Periodic Processes

Chapter 6

The Econometric Analysis of Seasonal Time Series

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(08.06.2010)

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Introduction

Periodic processes are processes in which the **coefficients change with the seasons** of the year. A deterministic seasonal process, in which the intercept changes seasonally, can be viewed as a special case of a periodic process.

Gersovitz and MacKinnon (1978) and Osborn (1988) argue that a process of this type arises when modeling the seasonal decisions of consumers, while Hansen and Sargent (1993) suggest that it could also arise from seasonal technology.

Some Periodic Processes

A periodic ARMA (p,q) process has the general form

$$\emptyset_s(L)y_{s\tau} = c_s + \theta_s(L)\varepsilon_{s\tau}\ , \qquad s = 1, \ldots, S,\ \ \tau = 1, \ldots, T_\tau$$

where

$$\emptyset_s(L) = 1 - \emptyset_{s1}L - \cdots - \emptyset_{sp}L^P$$

$$\theta_s(L) = 1 - \theta_{s1}L - \cdots - \theta_{sq}L^q$$

are polynomials in the conventional lag operator L. The polynomial orders (p,q) are defined by maximum AR and MA lags and $\varepsilon_{s\tau}$ is i.i.d process over both season and year.

- Heteroskedasticity over seasons is typically permitted, so that $E(\varepsilon_{s\tau}^2) = \sigma_s^2$.

- L operates on the season, so one-period lagged observation is $Ly_{s\tau} = y_{s-1,\tau}$

- Periodic ARMA processes have distinctive stationarity and invertibility properties compared with a conventional ARMA processes.

Periodic Heteroskedasticity

Considering a Periodic seasonal heteroskedastic AR(1) process as

$$y_{s\tau} = \emptyset y_{s-1,\tau} + \varepsilon_{s\tau}$$

assuming $y_{s\tau}$ corresponds to $s = S$, repeated substitution yields

$$y_{S\tau} = \emptyset^2 y_{S-2,\tau} + \varepsilon_{S\tau} + \emptyset\varepsilon_{S-1,\tau},$$

$$= \emptyset^S y_{S,\tau-1} + \varepsilon_{S\tau} + \emptyset\varepsilon_{S-1,\tau} + \ \ldots + \emptyset^{S-1}\varepsilon_{1\tau},$$

$$= \emptyset^{\tau S} y_{S0} + \sum_{j=0}^{\tau-1} \emptyset^{Sj} \ (\varepsilon_{S,\tau-j} + \emptyset\varepsilon_{S-1,\tau-j} + \cdots + \emptyset^{S-1}\varepsilon_{1,\tau-j})$$

If y_{S0} has the same variance as each sample period $y_{S\tau}$, then for $s = S$:

$$Var(y_{s\tau}) = \gamma_s(0) = \frac{\sigma_s^2 + \emptyset^2\sigma_{s-1}^2 + \cdots + \emptyset^{2\,(S-1)}\sigma_{s-(S-1)}^2}{1 - \emptyset^{2S}}$$

$Var(y_{s\tau})$is periodically varying since the weighting of each $\sigma_s^2 (s = 1, \dots, S)$depends on the season (s) in which $y_{s\tau}$ is observed.

The process autocovariances at lag k, $\gamma_s(k) = E(y_{s\tau} - \mu)(y_{s-k,\tau} - \mu)$ are also seasonally varying.

Solution: the effect of periodic heteroskedasticity can be removed by standardizing by division by the appropriate standard deviation. The standardized process has a zero mean, unit variance, and autocovariances that are independent of s.

$$(y_{s\tau} - \mu)/\sqrt{Var\,(y_{s\tau})}$$

Periodic MA(1)Process

The Periodic MA(1) process is

$$y_{s\tau} = \varepsilon_{s\tau} - \theta_s\, \varepsilon_{s-1,\tau}, \qquad s = 1, \ldots, S,$$

$$\begin{aligned} Var(y_{s\tau}) &= \gamma_s(0) = E\left(\varepsilon_{s\tau} - \theta_s\, \varepsilon_{s-1,\tau}\right)^2 \\ &= (1 + \theta_s^2)\sigma^2 \end{aligned}$$

and the autocovariance at lag 1:

$$\begin{aligned} \gamma_s(1) &= E\left(y_{s\tau}\, y_{s-1,\tau}\right) \\ &= E\left(\varepsilon_{s\tau} - \theta_s\, \varepsilon_{s-1,\tau}\right)\left(\varepsilon_{s-1,\tau} - \theta_{s-1}\, \varepsilon_{s-2,\tau}\right) \\ &= -\theta_s \sigma^2 \end{aligned}$$

- For $k > 1$, all $\gamma_s(k) = 0$

Thus, although the periodic MA(1) exhibits periodic variances and autocovariances, observations 1 year apart, $y_{s\tau}$ and $y_{s,\tau-1}$, are not correlated. This implies that the characteristic of seasonality in economic variables that the patterns in the observations tend to repeat each year, and hence that $y_{s,\tau-1}$ provides relevant information for the prediction of $y_{s\tau}$, cannot be delivered by a periodic MA(1) process. This remains true for any periodic MA process of order $S-1$ or less, and indicates why low order periodic MA processes have been of little interest in economics.

