A nonparametric test for seasonal unit roots

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The basic idea

We consider a nonparametric test for the null of seasonal unit roots in quarterly and monthly time series that builds on the RUR (range unit root) test by Aparicio, Escribano, and Sipols (2006, *Journal of Time Series Analysis*) , so we tentatively use the name RURS (for RUR–seasonal).

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Tests for seasonal unit roots

Given quarterly data, null hypotheses H_{0-} and H_{0i} are that the polynomial operator $\Phi(z)$ in

$$\Phi(B)x_t = \varepsilon_t$$

has roots -1 and $\pm i$. x_t may be superseded by deterministic cycles.

The most customary test is due to Hylleberg, Engle, Granger, Yoo (1990), the HEGY test: fully parametric, based on the AR model above.

Tests with hypothesis reversal: Caner (1998), Canova and Hansen (1995), Tam and Reinsel (1997).

Nonparametric tests for unit roots

Why consider nonparametric tests?
robust to *nonlinear transformations*, level shifts, outliers, and other *natural extensions* of the maintained model. Under regular conditions, they have typically *less power* than parametric tests.

What are nonparametric tests?
Parametric tests typically base statistics on sample moments (e.g. conditional correlation). Nonparametric tests use other characteristics, such as sign changes, extrema, range expansion, zero crossings.

Literature on nonparametric unit-root tests

- Granger and Hallman (1991, *Journal of Time Series Analysis*)
- Burridge and Guerre (1996, *Econometric Theory*)
- So and Shin (2001, *Journal of Econometrics*)
- Aparicio, Escribano, and Sipols (2006, *Journal of Time Series Analysis*, AES, the **RUR test**)

These authors treat unit roots at $+1$ only.

The maintained model

Consider the observed process (x_t),

$$x_t = x_t^* + \sum_{j=1}^{4} \gamma_j^* D_{jt} + c^* t,$$

with

$$\phi(B) x_t^* = \varepsilon_t,$$

where $\phi(z)$ is a finite-order polynomial.

The spectral representation

$$
\begin{aligned}
\Delta_4 x_t &= \mu + a_1 x_{t-1}^{(1)} + a_2 \left(x_{t-1}^{(2)} - \gamma_2 (-1)^t \right) \\
&\quad + a_3 \Delta_2 x_{t-2} + a_4 \Delta_2 x_{t-1} \\
&\quad - \left(a_3 + a_4 \right)^2 \left(\gamma_3 s_{t-2} + \gamma_4 s_{t-1} \right) + z_t,
\end{aligned}
$$

with the notation

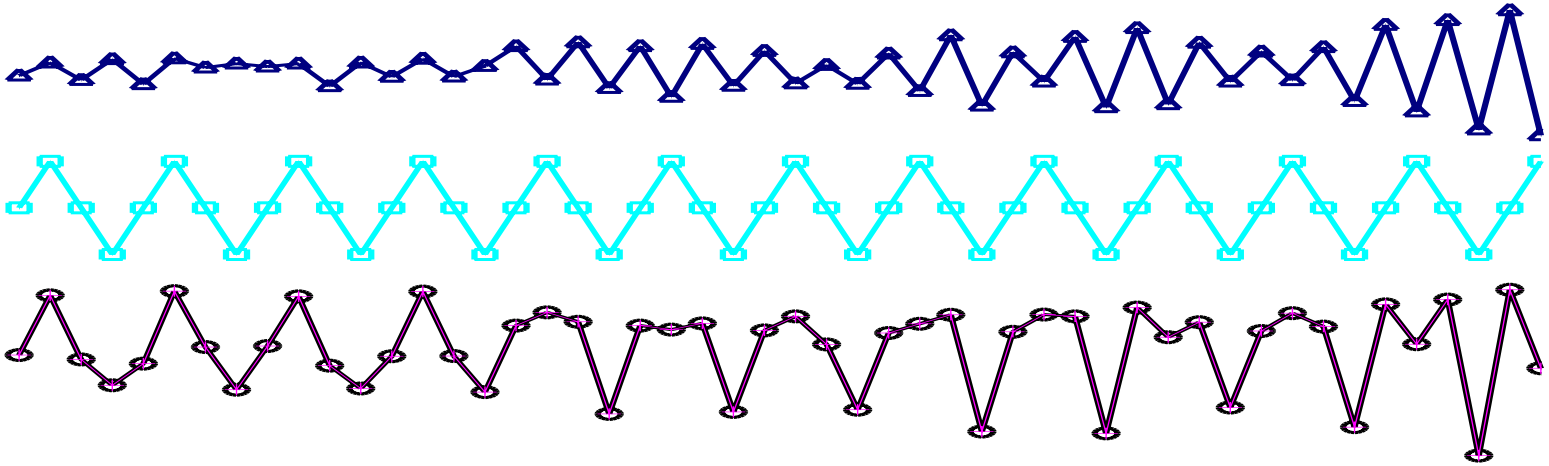
$$
\begin{aligned}
x_t^{(1)} &= x_t + x_{t-1} + x_{t-2} + x_{t-3}, \\
x_t^{(2)} &= x_t - x_{t-1} + x_{t-2} - x_{t-3}, \\
\Delta_m x_t &= x_t - x_{t-m}, \quad m = 2, 4, \\
s_t &= \sin \frac{\pi t}{2}.
\end{aligned}
$$

Equivalence of the representations

The decomposition form and the spectral representation are equivalent. The variable (x_t) is integrated

- at frequency $0 \Leftrightarrow a_1 = 0$ (hypothesis H_{0+}),
- at frequency $\pi \Leftrightarrow a_2 = 0$ (H_{0-}),
- at frequency $\frac{\pi}{2} \Leftrightarrow a_3 = a_4 = 0$ (H_{0i}).

