unit roots at all seasonal frequencies, then G must be nonsingular and γ_t must be time–varying
  2. When H_1 is unit roots at specific seasonal frequencies, G must be block diagonal with nonzero element in only selected blocks and a subset of γ_t must be time–varying

The Asymptotic Distribution I

$$\hat{\Omega}^f = \hat{\sigma}_e^2 \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 0 & \ddots \end{bmatrix}$$

- When, $\frac{G}{\sigma_e^2} = (\Omega^2)^{-1}$, then the asymptotic distribution is easy to evaluate [Hansen (1992)]
- Ω^f is the long–run convariance matrix of $f_i e_i$ [Canova and Hansen (1995)]
- Some notation:
  - W_m – a vector standard Brownian bridge of dimension m
  - $VM(m) = \int_0^1 W_m(r)' W_m(r) dr$ – generalized von Mises distribution with m df.

The Asymptotic Distribution II

Theorem 1 [Proof in Caner(1998)]

If $\Phi(L)$ is a finite AR polynomial in L with roots outside the unit circle and if e_t is iid, $Ee_t = 0$ and $Ee_t^2 = \sigma_e^2 < \infty$, then, under H_0

$$D \xrightarrow{d} VM(s-1)$$

The Asymptotic Distribution III

Individual Test Statistics:

1. $D_{j\pi/q} = \frac{2}{\hat{\sigma}_e^2 N^2} \sum_{t=1}^{N} \hat{F}_{jt}' \hat{F}_{jt}, \qquad j < q$
2. $D_{\pi} = \frac{2}{\hat{\sigma}_e^2 N^2} \sum_{t=1}^{N} \hat{F}_{qt}^2, \qquad j = q$

Theorem 2 [Proof in Caner(1998)]

Under the conditions in Theorem 1 for

1. $j < q, \qquad D_{j\pi/q} \xrightarrow{d} VM(2)$
2. $j = q, \qquad D_{\pi} \xrightarrow{d} VM(1)$

Introduction

- "A technique which obtains a probabilistic approximation to the solution of a problem by using statistical sampling techniques"
- An asymptotic distribution theory is derived and the finite-sample properties of the test are examined in a Monte Carlo simulation
- The test is compared with the Canova and Hansen test, but is superior in terms of both size and power
- A Monte Carlo exercise is conducted to examine and compare the finite-sample properties of the proposed test with those of the CH test

Monte Carlo Study I

- Two quarterly models are considered:
  1. $\Phi(L)y_t = \mu + \sum_{j=1}^{2} f_{jt}'\gamma_{jt} + e_t \qquad e_t \sim N(0,1)$
     $\gamma_t = \delta\gamma_{t-1} + u_t \qquad u_t \sim N(0, \sigma_u^2 G)$,
     where:
     $\gamma_0 = [1,1,1]$,
     $\gamma_t = (\gamma_{1t}, \gamma_{2t})'$,
     $0 < \delta < 1$.
     $\Phi(L)y_t$ is an AR(p) process.
  2. $y_t = \mu + \sum_{j=1}^{2} f_{jt}'\gamma_{jt} + \tau(L)e_t \qquad e_t \sim N(0,1)$
     $\gamma_t = \delta\gamma_{t-1} + u_t \qquad u_t \sim N(0, \sigma_u^2 G)$,
     where $\quad \tau(L) = 1 + \tau_1 L + \tau_2 L^2 + \cdots + \tau_l L^l$
- This model ensures a fair comparison between the D test and the CH test because the test captures an AR(p) type of autocorrelation

Monte Carlo Study II

- For both models, three different data–generating processes (DGP's) are used under the alternative hypothesis:

1. $DGP1 : G = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$
2. $DGP2 : G = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
3. $DGP3 : G = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Monte Carlo Study III

- Under DGP1, there is a unit root at the π frequency as $\sigma_u^2 \neq 0$
- Under DGP2, there is a pair of complex conjugate roots at the $\pi/2$ frequency when $\sigma_u^2 \neq 0$
- Under DGP3, there are no unit roots when $\sigma_u^2 = 0$ but there are unit roots at all seasonal frequencies if $\sigma_u^2 \neq 0$
- In the simulations, the order of the AR polynomial p and of the MA polynomial l are 1 and 2.
- Both the AR parameters of the first model and the MA parameters of the second model are chosen to understand the effect of autocorrelation

Monte Carlo Study IV

- The test statistics are calculated for unit roots at all, π (Semi-annual), and $\pi/2$ (Annual) seasonal frequencies
- The finite-sample properties are compared to those of the CH tests (with and without one lag of the dependent variable included)
- The underlying model of the CH test is the same as the first model and the second one, but they assume $p = 0$ or $p = 1$
- In this study the Bartlett kernel is used and, following Andrews (1991), the lag truncation number $z = 3, 4, 6$ is selected for $N =$ 50,100, 200, respectively
- The results of the exercise are presented in Tables 1 and 2. The percentage of rejection of the null is given at the 5% significance level
- Because the size of the D and CH tests that are calculated vary considerably, the size-adjusted power is calculated