Periodic AR(1)Process

The Periodic AR(1) process is

$$y_{s\tau} = \emptyset_s y_{s-1,\tau} + \varepsilon_{s\tau}, \qquad s = 1, \dots, S,$$

With substitution for lagged y,

$$y_{s\tau} = \emptyset_s \emptyset_{s-1} y_{s-2,\tau} + \varepsilon_{s\tau} + \emptyset_s \varepsilon_{s-1,\tau},$$

$$= \emptyset_S y_{S-1} \dots \emptyset_1 y_{s,\tau-1} + \varepsilon_{s\tau} + \emptyset_s \varepsilon_{s-1,\tau} + \emptyset_s \emptyset_{s-1} \varepsilon_{s-2,\tau}$$

$$+ \cdots + \emptyset_s \emptyset_{s-1} \dots \emptyset_{s-(S-1)} \varepsilon_{s-(S-1),\tau} \qquad (*)$$

The coefficient of $y_{s,\tau-1}$is the product of all S periodic AR(1) coefficients, namely $\psi = \emptyset_1 \emptyset_2 \dots \emptyset_S$.

The presence of the periodic MA process implies that $\text{Var}(y_{s\tau})$ and its autocovariances vary over s.

$$Var(y_{1\tau}) = \gamma_1(0) = (1/1-\psi^2)[\sigma_1^2 + \emptyset_1^2\sigma_4^2 + \emptyset_1^2\emptyset_4^2\sigma_3^2 + \emptyset_1^2\emptyset_4^2\emptyset_3^2\sigma_2^2]$$

Even with homoskedasticity in the disturbances, the periodic AR(1) process $y_{s\tau}$ exhibits periodic heteroskedasticity.

the autocovariance at lag 1 for PAR(1) satisfy

$$\gamma_s(1) = E\left(y_{s\tau}, y_{s-1,\tau}\right) = \emptyset_s\gamma_{s-1}(0)$$

and in the annual lag S

$$\gamma_s(S) = E\left(y_{s\tau}\ y_{s,\tau-1}\right) = \psi\gamma_s(0)$$

Notice that while $\gamma_s(1)$ is periodic through both $\emptyset_s$ and $\gamma_{s-1}(0)$, $\gamma_s(S)$ is periodic only through the variance $\gamma_s(0)$.

Consequently, the autocorrelation of $y_{s\tau}$ at lag S, namely $\rho_s(S) = \frac{\gamma_s(S)}{\gamma_s(0)} = \psi$, is constant over $s = 1, \ldots, S$.

Equation (*) also implies that the PAR(1) process gives rise to an annual pattern in the conditional expectations, with

$$E\left(y_{s\tau} \mid y_{s,\tau-1} \right) = \psi \; y_{s,\tau-1}$$

which applies for all s. Thus, in contrast to the periodic MA(1) process, the PAR(1) process gives rise to a type of seasonal habit persistence whereby an annual pattern in the observations will tend to be repeated when ψ is positive.

The VAR Representation

This representation is especially useful in the periodic case. The representation of the periodic process as a VAR effectively treats the observation $y_{s\tau}$ for the seasons as separate series. Example:

$$\begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ -\boldsymbol{\emptyset}_{\mathbf{2}} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\boldsymbol{\emptyset}_{\mathbf{3}} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -\boldsymbol{\emptyset}_{\mathbf{4}} & \mathbf{1} \end{bmatrix} \begin{bmatrix} \boldsymbol{y}_{\mathbf{1}\boldsymbol{\tau}} \\ \boldsymbol{y}_{\mathbf{2}\boldsymbol{\tau}} \\ \boldsymbol{y}_{\mathbf{3}\boldsymbol{\tau}} \\ \boldsymbol{y}_{\mathbf{4}\boldsymbol{\tau}} \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \boldsymbol{\emptyset}_{\mathbf{1}} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{y}_{\mathbf{1},\boldsymbol{\tau}-\mathbf{1}} \\ \boldsymbol{y}_{\mathbf{2},\boldsymbol{\tau}-\mathbf{1}} \\ \boldsymbol{y}_{\mathbf{3},\boldsymbol{\tau}-\mathbf{1}} \\ \boldsymbol{y}_{\mathbf{4},\boldsymbol{\tau}-\mathbf{1}} \end{bmatrix} + \begin{bmatrix} \boldsymbol{c}_{\mathbf{1}} \\ \boldsymbol{c}_{\mathbf{2}} \\ \boldsymbol{c}_{\mathbf{3}} \\ \boldsymbol{c}_{\mathbf{4}} \end{bmatrix} + \begin{bmatrix} \boldsymbol{\varepsilon}_{\mathbf{1}\boldsymbol{\tau}} \\ \boldsymbol{\varepsilon}_{\mathbf{2}\boldsymbol{\tau}} \\ \boldsymbol{\varepsilon}_{\mathbf{3}\boldsymbol{\tau}} \\ \boldsymbol{\varepsilon}_{\mathbf{4}\boldsymbol{\tau}} \end{bmatrix}$$

Or more compactly $\phi_0 Y_\tau = \phi_1 Y_{\tau-1} + C + U_\tau$

The one to one mapping between the polynomial coefficients of the PAR process and the elements of the VAR coefficient matrics is discussed in a number of the papers.

So the general vector representation for a PAR(p) process is the VAR(P):

$$\phi_0 Y_\tau = \phi_1 Y_{\tau-1} + \cdots + \phi_P Y_{\tau-P} + C + U_\tau$$

The more usual VAR(P) representation can be obtained by inverting ϕ_0, so that

$$\mathrm{Y}_\tau = \phi_0^{-1}\phi_1 \mathrm{Y}_{\tau-1} + \cdots + \phi_0^{-1}\phi_\mathrm{P} \mathrm{Y}_{\tau-\mathrm{P}} + \phi_0^{-1}\mathrm{C} + \phi_0^{-1}\mathrm{U}_\tau\ ,$$

$$= \mathrm{A}_1 \mathrm{Y}_{\tau-1} + \cdots + \mathrm{A}_\mathrm{P} \mathrm{Y}_{\tau-\mathrm{P}} + \tilde{\mathrm{C}} + \mathrm{V}_\tau$$

where $\mathrm{A}_\mathrm{i} = \phi_0^{-1}\phi_\mathrm{i} (\mathrm{i} = 1, \ldots, \mathrm{P})$, $\tilde{\mathrm{C}} = \phi_0^{-1}\mathrm{C}$ and $\mathrm{V}_\tau = \phi_0^{-1}\mathrm{U}_\tau$

Integration in PAR

Stationary condition for the PAR process is that the roots of determinantal polynomial lie outside the unit circle. In PAR context, 3 types of integrated processes exist for first order unit root nonstationarity.

- $y_t \sim I(1)$.This arises when each PAR operator $\emptyset_s(L)$ contains the common factor$\Delta_1 = (1 - L)$, but the matrix representation for $\Delta_1 y_{s\tau}$ is a stationary VAR process.

- $y_t \sim SI(1)$. This arises when each PAR operator $\emptyset_s(L)$contains the common factor$\Delta_S = (1 - L^S)$, with the matrix representation for $\Delta_S y_{s\tau}$ being a stationary VAR process.

- $y_t \sim PI(1)$. This arises when $|\phi(L)|$contains the factor $(1 - L^S)$, but $\Delta_S$is not common to each polynomial $\emptyset_s(L)(s = 1, \ldots, S)$, with the VAR for $\Delta_S y_{s\tau}$ being stationary.

The first 2 case imply stationary PAR processes in the appropriately differenced variable, namely $\Delta_1 y_t$ or$\Delta_S y_t$, while the third is a specific type of integration that can arise only in the periodic context.

Monte Carlo Analysis

Studies find that periodic models produce less accurate forecasts than nonperiodic models. As periodic processes have some attractive features, great strides have been made in establishing an appropriate toolkit for the statistical analysis of such processes. Empirical applications regarding this process are relatively few to date. While evidence of periodicity have been found, it would be foolhardy to conclude at the present time that the majority of important real macroeconomic variables are of this type.

It’s also true that the empirical analyses have generally been confined to quarterly series. The data requirements of periodic approaches with monthly data (notably those based on a VAR representation) are large.

It may be that methods based on restricting periodic processes, perhaps leading to specifications with relatively few additional parameters compared with nonperiodic models, will prove fruitful in the future. In this way, periodic features that are important may be taken into account without the sacrifice of too many degrees of freedom in relation to nonperiodic models.

Thank you for attention