A direct restriction test on a_j would yield a HEGY-type test.



Observed process at the bottom is the sum of two frequency components. It has a unit root at -1 but a deterministic cycle at $\pm i$. It is in $H_{0-} \cap H_{0i}^{C}$.

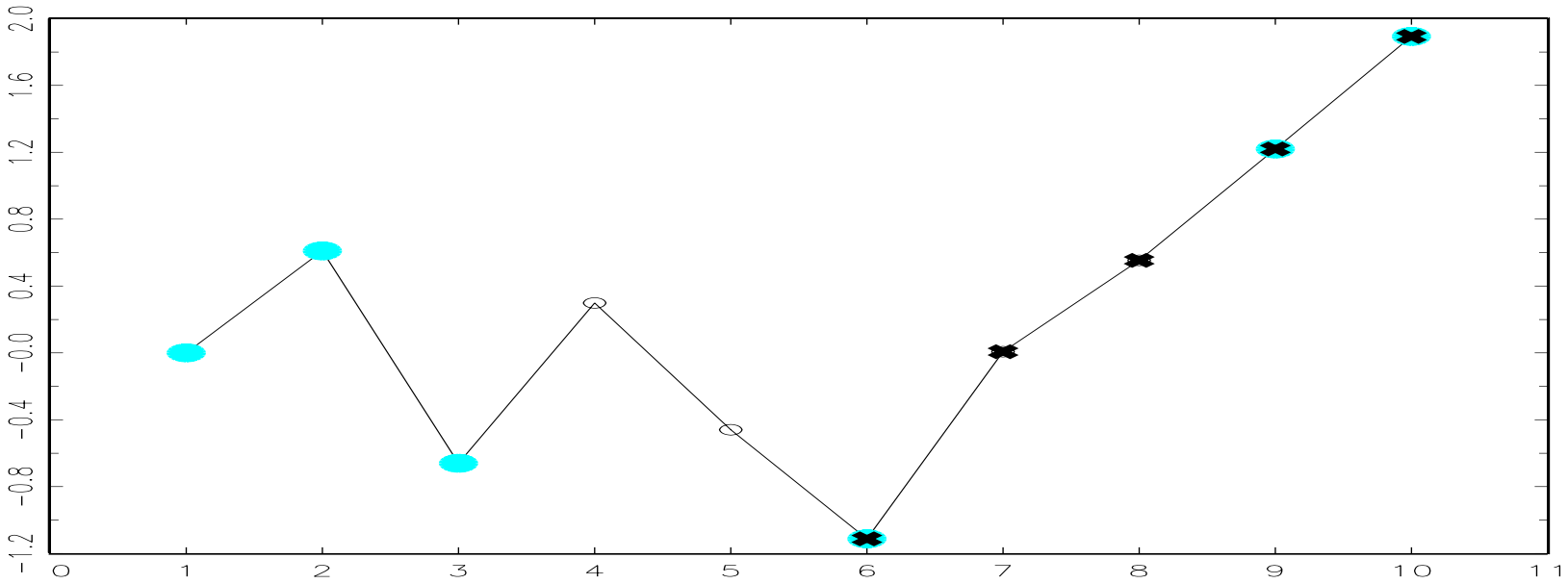
The record count

For a given realization $(x_t, t = 1, \ldots, n)$, define

$$x_{j,j} = \max_{t=1,\ldots,j} x_t,$$
$$x_{1,j} = \min_{t=1,\ldots,j} x_t.$$

Then, $x_{j,j} - x_{1,j}$ defines the sequence of *ranges* of the series. Any time it increases over $j = 1, \ldots, n$, this is called a *record*. The number of records until n is denoted as $R^{(x)}(n)$ or $R(n)$.

An example for counting records



Filled-in circles mark records from the left, crosses mark records from the right.

The RUR test

AES show that:

- $R(n) = O(n^{1/2})$ for a random walk with independent increments;
- $R(n) = O(\log n)$ for many stationary processes.

Thus, the RUR (range unit root) statistic

$$J_0^{(n)} = n^{-1/2} R^{(x)}(n)$$

can be the basis for a consistent test, if the null is a random walk and the alternative is stationarity.

RUR test: good and bad news

Good news. The statistic is invariant to monotonic transformations of the observed variable. x and $\log x$ yield the same RUR statistic. It is reasonably robust to level shifts and to distributional assumptions. Under ideal conditions, its power is acceptable.

Bad news. The test has less power than the DF test. It is sensitive to correlation in the increments of the random walk under the null. It is not even asymptotically similar for a general I(1) null.