Size and Power of the Test: $AR(p)$ Process I

- The size and power properties of the D test are compared with the CH test under the first model
- In Table 1, when analyzing the size δ is selected to be 0.8 because this value corresponds to a "near" seasonal unit root
- In calculating the power of the tests in Tables 1 and 2, δ is set equal to 1

Size and Power of the Test: $AR(p)$ Process II

Table 1. Size and Power Comparison Between the CH and D tests: AR Model

DGP	SS	D: J	D: π	D: $\pi/2$	CH_0: J	CH_0: π	CH_0: $\pi/2$	CH_1: J	CH_1: π	CH_1: $\pi/2$
$y_t + .8y_{t-1} = \mu + f_t'\gamma_t + e_t$										
Size	200	12.9	8.8	13.5	8.9	24.9	.7	6.2	3.7	7.0
Size	100	13.4	11.2	11.7	10.1	29.6	.5	3.4	7.5	7.3
Size	50	14.8	15.0	10.1	13.0	34.6	.7	2.3	3.9	5.1
DGP3	200	99.4	35.3	89.0	95.8	69.2	80.1	91.0	43.3	95.1
DGP3	100	88.5	23.3	77.2	77.9	58.9	54.9	83.1	25.9	83.3
DGP3	50	54.6	30.4	48.5	53.2	45.0	45.1	52.8	17.0	52.6
DGP2	200	97.9	1.5	98.2	90.6	4.3	97.4	94.7	14.0	95.5
DGP2	100	84.3	4.7	86.3	64.4	6.6	85.9	79.2	16.0	80.1
DGP2	50	59.1	12.5	51.4	20.5	9.2	50.4	47.0	8.7	51.9
DGP1	200	48.0	56.0	.9	68.1	71.5	.0	5.3	12.9	2.7
DGP1	100	35.7	45.0	1.6	54.9	57.4	.1	4.6	11.1	3.2
DGP1	50	24.8	31.2	2.4	40.3	40.8	8.7	3.9	12.6	20.0
$y_t + .8y_{t-2} = \mu + f_t'\gamma_t + e_t$										
Size	200	17.3	16.7	12.0	40.9	.6	55.5	49.9	1.8	63.8
Size	100	21.0	14.3	16.6	47.7	.6	63.7	50.4	1.0	64.1
Size	50	23.6	10.7	20.4	44.7	1.5	61.1	45.7	1.5	58.8
DGP3	200	94.0	83.0	47.5	94.1	64.1	86.3	93.3	61.5	85.5
DGP3	100	81.1	61.9	41.6	78.9	42.1	74.3	77.9	44.2	73.0
DGP3	50	34.5	40.1	19.7	62.0	38.6	61.0	65.1	32.9	59.2
DGP2	200	65.1	.4	72.4	84.9	.1	85.7	84.8	.0	85.9
DGP2	100	51.8	.4	59.1	75.0	.3	74.9	74.7	2.0	75.6
DGP2	50	13.7	.8	14.6	60.4	8.2	62.2	59.4	.2	56.8
DGP1	200	85.3	91.0	2.6	56.4	89.4	7.3	55.9	90.1	6.7
DGP1	100	66.3	73.1	10.5	31.7	67.8	10.7	30.4	68.9	16.5
DGP1	50	29.5	39.9	15.5	19.1	41.0	16.8	24.1	40.9	17.0

Size and Power of the Test: $AR(p)$ Process III

- In Table 1, the size of the tests is slightly above the nominal size of 5% in most of the cases.
- The CH tests have large size distortions for $AR(2)$ parameterization
- However, for example, for $N = 200$ in an $AR(2)$ framework, the size of the joint D test is 17%, whereas the joint CH tests reject the true null in 41-50% of the trials
- The D tests have good power under different alternatives:
  - For $N = 100$, the power of the joint test is 84% when there are seasonal unit roots present at the $\pi/2$ frequency ($DGP2$)
  - For $N = 200$, in an $AR(2)$ process, the power of the joint test is 85% when there is a seasonal unit root at the π frequency ($DGP1$)

Size and Power of the Test: $AR(p)$ Process IV

- The CH tests have mixed results under an AR structure:
  - For $N = 100$, in an $AR(1)$ process the joint test has 64- 79% power against $DGP2$. CH_1 in Tables 1-2 perform quite poorly in an $AR(1)$ structure. The power is near the nominal size of the tests
  - Both CH tests also have trouble in an $AR(2)$ structure when only a seasonal unit root at the π frequency is present ($DGP1$) For $N = 200$, the joint tests have 56% power under $DGP1$.
- Overall, the CH tests do not perform well near seasonal unit roots because they suffer from size distortion
- While the proposed tests have good size and power

