Some Nonlinear Seasonal Models

Chapter 7 from Ghysels and Osborn: The Econometric Analysis of Seasonal Time Series

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Introduction

- Nonlinear models:
  - Provide tools to model nonlinear relationship between variables (e.g. seasons)
  - Examples of nonlinear dynamics: time-changing (seasonal) variance, asymmetric cycles, higher-moment structures
- (Seasonal) Nonlinearity can be primarily found in high-frequency data like intradaily seasonal patterns
  - S&P’s composite stock-price index
  - Exchange rates
- Different seasonal models with different types of nonlinearity:
  - Stochastic seasonal unit roots – *varying impact of seasonal shocks*
  - Seasonal (G)ARCH models – *structure of seasonal variance*
  - Periodic GARCH models – *time-varying seasonal coefficients*
  - Periodic Markov switching models – *seasonal mean shifts*

Stochastic Seasonal Unit Root

- Motivation: <u>not</u> all macroeconomic shocks may have the <u>same</u> impact
- Generalization of linear processes by allowing for random parameters
- First-order seasonal random coefficient autoregressive process:

$$(1-\tilde{\phi}_{s\tau} L^s)\tilde{y}_{s\tau} = \varepsilon_{s\tau} \qquad s = 1,\ldots,S \qquad \tau = 1,2,\ldots,T_\tau$$

where $\tilde{\phi}_{s\tau} = 1+\tilde{\alpha}_{s\tau}$ and $\tilde{\alpha}_{s\tau} = \rho\tilde{\alpha}_{s,\tau-1} + \xi_{s\tau}$

- $0 \leq \rho \leq 1$, ε and ξ are i.i.d. and normally distributed with σ^2, ω^2
- The randomized seasonal autoregressive process is then:

$$\Delta_S \tilde{y}_{s\tau} = \tilde{\alpha}_{s\tau}\tilde{y}_{s,\tau-1} + \varepsilon_{s\tau}$$

- It is also called a heteroskedastic seasonally integrated process
  – Conditional on its own normally distributed past $N(\rho\tilde{\alpha}_{s,\tau-1}\tilde{y}_{s,\tau-1}, \sigma^2 + \omega^2 \tilde{y}_{s,\tau-1}^2)$

heteroskedasticity

- If $\omega^2 = 0$, the process is a regular seasonal random walk with homoskedastic innovations
- Hence, the test hypotheses are $H_0 : \omega^2 = 0$

  $H_A : \omega^2 \rangle 0$

- Taylor-Smith test is used for determination of heteroskedastic seasonal integration

$$S_\rho = \frac{1}{\sigma^4} \sum_{s=1}^{S} \sum_{j}^{T_\tau} \rho^{-2j} \left[\left(\sum_{\tau=j}^{T_\tau} \rho^\tau y_{s,\tau-1} \Delta_S y_{s\tau} \right)^2 - \sigma^2 \sum_{\tau=j}^{T_\tau} \rho^{2\tau} y_{s,\tau-1} \right]$$

  – $y_{s\tau} = \phi(L)\left[\tilde{y}_{s\tau} - \mu_s^* - \beta_s^*(S\tau + s)\right]$
  – ρ is unidentified under the null hypothesis, hence, the two polar cases S_0 and S_1 are computed
  – the limiting distributions for S_0 and S_1 are nonstandard

- However, this process is <u>not a covariance stationary process</u>

Seasonal (G)ARCH Models

- Application: - financial time series
  - stock-market dividend yields
  - Foreign-exchange volatility
  - Lead-lag relations between two or more simultaneously traded markets

  - volatility of few macroeconomic time series
- The GARCH(p,q) process:

$$X_t = \sigma_t \varepsilon_t$$

$$\sigma_t^2 = \omega + \phi(L)\varepsilon_{t-1}^2 + \theta(L)\sigma_{t-1}^2$$

  – if we define $v_t = \varepsilon_t^2 - \sigma_t^2$, one obtains:

$$\Leftrightarrow \varepsilon_t^2 = \omega + [\phi(L) + \theta(L)]\varepsilon_{t-1}^2 - \theta(L)v_{t-1} + v_t$$

ARMA[max(p,q),q] model

- Analogously, the seasonal GARCH(p,q) process:

$$\varepsilon_t^2 = \omega + [\phi(L) + \theta(L)]\varepsilon_{t-s}^2 - \theta(L)v_{t-s} + v_t$$