The idea of the nonparametric RURS test

Assume (x_t) follows a *seasonal random walk* $x_t = x_{t-4} + \varepsilon_t$. Consider the transformed variables

$$
\begin{aligned}
x_t^{[1]} &= x_t + x_{t-1} + x_{t-2} + x_{t-3}, \quad t = 1, \ldots, n, \\
x_t^{[2]} &= (-1)^t (x_t - x_{t-1} + x_{t-2} - x_{t-3}), \quad t = 1, \ldots, n, \\
x_t^{[3]} &= (-1)^t \Delta_2 x_{2t}, \quad t = 1, \ldots, \frac{n}{2}, \\
x_t^{[4]} &= (-1)^t \Delta_2 x_{2t-1}, \quad t = 1, \ldots, \frac{n}{2},
\end{aligned}
$$

which then yield pure random walks.

The RURS statistics

The statistics

$$
\begin{aligned}
J_1 &= n^{-1/2} R_n^{x[1]}, \\
J_2 &= n^{-1/2} R_n^{x[2]}, \\
J_3 &= (\frac{n}{2})^{-1/2} R_{\frac{n}{2}}^{x[3]}, \\
J_4 &= (\frac{n}{2})^{-1/2} R_{\frac{n}{2}}^{x[4]}
\end{aligned}
$$

are the RURS (range unit roots seasonal) statistics. J_1 tests H_{0+}, J_2 tests H_{0-}, J_3 and J_4 test H_{0i}.

Assumption

There is a representation

$$(1-B)^{m_1}(1+B)^{m_2}(1+B^2)^{m_3}x_t = \tilde{x}_t,$$

such that (x_t) *is stationary, where* $m_j \in \{0,1\}$ *for* $j = 1,\ldots,3$. *The representation is unique if* m_j *is defined as the minimum value that achieves stationarity in* $\tilde{x}_t$.

Assumption

If at least one of the values m_j *is 0, the stationary process* $(\tilde{x}_t)$ *considered in Assumption 1 fulfills a Berman condition. A stationary process with autocorrelations* c_k *is said to fulfill a Berman condition if* $c_k \log k \to 0$ *as* $k \to \infty$.

Asymptotic properties of the RURS statistic

Theorem

Under assumptions 1 and 2, the following two properties hold:

(a) *If (x_t) is a seasonal random walk with regular i.i.d. increments and thus is an element of $H_{0+} \cap H_{0-} \cap H_{0i}$, the distribution of all statistics $J_j, j = 1, \ldots, 4$ converges to the law indicated by AES;*

(b) *If (x_t) is in the alternative of any of the three hypotheses H_{0+}, H_{0-}, H_{0i}, the corresponding test statistic J_j will converge to 0 as $n \to \infty$.*

Handling RUR non-similarity

Assume (x_t) is I(1) but not a random walk. We suggest to eliminate serial correlation under the null by regressing Δx_t on p BIC–selected lags ('augmentation')

$$\Delta x_t = \mu + \sum_{k=1}^{p} \gamma_k \Delta x_{t-k} + u_t, t \geq p+2.$$

Accumulate estimation residuals $\hat{u}_t$ according to

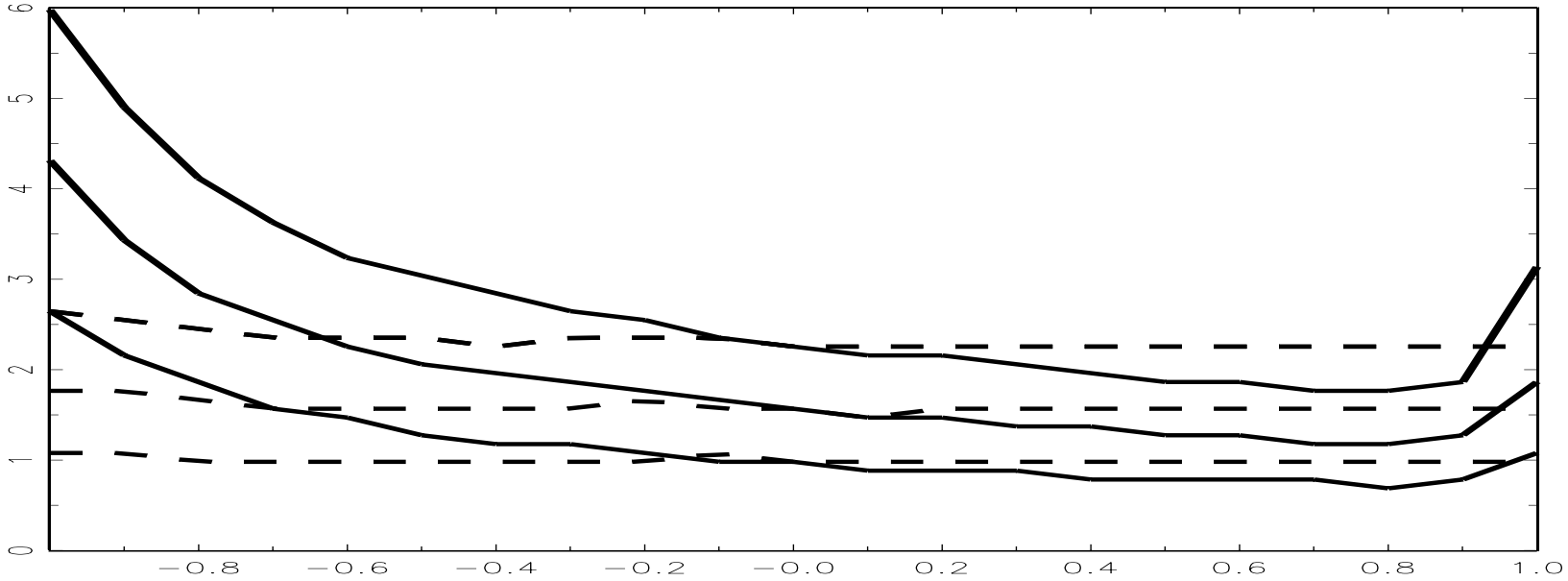
$$\tilde{x}_t = \sum_{j=p+2}^{t} \hat{u}_j + x_{p+1},$$