Size and Power of the Test: $MA(1)$ Process I

Table 2. Comparison of Size and Power: The CH and D tests in an MA Model

DGP	SS	D J	D π	D $\pi/2$	CH_0 J	CH_0 π	CH_0 $\pi/2$	CH_1 J	CH_1 π	CH_1 $\pi/2$
		$y_t = \mu + \sum_{j=1}^{2} f'_{jt}\gamma_{jt} + \varepsilon_t + \tau\varepsilon_{t-1}$								
Size	200	9.5	8.8	6.0	8.0	9.3	6.6	8.9	11.6	7.4
Size	100	9.8	9.5	5.4	4.1	6.4	5.5	9.0	14.6	6.8
Size	50	8.4	6.9	7.4	1.7	4.7	4.4	4.7	8.7	6.7
DGP3	200	92.3	51.9	68.5	98.7	83.0	92.9	98.8	78.5	93.6
DGP3	100	73.4	52.4	43.7	91.8	70.3	75.2	91.6	67.0	71.1
DGP3	50	67.2	48.9	30.3	70.4	60.1	42.2	65.0	54.5	39.9
DGP2	200	88.1	.0	93.6	89.1	0.0	93.9	86.2	.0	96.2
DGP2	100	62.9	.0	75.8	69.7	0.0	80.3	64.8	.0	77.7
DGP2	50	24.4	.1	40.5	34.4	0.0	48.2	31.5	.0	45.7
DGP1	200	56.9	61.0	3.2	73.3	82.9	2.2	71.9	81.0	1.6
DGP1	100	43.3	53.5	0.6	65.6	73.5	1.4	61.5	70.2	1.4
DGP1	50	41.8	50.1	3.6	43.0	56.1	4.0	40.3	53.7	3.1
		$y_t = \mu + \sum_{j=1}^{2} f'_{jt}\gamma_{jt} + \varepsilon_t + \tau\varepsilon_{t-2}$								
Size	200	14.1	17.4	9.9	3.0	9.4	1.1	1.6	7.2	1.8
Size	100	14.4	18.0	6.7	2.6	8.6	.6	1.5	9.3	1.0
Size	50	5.6	17.4	0.7	1.6	11.3	.1	1.4	10.8	.2
DGP3	200	97.8	42.2	93.0	99.7	73.4	98.9	99.5	73.1	99.5
DGP3	100	92.2	29.8	90.3	94.2	55.2	90.7	93.9	51.9	94.1
DGP3	50	64.5	13.9	78.0	71.4	34.6	76.0	69.5	29.5	74.7
DGP2	200	91.1	.8	93.9	97.2	2.8	99.3	95.7	2.7	98.8
DGP2	100	78.5	1.6	89.4	88.3	3.7	94.2	87.1	3.4	93.3
DGP2	50	50.3	1.4	77.4	55.6	4.3	75.1	52.6	4.2	72.2
DGP1	200	63.5	72.3	.0	64.8	75.8	.0	64.9	72.9	.0
DGP1	100	45.4	54.1	.0	45.9	54.7	.0	41.8	53.1	.0
DGP1	50	31.5	34.5	.0	24.1	31.6	.1	15.9	27.6	.1

Size and Power of the Test: $MA(1)$ Process II

- Table 2 shows that the proposed D tests have good size. The test at the $\pi/2$ frequency performs well even in the small samples. For example, for $N = 50$ in an $MA(1)$ process, the size is 7%.
- Even though the test at the π frequency performs well in an $MA(1)$ model, the size rises above the nominal level and is around 18% in an $MA(2)$ setup.
- The CH tests also have good size properties
- For example, the size of the joint CH test with no lags of the dependent variable (CH_0) is 2-11%
- The sizes of both tests do not seem to be affected by the sample size
- Both the D and CH tests have good power under different alternatives
- However, the asymptotic rejection frequency of the D tests is better than that of the CH tests

The Robustness Experiments

- The size and the power of the test were not affected by the changes in σ_u^2. Smaller δ and AR coefficients resulted in better size properties for the test
- Monte Carlo designs with longer AR polynomials such as 3 and 4 were tried, generating results that were very similar to the case of $AR(2)$ design in Table 1

Conclusions

1. The paper proposes a locally best test for detecting the seasonal unit roots in time series models
2. The null hypothesis of the proposed test is seasonal stationarity, whereas the seasonal unit root hypothesis forms the alternative
3. The derived asymptotic distribution is non standard and covers serially correlated processes
4. The main difference between the proposed test and the CH test is the handling of autocorrelation under the respective null and alternative hypotheses
5. The proposed test has better size and power properties than the CH test in an AR type of autocorrelation
6. The CH test suffers from size distortion in an AR model, whereas the proposed test has good size and power