- where: $\phi(L) = \sum_{j=1}^{p} b_j L^j$ and $\theta(L) = \sum_{j=1}^{q} a_j L^j$
- If $\sum_{j=1}^{p} b_j + \sum_{j=1}^{q} a_j \langle 1$ then this GARCH(p,q) process has a unique strictly and <u>covariance stationary solution</u>
  - For a GARCH(1,1) process being strictly stationary it is enough to fulfill the following condition: $a_1 + b_1 = 1$
- Maximum-Likelihood-based estimation of coefficients
- The effects of Filtering on ARCH models
  - Seasonal filtering may lead to bias in the autocorrelation function
  - In order to present these biases one may define the weak GARCH(p,q) process, linear filters, and derive the (filtered) autocovariance and autocorrelation functions
  - Then, one can derive conditions to have unbiased autocorrelation function

- The **weak GARCH(p,q) process**:

$$\varepsilon_t^2 = \omega + \sum_{j=1}^{\max(p,q)} (\phi_j + \theta_j)\varepsilon_{t-j}^2 - \sum_{j=-1}^{q} \theta_j v_{t-j} + v_t$$

  - where $\sigma_t^2 = E_{Lt}(\varepsilon_{t+1}^2)$ is the conditional variance with $\varepsilon_{t+1}^2 = \sigma_t^2 + v_{t+1}$ and $E_{Lt}(.)$ is the linear projection on the space spanned by 1, $(\varepsilon_{t-j}, \varepsilon_{t-j}^2): j \geq 0$
  - v_{t+1} is a Martingale difference sequence with respect to the linear span filtration
- Suppose the following **linear filter** that filters the nonseasonal (ns) components in the data: $z_t^{ns} = z_t^F = v(L)z_t = \sum_{k=-\infty}^{+\infty} v_k L^k z_t$
  - z is a variable with seasonal (s) and nonseasonal components:
    $z = z^S + z^{NS}$
  - L is the lag operator
- **Autocovariance function**:
  - $\gamma_2(j) = E_L(\varepsilon_t^2 \varepsilon_{t-j}^2)$
  - Applying the linear filter to the residuals one obtains the **filtered autocovariance function**:

$$\gamma_2^F(j) = E_L(\varepsilon_t^F)^2(\varepsilon_{t-j}^F)^2 = E_L(v(L)\varepsilon_t)^2(v(L)\varepsilon_{t-j})^2$$

- In case of weak GARCH(1,1) one can derive the following autocovariance (γ) and **autocorrelation** (ρ) **function**s:

$$\gamma_2(0) = \frac{\left[1+\theta^2 - 2(\phi+\theta)\theta\right]}{\left[1-(\phi+\theta)^2\right]}\sigma_v^2$$

$$\gamma_2(j) = \frac{\phi\left[1-(\phi+\theta)\theta\right]}{\left[1-(\phi+\theta)^2\right]}(\phi+\theta)^{j-1}\sigma_v^2$$

$$\rho_2(j) = \frac{\phi\left[1-\phi\theta-\theta^2\right]}{\left[1-2\phi\theta-\theta^2\right]}(\phi+\theta)^{j-1}$$

- Let define $\lambda \equiv \phi + \theta$ and apply the linear filter v(L) ≡ (1-L^S) to this process, then one obtains the **filtered autocorrelation function**:

$$\rho_2^F(j) \equiv \frac{2+\lambda+\lambda^S}{2+6\lambda\rho_2(S)}\rho_2(j)$$

- As a final step the condition on parameters ϕ and θ is obtained to have an unbiased autocorrelation function

- If the parameters ϕ and θ solve the following equation:

$$2\lambda^S\theta-(1+\theta^2-6\phi\theta)\lambda^{S-1}+6\phi\lambda^{S-2}+2\lambda\theta-(1+\theta^2)=0$$

  the autocorrelation function is **unbiased**

- If the following inequality holds:

$$2\lambda^S\theta-(1+\theta^2-6\phi\theta)\lambda^{S-1}+6\phi\lambda^{S-2}+2\lambda\theta-(1+\theta^2)>0$$

  - the autocorrelation function is **upward biased**;

$$2\lambda^S\theta-(1+\theta^2-6\phi\theta)\lambda^{S-1}+6\phi\lambda^{S-2}+2\lambda\theta-(1+\theta^2)<0$$

  - the autocorrelation function is **downward biased**

Periodic GARCH Models

- Time-varying coefficient model for conditional heteroskedasticity instead of fixed parameter structure
- Motivation: intraday pattern in market activity with regular opening and closure of financial markets (DEM/USD and DEM/GBP)
- Consider a modified Borel σ-field filtration in which the usual Ω_{t-1} is augmented by a process defining the stage of the periodic cycle at each point in time, say Ω^{S}_{t-1}
- Then the P-GARCH process is:

$$E\left[\tilde{\varepsilon}_t \mid \Omega^s_{t-1}\right] = 0$$

$$E\left[\tilde{\varepsilon}_t^2 \mid \Omega^s_{t-1}\right] \equiv \tilde{\sigma}_t^2 = \omega_{s(t)} + \sum_{j=1}^{p} \phi_{is(t)} \tilde{\varepsilon}^2_{t-i} + \sum_{j=1}^{q} \theta_{js(t)} \tilde{\sigma}^2_{t-j}$$