such that $\tilde{x}_t$ is ideally a pure random walk without drift. This is done for all constructed variables $x^{(1)}$, $x^{[2]}$, $x^{[3]}$, $x^{[4]}$.

Handling deterministic terms

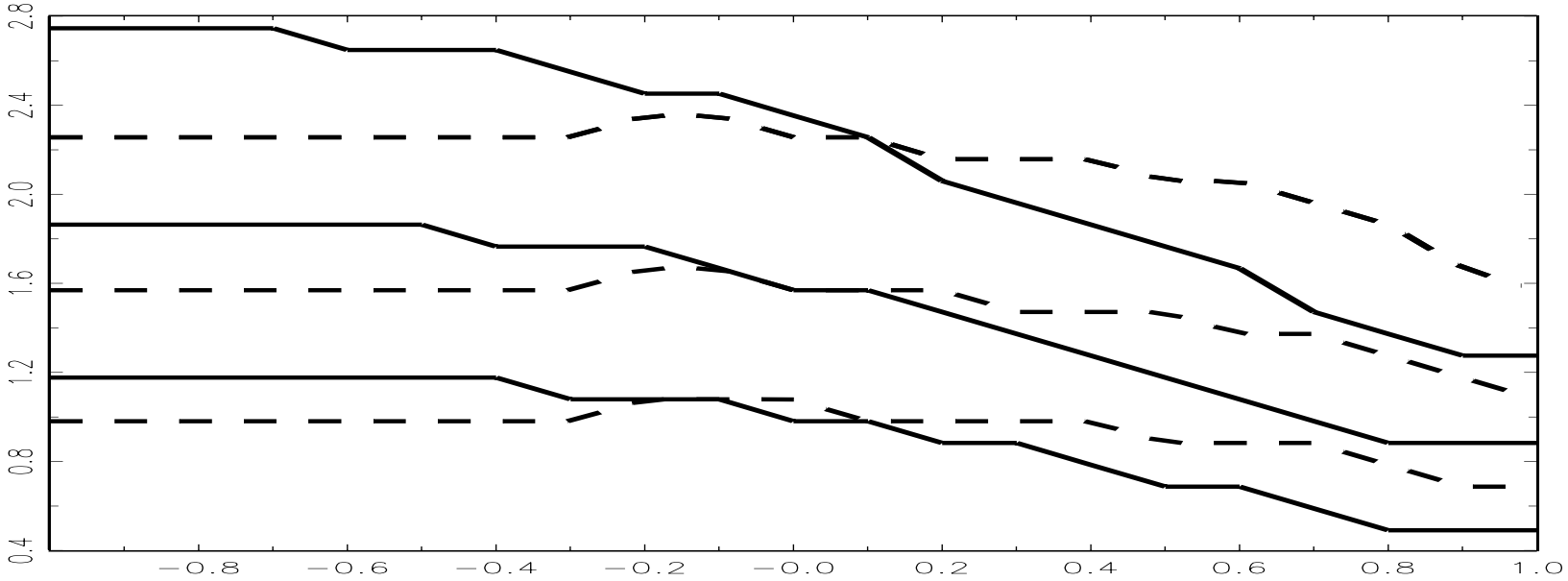
AES suggest a two-sided RUR test: left tail supports the stationary alternative, mode supports random walk null, right tail supports 'drifting' alternatives.
The RURS construction eliminates drifts and deterministic seasonal patterns. The RURS test is conducted as a *one-sided* test.

Correction for autocorrelation works: AR disturbances



10%, 50%, and 90% quantiles for uncorrected (solid) and for augmentation-corrected (dashed) RURS statistic $J_2^{(n)}$ if calculated from trajectories of length $n = 100$ from the data-generation process $\Delta_4 x_t = \phi \Delta_4 x_{t-1} + \varepsilon_t$. ϕ values on the abscissa.

Correction for autocorrelation works: MA disturbances



10%, 50%, and 90% quantiles for uncorrected (solid) and for augmentation-corrected (dashed) RURS statistic $J_2^{(n)}$ if calculated from trajectories of length $n = 100$ from the data-generation process $\Delta_4 x_t = \varepsilon_t + \theta\varepsilon_{t-1}$. θ values on the abscissa.

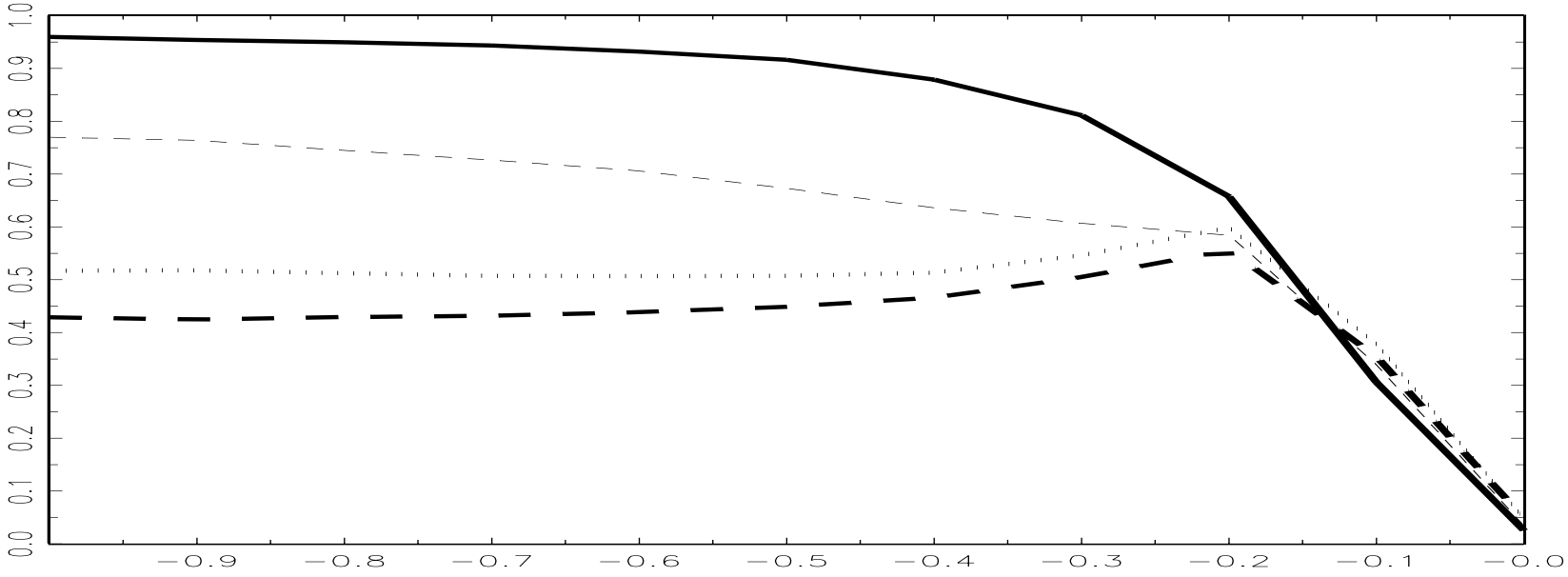
The RURS-fb test

AES suggest to increase test power by running the record count forward $(t = 1, \ldots, n)$ and backward $(t = n, \ldots, 1)$. Consider the RUR-fb statistic