- s(t) refers to the stage of the periodic cycle at time t
- $\omega_{s(t)}$ may capture the nontrading-day effects

- $\phi_{is(t)}$ – measure of the immediate, or direct, impact of any new arrivals
- $\theta_{js(t)}$ – smooth long-term evolution in the volatility process
- But, in many empirical applications the long-term effect $\theta_{js(t)} = \theta_j$ is constant across all stages of the cycle
- A straightforward P-GARCH model:
  - s(t) = 1+[(t-1)mod S]
  - S is the length of the cycle (e.g. S = 5 if stock markets)

- Periodic ARMA representation of this process with time-varying but periodic correlation structure:

$$\tilde{\varepsilon}_t^2 = \omega_{s(t)} + \sum_{i=1}^{\max(p,q)} (\phi_{is(t)} + \theta_{is(t)}) \tilde{\varepsilon}_{t-i}^2 - \sum_{j=-1}^{q} \theta_{js(t)} \tilde{v}_{t-j} + \tilde{v}_t$$

with $\tilde{v}_t \equiv \tilde{\varepsilon}_t - E\left[\tilde{\varepsilon}_t^2 \mid \Omega_{t-1}^s\right] \equiv \tilde{\varepsilon}_t^2 - \tilde{\sigma}_t^2$

- ML- and quasi-ML-estimation of coefficients

Periodic Markov Switching Models

- Modeling seasonal mean shifts caused by changes in the regime
- Application:
  - Aggregate macroeconomic time series
  - historical wheat prices
- Consider a univariate time series process with the following stochastic structure:

$$\{y_t - \mu[(i_t, s_t)]\} = \phi\{y_{t-1} - \mu[(i_{t-1}, s_{t-1})]\} + \varepsilon_t$$

where the intercept shift function is:

$$\mu[(i_t, s_t)] = \alpha_0 + \alpha_1 i_t + \sum_{s=1}^{S-1} \delta_{st} \alpha_s$$

- (i_t, s_t) denotes the state of world which is a binary stochastic switching regime process $\{i_t\}$ and the seasonal indicator process $\{s_t\}$
- $s_t = t \bmod(S)$ where S is the frequency of sampling throughout the year

- Transition matrix determines the probability of being in state k = 0,1 in period t conditional on being in state k = 0,1 in period t-1:

	0	1
0	$q(s_t)$	$1- q(s_t)$
1	$1-p(s_t)$	$p(s_t)$

  – If p and q are constant we obtain the standard homogeneous Markov chain model

- Let $\{i_t\}$ be an AR(1) process: $i_t = [1 - q(s_s)] + \lambda(s_t) i_{t-1} + v_t(s_t)$
  – where $\lambda(s_t) \equiv -1 + p(s_t) + q(s_t) = \lambda^s \quad for \quad s_t = s$

    for $i_{t-1}=1$ $$v_t(s_t) = \begin{cases} [1 - p(s_t)] & with \quad probability \quad p(s_t) \\ -p(s_t) & with \quad probability \quad 1 - p(s_t) \end{cases}$$

    for $i_{t-1}=0$ $$v_t(s_t) = \begin{cases} -[1 - q(s_t)] & with \quad probability \quad q(s_t) \\ q(s_t) & with \quad probability \quad 1 - q(s_t) \end{cases}$$

  – this is a periodic AR(1) model with seasonally varying parameter values

- Let define the process $\{y_t\}$ as follows:

$$y_t = \alpha_0 + \alpha_1 i_t + \sum_{s=1}^{S-1} \delta_{st}\alpha_s + (1-\phi L)^{-1}\varepsilon_t$$

$$\Leftrightarrow y_t - \sum_{s=1}^{S-1} \delta_{st}\alpha_s = \alpha_0 + \alpha_1 i_t + (1-\phi L)^{-1}\varepsilon_t$$

where ε_t is Gaussian and independent and the left-hand-side of the equation is the sum of two independent unobserved processes, $\{i_t\}$ and $(1-\phi L)^{-1}\varepsilon_t$

- The left-hand-side of this equation have the following linear time series representation:

$$s_y(z) = \alpha_1^2 s_i(z) + \frac{1}{(1-\phi z)(1-\frac{\phi}{z})}\frac{\sigma^2}{2\pi}$$

which has hidden periodic properties and inherits the nonlinear predictable features of $\{i_t\}$

Thank you for your attention!