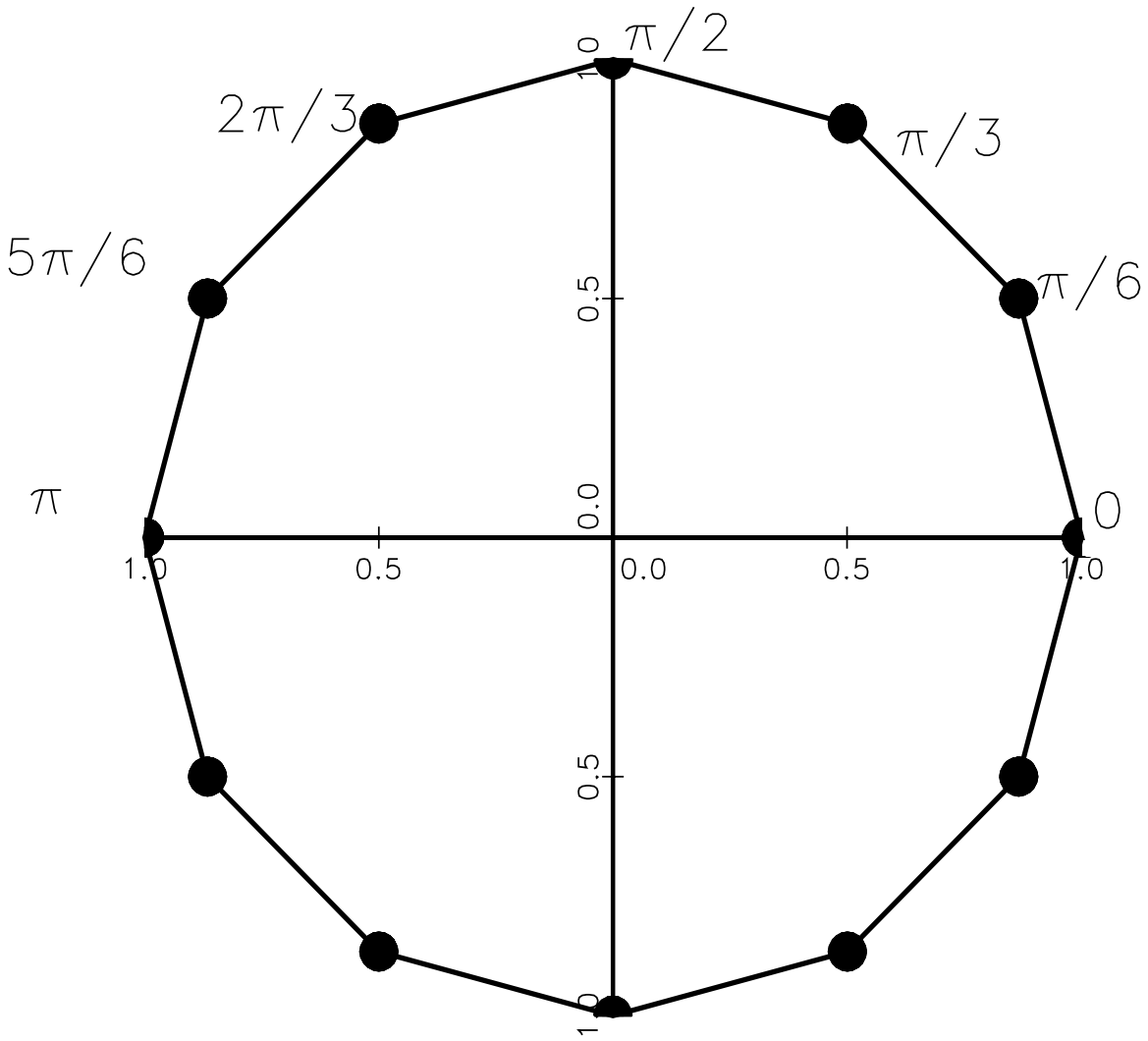
$$J_* = 2^{-\frac{1}{2}}(J_0 + J_0')$$

with a modified limit law under the random-walk null.
The same can be done for the RURS test: RURS-fb test.
RURS-fb achieves some improvement in test power over the one-sided RURS test.

The power of the RURS test



Empirical rejection frequency of unit-root hypothesis H_{0-} using RURS-fb tests for 500 observations at nominal 5% significance level. Upper bound for augmentation varies. Solid curve: no augmentation, short dashes: $N = n^{1/4}$, dots: $N = n^{1/3}$, long dashes: $N = 2n^{1/3}$. Generating model $\Delta_4 x_t = a_2 x_{t-1}^{(2)} + \varepsilon_t$ with N(0,1) errors ε_t, a_2 on the abscissa axis, 20000 replications.


π/2
2π/3
π/3
5π/6
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0
1.0
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1.0
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1.0

Well-behaved and other cases

Problem: The transformation to random walks works for 3 out of 7 cases only: ± 1 ($\omega = 0, \pi$), and $\pm i$ ($\omega = \pi/2$).
At other frequencies, the asymptotic distribution of the RURS statistic is unknown. The expansion rate is the same.

Solution: either use significance points tuned to the 4 problematic frequencies or generate random walks in the accumulation step.
Both solutions yield comparable test power.

Example: testing at frequency $\pi/6$

Assume a monthly SRW $x_t = x_{t-12} + \varepsilon_t$. The transformation

$$y_t = (1+\sqrt{3}B+2B^2+\sqrt{3}B^3+B^4-B^6-\sqrt{3}B^7-2B^8-\sqrt{3}B^9-B^{10})x_t$$

yields a pure unit-root process of the form $(1-\sqrt{3}B+B^2)y_t = \varepsilon_t$ at angular frequency $\pi/6$.
Consider the auxiliary regression

$$(1-\sqrt{3}B+B^2)y_t = \mu + \sum_{j=1}^{p}(1-\sqrt{3}B+B^2)y_{t-j} + u_t, t \geq p+3.$$

Purged trajectories evolve from accumulating estimation residuals

$$\tilde{y}_t = \sqrt{3}\tilde{y}_{t-1} - \tilde{y}_{t-2} + \hat{u}_t, t \geq p+3.$$

The same can be done for frequencies $\frac{\pi}{3}$, $\frac{2\pi}{3}$, $\frac{5\pi}{6}$.

Simulated quantiles for the monthly RURS test

Table: Empirical significance points for the monthly RURS test statistics for $n = 100$.

statistic	frequency	n	1%	5%	10%	median
J_0	0	100	0.995	1.194	1.293	2.059
J_1	$\pi/6$	100	0.697	0.995	1.095	1.806
J_2	$\pi/3$	100	0.498	0.796	0.995	1.678
J_3	$\pi/2$	100	0.885	1.180	1.327	1.959
J_4	$2\pi/3$	100	0.597	0.896	0.995	1.816
J_5	$5\pi/6$	100	0.498	0.697	0.896	1.601
J_6	π	100	0.995	1.194	1.393	2.058

Note: Empirical quantiles from 10,000 replications. Generating model is the SRW $x_t = x_{t-12} + \varepsilon_t$ with i.i.d. N(0,1) errors.

Simulated quantiles for the monthly RURS-fb test

Table: Empirical significance points for the monthly RURS-fb test statistics for $n = 200$.

statistic	frequency	n	1%	5%	10%	median
J_0	0	200	1.439	1.652	1.812	2.345
J_1	$\pi/6$	200	1.439	1.652	1.759	2.345
J_2	$\pi/3$	200	1.439	1.652	1.759	2.345
J_3	$\pi/2$	200	1.410	1.632	1.773	2.337
J_4	$2\pi/3$	200	1.439	1.652	1.759	2.345
J_5	$5\pi/6$	200	1.439	1.652	1.759	2.345
J_6	π	200	1.439	1.652	1.812	2.345

Note: Empirical quantiles from 10,000 replications. Generating model is the SRW $x_t = x_{t-12} + \varepsilon_t$ with i.i.d. N(0,1) errors.

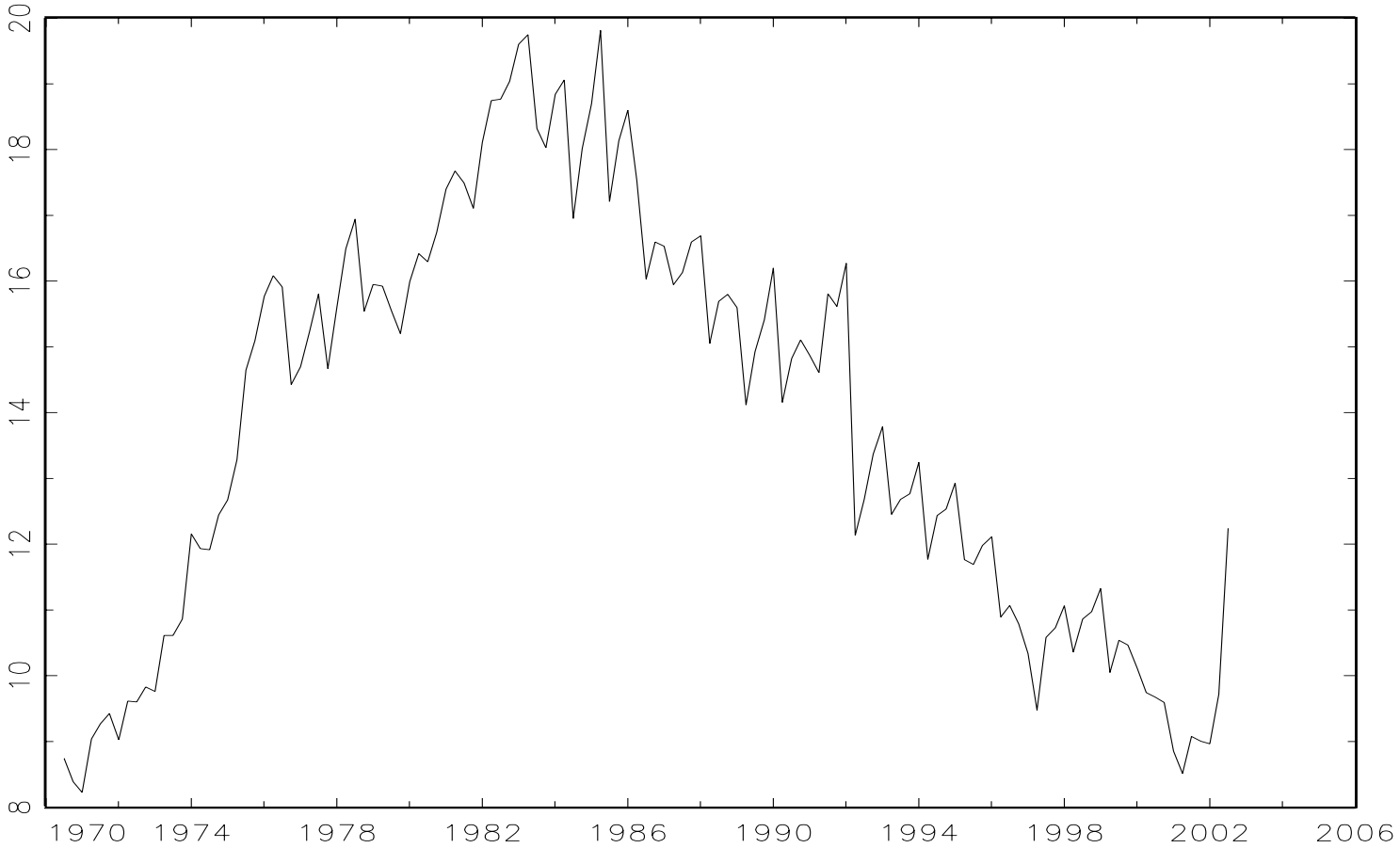

Figure: Belgian barley prices, 1971:1–2003:1, quarterly observations.

RURS and HEGY yield different results

Traditional parametric HEGY tests reject unit roots at -1 and at $\pm i$ and indicate purely deterministic seasonal variation.
RURS statistics are $J_1 = 2.03$, $J_2 = 1.85$, and $J_3 = J_4 = 1.00$.
Unit roots at ± 1 are supported, while $\pm i$ is rejected at the 5% level. The RURS test finds a deterministic annual cycle and a persistently changing semi-annual pattern.

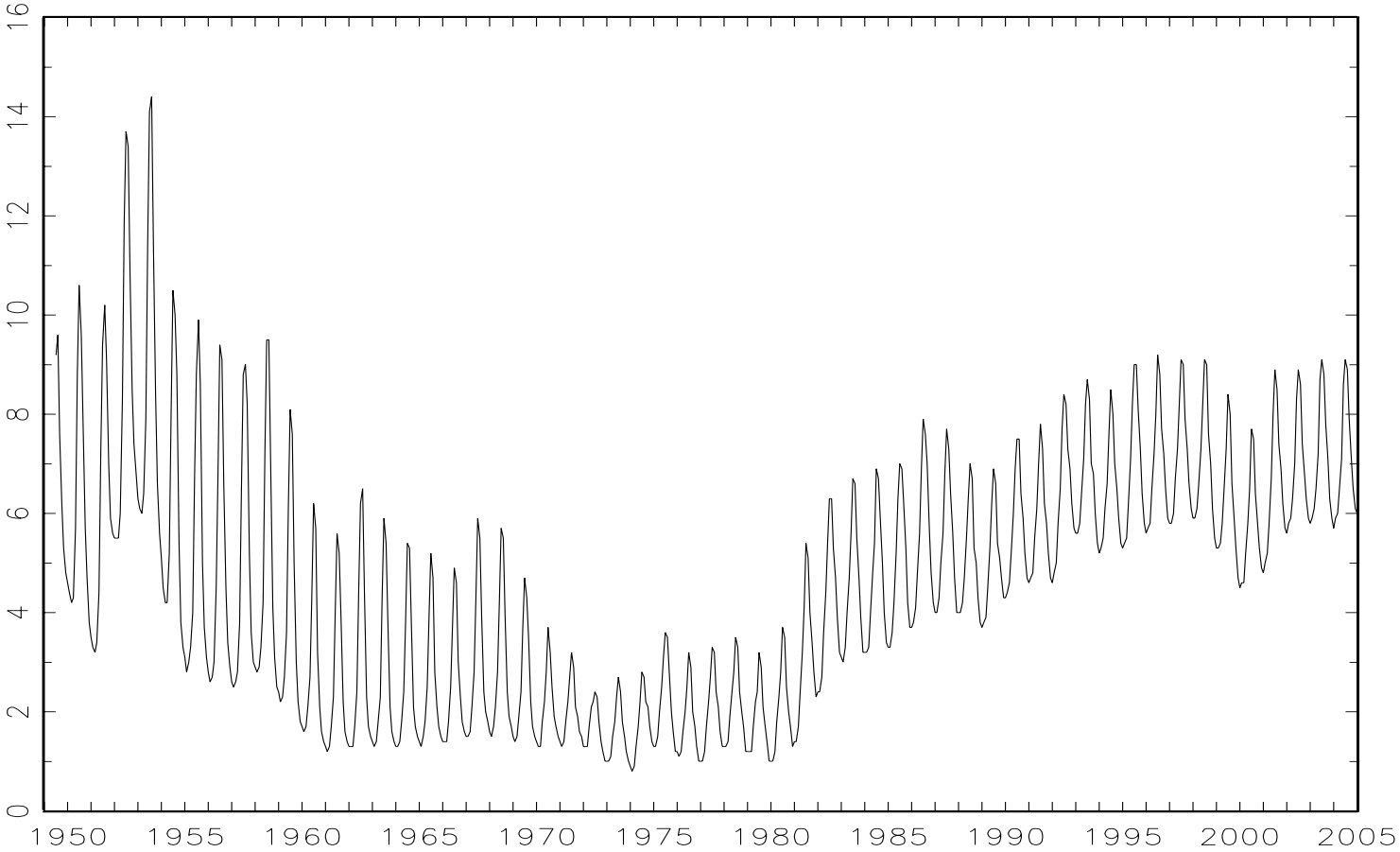

Figure: Austrian unemployment rate, 1950:1–2005:12, monthly observations.

Test results for the unemployment rate

RURS rejects at most frequencies at 5% , excepting the frequency $2\pi/3$. Seasonality of the unemployment rate is primarily deterministic, most apparent pattern changes can be described by an unspecified nonlinear transformation of a stationary variable with an added deterministic cycle.

RURS-fb accepts seasonal unit roots at most frequencies, including the long-run frequency, but not at -1 and at $\pm i$. There is a random-walk type trend in the data, main backbone of the annual seasonal cycle is deterministic, details of variation from month to month experience persistent changes.

HEGY test does not reject unit roots at the low frequencies but rejects at $\pi/2$ and at all higher frequencies. It sees the unemployment rate as driven by longer stochastic waves and by short-run cycles of fixed structure.

- The RURS test is a useful additional tool for testing unit roots in seasonal variables.
- The test is invariant to monotonic transformations and deterministic seasonal variation. It is indeed robust to breaks and outliers.
- In standard situations, RURS performance is much worse than parametric test performance.
- The test is interesting if nonlinear transformations of random walks etc. are suspected (unemployment rates).
- The augmentation step should be performed with care only. A strict upper bound for p, for example $n^{1/4}$, is recommended.

Thank you for your attention